Section A: Pure Mathematics [50 marks]

	Solution
P2 Q1	Let P_n be the statement $$2\left(7^n\right)+3\left(5^n\right)-5 \text{ is divisible by } 24$$ for all $n \in \mathbb{Z}^+$. When n = 1, $2(7)+3(5)-5=24=24(1)$. $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $$2\left(7^k\right)+3\left(5^k\right)-5=24a \text{ for some integer } a.$$ We want to prove that P_{k+1} is true, i.e. $$2\left(7^{k+1}\right)+3\left(5^{k+1}\right)-5=24b \text{ for some integer } b.$$ $$\begin{aligned}2\left(7^{k+1}\right)+3\left(5^{k+1}\right)-5&=14\left(7^k\right)+15\left(5^k\right)-5\\&=12\left(7^k\right)+12\left(5^k\right)+2\left(7^k\right)+3\left(5^k\right)-5\\&=12\left[\left(7^k\right)+\left(5^k\right)\right]+24a.\end{aligned}$$ Since 7^k and 5^k are both odd, $\left(7^k\right)+\left(5^k\right)$ is even. Thus, $\left(7^k\right)+\left(5^k\right)=2c$ for some integer c. Therefore, $$\begin{aligned}12\left[\left(7^k\right)+\left(5^k\right)\right]+24a&=12(2c)+24a\\&=24(c+a) \text{ where } c+a \in \mathbb{Z}.\end{aligned}$$ $\therefore\ P_k \Rightarrow P_{k+1}$. Since P_1 is true, and $P_k \Rightarrow P_{k+1}$, by mathematical induction, P_n is true for all positive integers n.

	Solution
P2 Q2	**Method 1** $x=\sin t \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t}=\cos t=\sqrt{1-x^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\sin t=-x$ Then, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}\right) \quad \left[\text{Note: } \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\frac{\mathrm{d}x}{\mathrm{d}t}\right]$ $=\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\frac{\mathrm{d}x}{\mathrm{d}t}\right]\times\frac{\mathrm{d}x}{\mathrm{d}t}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)$ (by product rule) $=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+(-x)\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\left(\sqrt{1-x^2}\right)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}$ $=\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}$ Substituting into the original differential equation: $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+4y=0 \quad \Rightarrow \quad \frac{\mathrm{d}^2y}{\mathrm{d}t^2}+4y=0$ (shown) **Method 2** $x=\sin t \Rightarrow t=\sin^{-1}x \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{1-x^2}$ Thus, $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}$ (1) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}t}$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{1-x^2}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\times\sqrt{1-x^2}$ $=\left(\sqrt{1-x^2}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-\frac{x}{\sqrt{1-x^2}}\frac{\mathrm{d}y}{\mathrm{d}x}\right)\sqrt{1-x^2}$ $=\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}$ (2) Substituting (1) and (2) into the DE: $\left(1-x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+4y=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+4y=0$ (shown) **Method 2** $x=\sin t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=\cos t \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x}=\sec t$ Thus, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\sec t\frac{\mathrm{d}y}{\mathrm{d}t}$ (1)

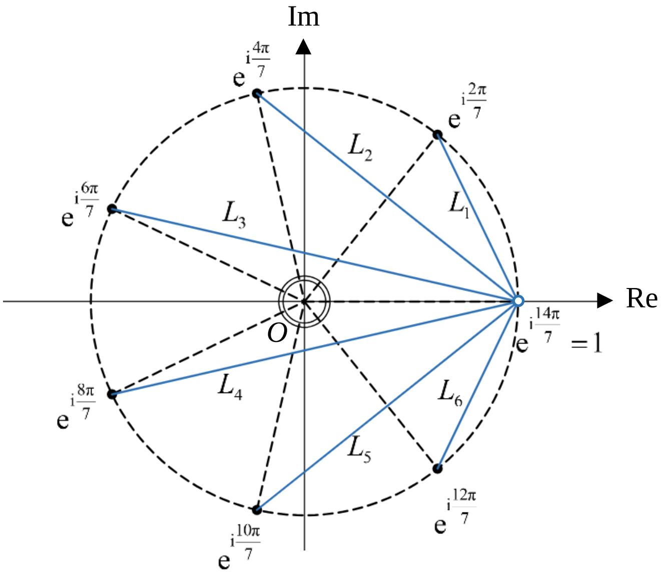
	$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$ $= \frac{d}{dt}\left(\sec t \frac{dy}{dt}\right) \times \sec t$ $= \left(\sec t \frac{d^2y}{dt^2} + \sec t \tan t \frac{dy}{dt}\right)\sec t$ $= \sec^2 t \frac{d^2y}{dt^2} + \sec^2 t \tan t \frac{dy}{dt}$ (2) Substituting (1) and (2) into the DE: $\left(1-\sin^2 t\right)\left(\sec^2 t \frac{d^2y}{dt^2} + \sec^2 t \tan t \frac{dy}{dt}\right) - \sin t \sec t \frac{dy}{dt} + 4y = 0$ $\cos^2 t\left(\sec^2 t \frac{d^2y}{dt^2} + \sec^2 t \tan t \frac{dy}{dt}\right) - \frac{\sin t}{\cos t} \cdot \frac{dy}{dt} + 4y = 0$ $\frac{d^2y}{dt^2} + \tan t \frac{dy}{dt} - \tan t \frac{dy}{dt} + 4y = 0$ $\frac{d^2y}{dt^2} + 4y = 0$ (shown) Auxiliary equation for reduced DE is $m^2 + 4 = 0 \Rightarrow m = \pm 2\mathrm{i}$ General solution is $y = e^{(0)t}\left(A\cos 2t + B\sin 2t\right)$ $= A\left(1-2\sin^2 t\right) + B\left(2\sin t \cos t\right)$ $= A\left(1-2x^2\right) + B\left(2x\sqrt{1-x^2}\right)$ $\left[\text{Since } x = \sin t \text{ and } \sqrt{1-x^2} = \cos t\right]$ $= A\left(1-2x^2\right) + Cx\sqrt{1-x^2}$, where $C = 2B$ and A are arbitrary constants.

	Solution
P2 Q3	**(a)** $\det(\mathbf{M}-\lambda\mathbf{I})=\begin{vmatrix}1-\lambda & 1\\ 3 & -1-\lambda\end{vmatrix}=\lambda^2-1-3=(\lambda+2)(\lambda-2)$ so the eigenvalues of **M** are –2 and 2. Let $\lambda_1=-2$, $\left(\begin{array}{cc\|c}3&1&0\\3&1&0\end{array}\right)\xrightarrow[\frac{1}{3}R_1]{R_2-R_1}\left(\begin{array}{cc\|c}1&\frac{1}{3}&0\\0&0&0\end{array}\right)$. Corresponding eigenvector $\mathbf{e}_1=\begin{pmatrix}1\\-3\end{pmatrix}$ Let $\lambda_2=2$, $\left(\begin{array}{cc\|c}-1&1&0\\3&-3&0\end{array}\right)\xrightarrow[-R_1]{R_2+3R_1}\left(\begin{array}{cc\|c}1&-1&0\\0&0&0\end{array}\right)$. Corresponding eigenvector $\mathbf{e}_2=\begin{pmatrix}1\\1\end{pmatrix}$ So the eigenvalues of **M** are –2 and 2 and their corresponding eigenvectors are $\begin{pmatrix}1\\-3\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$. **(b)** Hence, we have $\mathbf{M}=\begin{pmatrix}1&1\\-3&1\end{pmatrix}\begin{pmatrix}-2&0\\0&2\end{pmatrix}\begin{pmatrix}1&1\\-3&1\end{pmatrix}^{-1}$ Thus $\mathbf{M}^n=\left[\begin{pmatrix}1&1\\-3&1\end{pmatrix}\begin{pmatrix}-2&0\\0&2\end{pmatrix}\begin{pmatrix}1&1\\-3&1\end{pmatrix}^{-1}\right]^n=\begin{pmatrix}1&1\\-3&1\end{pmatrix}\begin{pmatrix}-2&0\\0&2\end{pmatrix}^n\begin{pmatrix}1&1\\-3&1\end{pmatrix}^{-1}$ $=\frac{1}{4}\begin{pmatrix}1&1\\-3&1\end{pmatrix}\begin{pmatrix}(-2)^n&0\\0&2^n\end{pmatrix}\begin{pmatrix}1&-1\\3&1\end{pmatrix}$ $=\frac{1}{4}\begin{pmatrix}3(2^n)+(-2)^n & 2^n-(-2)^n\\ 3(2^n)-3(-2)^n & 2^n+3(-2)^n\end{pmatrix}$ **(c)** **<u>Method 1</u>** $\mathbf{M}^n=2^{n-2}\begin{pmatrix}3+(-1)^n & 1-(-1)^n\\ 3-3(-1)^n & 1+3(-1)^n\end{pmatrix}$. As $(-1)^n=\begin{cases}-1 & \text{if } n \text{ is odd}\\ 1 & \text{if } n \text{ is even}\end{cases}$, $3-3(-1)^n=1-(-1)^n=0$ when n is even So $\mathbf{M}^n$ is a diagonal matrix for all even values of n. **<u>Method 2</u>** As $(-2)^n=2^n$ when n is even, $\mathbf{M}^n=\begin{pmatrix}1&1\\-3&1\end{pmatrix}\begin{pmatrix}2^n&0\\0&2^n\end{pmatrix}\begin{pmatrix}1&1\\-3&1\end{pmatrix}^{-1}$ for even values of n $=2^n\mathbf{QIQ}^{-1}=2^n\mathbf{I}$ which is a diagonal matrix. So $\mathbf{M}^n$ is a diagonal matrix for all even values of n.

	Solution
P2 Q4	**(a)** By the reflective property of the parabola, any incoming light ray parallel to its axis will reflect off the parabola and pass through the focus of the parabola. If this is also a focus of the ellipse, then by the reflective property of the ellipse, the light ray will reflect off the ellipse and pass though the other focus of the ellipse, which is the point *S*. Hence, incoming light rays which are parallel to the axis will all reflect off both mirrors and pass through point *S*. **(b)** $y^2 = 37.6x = 4(9.4)x$. Hence the focus *F* of the parabola is at $(9.4, 0)$. **(c)** The centre *K* of C_2 is at the midpoint of *SF*, which is $(4.6, 0)$. The semimajor axis is $a = 9.6 - 4.6 = 5$ The distance of the foci from the centre, $c = 9.4 - 4.6 = 4.8$. Since $c^2 = a^2 - b^2$, we have $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4.8^2} = 1.4$ Hence, the equation of the ellipse C_2 is $\frac{(x-4.6)^2}{5^2} + \frac{y^2}{1.4^2} = 1$. **(d)** Consider a light ray travelling along the line $y = 4.2$. When it touches the parabola, $y^2 = 37.6x \Rightarrow x = \frac{4.2^2}{37.6} = 0.469149$. Hence, the point where it touches the parabola is $(0.469149, 4.2)$. Since it reflects off the parabola and passes through its focus at $(9.4, 0)$, the reflected light ray has equation $\frac{y}{x-9.4} = \frac{4.2}{0.469149 - 9.4}$ $y = -0.470280(x - 9.4)$ Substitute this into the equation of the ellipse: $\frac{(x-4.6)^2}{5^2} + \frac{0.470280^2 (x-9.4)^2}{1.4^2} = 1$ $0.15284x^2 - 2.48936x + 9.81680 = 0$ Using GC, $x = 9.5898$ or $x = 6.6977$ (reject since $x > 9.4$). Then $y = -0.470280(x - 9.4) = -0.0893$. By symmetry, the light ray travelling along the line $y = -4.2$ will reflect off the parabola and intersect the ellipse at $y = 0.0893$. Hence, the range of *y*-coordinates of the ellipse must include $-0.0893 \le y \le 0.0893$.

	Solution
P2 Q5	**(a)(i)** $z_k = \mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}}$, where $k = 1, 2, \ldots, n$. **(a)(ii)** **<u>Method 1</u>** Note that $z_{n-k} = \mathrm{e}^{\mathrm{i}\frac{2(n-k)\pi}{n}} = \mathrm{e}^{\mathrm{i}2\pi}\mathrm{e}^{\mathrm{i}\frac{-2k\pi}{n}} = \mathrm{e}^{\mathrm{i}\frac{-2k\pi}{n}} = z_k{}^*$, so $z_k z_{n-k} = \lvert z_k \rvert^2 = 1$. If n is odd, $\frac{n-1}{2} \in \mathbb{Z}^+$. Then for all $m \in \mathbb{Z}^+, 1 \le m \le \frac{n-1}{2}$, $\frac{2}{1+z_m} + \frac{2}{1+z_{n-m}} = 2\left[\frac{1+z_m+1+z_{n-m}}{(1+z_m)(1+z_{n-m})}\right]$ $= 2\left[\frac{2+z_m+z_{n-m}}{1+z_m+z_{n-m}+(z_m)(z_{n-m})}\right]$ $= 2\left[\frac{2+z_m+z_{n-m}}{2+z_m+z_{n-m}}\right]$ $= 2$ So $\frac{2}{1+z_1} + \ldots + \frac{2}{1+z_{n-1}} + \frac{2}{1+z_n} = \left(\frac{n-1}{2}\right)(2) + \frac{2}{1+z_n} = n-1+\frac{2}{1+1} = n$ **<u>Method 2</u>** $\frac{2}{1+z_1} + \frac{2}{1+z_2} + \frac{2}{1+z_3} + \ldots + \frac{2}{1+z_{n-1}} + \frac{2}{1+z_n} = \sum_{k=1}^{n} \frac{2}{1+z_k} = \sum_{k=1}^{n} \frac{2}{1+\mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}}}$ $= \sum_{k=1}^{n} \frac{2\mathrm{e}^{-\mathrm{i}\frac{k\pi}{n}}}{\mathrm{e}^{-\mathrm{i}\frac{k\pi}{n}} + \mathrm{e}^{\mathrm{i}\frac{k\pi}{n}}} = \sum_{k=1}^{n} \frac{2\left[\cos\left(\frac{k\pi}{n}\right) - \mathrm{i}\sin\left(\frac{k\pi}{n}\right)\right]}{2\cos\left(\frac{k\pi}{n}\right)} = \sum_{k=1}^{n}\left[1 - \mathrm{i}\tan\left(\frac{k\pi}{n}\right)\right]$ $= n - \mathrm{i}\left[\underbrace{\tan\left(\frac{\pi}{n}\right) + \tan\left(\frac{2\pi}{n}\right) + \ldots + \tan\left(\frac{(n-2)\pi}{n}\right) + \tan\left(\frac{(n-1)\pi}{n}\right)}_{\text{even number of terms since } n \text{ is odd}} + \tan\left(\frac{n\pi}{n}\right)\right]$ $= n - \mathrm{i}\left[\tan\left(\frac{\pi}{n}\right) + \tan\left(\frac{(n-1)\pi}{n}\right) + \tan\left(\frac{2\pi}{n}\right) + \tan\left(\frac{(n-2)\pi}{n}\right) + \ldots + \tan\pi\right]$ $= n - \mathrm{i}\left[\tan\left(\frac{\pi}{n}\right) - \tan\left(\frac{\pi}{n}\right) + \tan\left(\frac{2\pi}{n}\right) - \tan\left(\frac{2\pi}{n}\right) + \ldots + 0\right]$ $= n - \mathrm{i}(0+0+\ldots+0)$ $= n.$

(b)(i)&(ii)



(b)(iii)

$$\sum_{r=0}^{6} z^r = 1+z+z^2+z^3+z^4+z^5+z^6$$

$$= \frac{z^7-1}{z-1} \qquad \text{(using the sum of a geometric series)}$$

By re-writing the numerator in terms of its factors, we have

$$\sum_{r=0}^{6} z^r = \frac{(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6)(z-1)}{z-1}$$

$$=(z-z_1)(z-z_2)(z-z_3)(z-z_4)(z-z_5)(z-z_6) \quad \text{(shown)}$$

Product of lengths of all six loci

$=|z_1-1||z_2-1||z_3-1||z_4-1||z_5-1||z_6-1|$

$=|(z_1-1)(z_2-1)(z_3-1)(z_4-1)(z_5-1)(z_6-1)|$

$=|(1-z_1)(1-z_2)(1-z_3)(1-z_4)(1-z_5)(1-z_6)|$

$=\left|\sum_{r=0}^{6} 1^r\right|$ (substituting $z=1$ into the result above)

$=|1+1^0+1^2+1^3+1^4+1^5+1^6|$

$=7$

(b)(iv)

Since $\left|\frac{z_a-z_b}{z_c-z_b}\right|=1 \Rightarrow |z_a-z_b|=|z_c-z_b|$,

$z_a z_b z_c$ forms an isosceles triangle with vertex z_b and vertex angle θ.

$\theta=\frac{\pi}{7}$, $a+3=b=c-3$. (or $a=1,\ b=4,\ c=7$)

$\theta=\frac{3\pi}{7}$, $a+2=b=c-2$, $b=3, 4, 5$.

$\theta=\frac{5\pi}{7}$, $a+1=b=c-1$, $b=2, 3, 4, 5, 6$.

Section B: Probability and Statistics [50 marks]

	Solution
P2 Q6	

(a) Let the median of the differences in reaction times be *m*.

Test $\text{H}_0: m = 0$ against $\text{H}_1: m \neq 0$ at 2% level of significance

Participant	1	2	3	4	5	6	7	8	9	10
Condition A	21	24	20	22	25	24	23	27	28	32
Condition B	18	22	19	24	23	24	18	25	27	31
Sign of A – B	+	+	+	-	+	NA	+	+	+	+

Let S_- be the number of "–" out of 9 pairs = 1

S_+ be the number of "+" out of 9 pairs = 8

Under H$_0$, Test Statistic: $S = \min\{S_+, S_-\} = S_- \sim \text{B}(9, 0.5)$

Since p-value $= 2 \times \text{P}(S \leq 1) = 0.0391 > 0.02$, we do not reject H_0. Hence there is insufficient evidence at 2% significance level that there is a difference in the reaction times of participants under the two conditions.

(b) If the investigation is about whether the reaction times of participants are shorter under condition B, then we would be testing

$\text{H}_0: m = 0$, against $\text{H}_1: m > 0$

$\Rightarrow$ p-value $= \text{P}(S_- \leq 1) = 0.0195 < 0.02$

Thus we would reject H_0 instead and the conclusion would change.

(c) The distribution of the differences is not known to be symmetrical about the median, hence a Wilcoxon test might be inappropriate.

OR

The data obtained in this case have too many tied ranks (e.g. 4 participants with difference of 2), thus using Wilcoxon matched pair signed rank test would not be appropriate.

	Solution
P2 Q7	**(a)** Let X be the number of views the company's website receives. For 3 hours, $X \sim \text{Po}(15)$. $\text{P}(X > 20) = 1 - \text{P}(X \le 20)$ $= 0.0830$ (to 3 sf) **(b)** The session durations of users of the webpage are independent of each other. (OR Whether a view is a quality one is independent of another.) The probability that a user spends more than 3 minutes on the webpage is a constant, *p*, for every user. (OR The probability of having a quality view is a constant.) **(c)** $\text{E}(Y) = 4 \Rightarrow p = 0.25$ $\therefore Y \sim \text{Geo}(0.25)$ Required probability $= (0.75)^6$ $= 0.178$ (to 3 sf) **(d)** $\text{P}(10^{\text{th}}$ view is 2nd quality view | 3^{rd} view is 1st quality view) $=$ P(only the last view amongst 7 views is a quality view) $= (0.75)^6(0.25)$ $= 0.0445$ (to 3 sf) Or Required probability = $\dfrac{(0.75)^8(0.25)^2}{(0.75)^2(0.25)} = (0.75)^6(0.25)$ = 0.0445

	Solution
P2 Q8	**(a)** $\int_0^1 \frac{1}{\sqrt{x-x^2}}\,dx = \int_0^1 \frac{1}{\sqrt{\left(\frac{1}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\,dx$ $= \left[\sin^{-1}\frac{x-\frac{1}{2}}{\frac{1}{2}}\right]_0^1$ $= \sin^{-1}(1)-\sin^{-1}(-1)$ $= \frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi$ (shown) **(b)(i)** $k=\frac{1}{\pi}$ **(b)(ii)** $E(X)=\frac{1}{\pi}\int_0^1 \frac{x}{\sqrt{x-x^2}}\,dx$ $=\frac{1}{\pi}\left[\frac{1}{2}\int_0^1 \frac{1}{\sqrt{x-x^2}}\,dx-\frac{1}{2}\int_0^1 \frac{1-2x}{\sqrt{x-x^2}}\,dx\right]$ $=\frac{1}{\pi}\left\{\frac{\pi}{2}-\left[\sqrt{x-x^2}\right]_0^1\right\}$ $=\frac{1}{\pi}\left(\frac{\pi}{2}\right)=\frac{1}{2}$ Alternatively: Observe that the domain for the p.d.f. is finite from 0 to 1 and the p.d.f is symmetrical about 0.5, hence we have the median to be 0.5 and for a symmetrical p.d.f. over a finite domain, the mean will be equal to the median. Hence E(X) = 0.5. **(b)(iii)** Let the lower quartile be q. Then $P(X<q)=\frac{1}{4}$ $\Rightarrow \frac{1}{\pi}\int_0^q \frac{1}{\sqrt{x-x^2}}\,dx=\frac{1}{4}$ ---(1)

	$\left[\sin^{-1}\dfrac{x-\frac{1}{2}}{\frac{1}{2}}\right]_0^q = \dfrac{\pi}{4}$ $\sin^{-1}\dfrac{q-\frac{1}{2}}{\frac{1}{2}} - \sin^{-1}(-1) = \dfrac{\pi}{4}$ $\sin^{-1}\dfrac{q-\frac{1}{2}}{\frac{1}{2}} + \dfrac{\pi}{2} = \dfrac{\pi}{4}$ $\dfrac{q-\frac{1}{2}}{\frac{1}{2}} = \sin\left(-\dfrac{\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$ $q = \dfrac{2-\sqrt{2}}{4}$ (or $q = 0.146$)

	Solution
P2 Q9	**(a)** Let X and Y denote the increase in cognitive focus score after consuming Brand A and Brand B respectively. $H_0: \quad \mu_X - \mu_Y = 0 \qquad H_1: \quad \mu_X - \mu_Y \neq 0$ Assumptions: (1) X and Y are normally distributed. (2) Both populations share a common population variance. Note: the readings do not come from the same pair of participants. Do not use paired sample t-test! Under H_0, Test statistic: $T = \dfrac{(\overline{X} - \overline{Y}) - 0}{S_p\sqrt{\frac{1}{6} + \frac{1}{6}}} \sim t(10)$. where $\bar{x} = 7.3667$, $\bar{y} = 7$, $s_p{}^2 = 0.42348^2$ Using GC, p-value = 0.165 > 0.02 . We do not reject H_0 and conclude that there is insufficient evidence at 2% significance level that there is a difference between the effects of the two brands of supplement, i.e. there is no significant evidence of a difference in the increase of cognitive focus score. **(b)** Let $D = X - Y$ d = 1.4, –0.6 , 0.1, 0.6, 0.9, –0.2 From GC, $\bar{d} = 0.36667 \qquad s_D{}^2 = 0.73937^2$ A 98% confidence limit for D is $= 0.36667 \pm t_{(5,0.99)} \dfrac{s_D}{\sqrt{6}}$ $= 0.36667 \pm 3.3649 \dfrac{0.73937}{\sqrt{6}}$ A 98% confidence interval $= (-0.649,\ 1.382)$ <u>Alternative</u> Let $D = Y - X$ $d = -1.4,\ 0.6,\ -0.1,\ -0.6,\ -0.9,\ 0.2$ From GC, $\bar{d} = -0.36667 \qquad s_D{}^2 = 0.73937^2$ A 98% confidence limit for D is $= -0.36667 \pm t_{(5,0.99)} \dfrac{s_D}{\sqrt{6}}$ $= -0.36667 \pm 3.3649 \dfrac{0.73937}{\sqrt{6}}$ A 98% confidence interval $= (-1.382,\ 0.649)$ **(c)** $H_0: \quad \mu_D = 0 \qquad H_1: \quad \mu_D \neq 0$ Since $0 \in (-0.649,\ 1.382)$, we do not reject H_0 and conclude that there is insufficient evidence, at the 2% significance level, that there is a significant difference between the effects of Brand A and Brand B (i.e. no significant evidence of an increase in cognitive focus score). **(d)** The paired sample t-test in **(c)** is more suitable as it eliminates factors of variation across different individuals such as individual's physiological state, body susceptibility towards supplement, genetic makeup etc.

	Solution
P2 Q10	**(a)** Let X be the no. of emergency calls received at the fire station in a day.

H_0 : X follows a Poisson distribution.

H_1 : X does not follow a Poisson distribution.

Since the parameters are not provided, we find $\bar{x} = \frac{\sum fx}{\sum f} = 2.1667$.

Thus, an estimate of $\lambda = \mathrm{E}(X) \approx 2.1667$.

Under H_0, $\mathrm{P}(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$, $e_i = \mathrm{P}(X = x) \times 60$

x	0	1	2	3	4	5	≥ 6
O_i	7	14	16	12	7	4	0
E_i	6.8733	14.892	16.134	11.652	6.3117	2.7351	1.4015

Since we need all $E_i > 5$,

X	0	1	2	3	≥ 4
O_i	7	14	16	12	11
E_i	6.8733	14.892	16.134	11.652	10.4483

Degree of freedom = 5 – 1 – 1 = 3

Test statistic: Under H_0, $\chi^2 = \sum_{i=1}^{n} \frac{(F_i - E_i)^2}{E_i} \sim \chi_3^2$

Value of χ^2 = 0.096425

p-value = 0.992

Since p-value is almost 1, there is little to no evidence to conclude that the data does not have a Poisson distribution with $\lambda = 2.1667$ i.e. there is strong evidence to support the analyst's claim of a Poisson distribution.

(b) The <u>mean and variance</u> of emergency calls to a fire department of the data provided <u>are approximately equal</u>, thus resulting in the data being a good fit to the Poisson distribution.
(Note: $s^2 = 1.9718 \approx \mathrm{E}(X) = 2.1677$)

(c) Should the proportion of calls in each category remains the same, the new observed frequencies = $20\,O_i$ and the new expected frequencies = $20\,E_i$.

Since all new expected frequencies are now larger than 5, we do not need to collapse categories and new df = 7 – 1 – 1= 5

x	0	1	2	3	4	5	≥ 6
O_i	140	280	320	240	140	80	0
E_i	137.47	297.85	322.67	233.05	126.23	54.703	28.03

$\chi_{\text{new}}{}^2 = \sum_{i=1}^{7} \frac{(20F_i - 20E_i)^2}{20E_i} = 42.576$

Since $\chi_{\text{new}}{}^2 > \chi^2_{(0.9,5)} = 9.236$, we reject H_0 and conclude that at 10% significance level, there is sufficient evidence that the data does not have a Poisson distribution with $\lambda = 2.1667$ i.e. there is evidence to refute the analyst's claim.

<u>Alternatively,</u>

p-value = 4.50×10^{-8} < 0.1, we reject H_0 …