TOPIC 2 KINEMATICS

CONTENT

1. Distance and Displacement
2. Speed and Velocity
3. Displacement-Time graphs
4. Acceleration
5. Velocity-Time graphs
6. Free fall in a Gravitational field w/o air resistance
7. Deriving the equations of Kinematics
8. Free fall in a Gravitational field w air resistance
9. Projectile Motion

LEARNING OUTCOMES

(a) **define and use displacement, speed, velocity and acceleration**
(b) **use graphical methods to represent distance, displacement, speed, velocity and acceleration use SI base units to check the homogeneity of physics equations**
(c) **identify and use the physical quantities from the gradients of displacement-time graphs and areas under and gradients of velocity-time graphs, including cases of non-uniform acceleration**
(d) **derive, from the definitions of velocity and acceleration, equations which represent uniformly accelerated motion in a straight line**
(e) **solve problems using equations which represent uniformly accelerated motion in a straight line, including the motion of bodies falling in a uniform gravitational field without air resistance**
(f) **describe qualitatively the motion of bodies falling in a uniform gravitational field with air resistance**
(g) **describe and explain motion due to a uniform velocity in one direction and a uniform acceleration in a perpendicular direction.**

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0. OVERVIEW OF KINEMATICS

To begin learning about classical mechanics, our initial concentration will be solely on an object's movement, disregarding any forces that may impact or alter it. This particular branch of classical mechanics is called **kinematics** which is the study of motion, a continuos change of position with time.

In this chapter we'll discuss how to quantify that using three key quantities: displacement, velocity and acceleration for objects moving with respect to a coordinate axis in one and two dimensions.

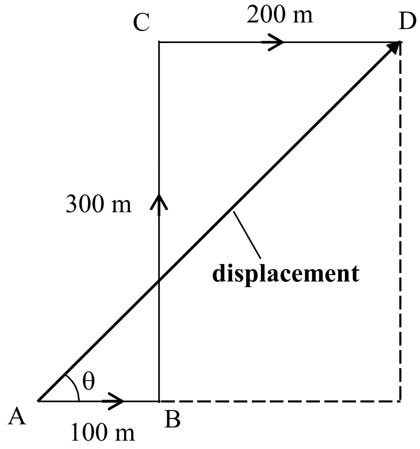
1. DISTANCE AND DISPLACEMENT

We start with a few definitions:

- **Distance** is the total length of the path travelled by a body regardless of direction travelled. It is a scalar quantity meaning that it only has magnitude but no direction.

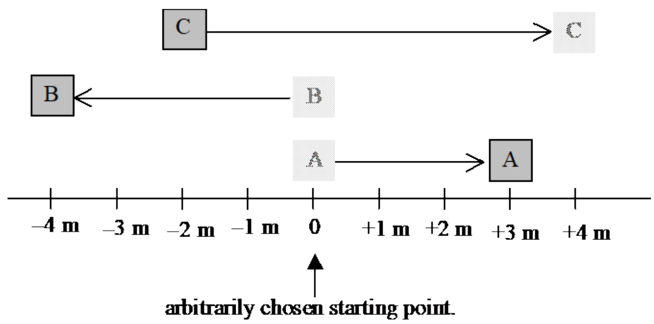
- **Displacement**: the distance travelled in a straight line in a specified direction, from some reference point. It is a **vector** quantity, meaning that it has both magnitude and direction.

Example 1:
From point A, a boy travels east for 100 m to point B, then travels north for 300 m to point C. He then travels a further 200 m towards the east, reaching point D. What is the total distance he travelled, and what is his final displacement from his starting position?



1.1 Displacement along a straight-line path

To define displacement, we fix need a reference starting position which can be chosen arbitrarily. Following which, we decide on a sign convention, e.g. take “to the right” to be positive.



Change in displacement of object A = +3 m.
Change in displacement of object B = - 4 m.
Change in displacement of object C = +6 m.
(Final displacement of C is +4 m.)

2. SPEED AND VELOCITY

- **Speed** is defined as the **rate of change of distance travelled** or as the distance travelled **per** unit time.

- The use of “per” in the definition is important because it signifies that speed is a **ratio**:

$$\boxed{\text{Speed} = \frac{\text{Distance travelled}}{\text{time taken}}}$$

- Speed is a scalar quantity.

- **Velocity** is defined as the **rate of change of displacement** or as the change of displacement **per** unit time.

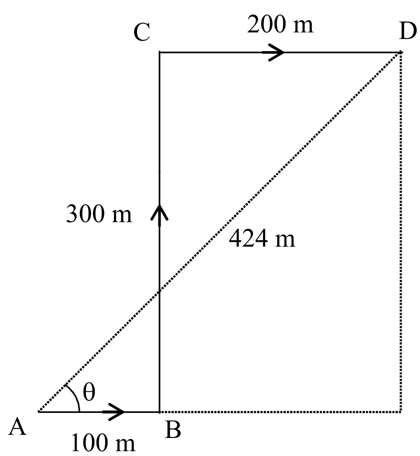
$$\boxed{\text{Velocity} = \frac{\text{Change of displacement}}{\text{time taken}}}$$

- Velocity is a vector quantity.
- The S.I. units for both speed and velocity are metres per second (m s^{-1}).

<u>Example 2:</u>
In the previous example, the boy took 100 s to complete the whole journey ABCD. What is his average velocity over the whole journey?

<u>Answer:</u>

Average velocity $= \dfrac{\text{change of displacement}}{\text{time taken}}$



2.1 Average velocity vs Instantaneous velocity

- **Average speed** is defined as the total distance travelled over the total time taken.
- Average speed = $\dfrac{\text{distance travelled}}{\text{time taken}} = \dfrac{\Delta D}{\Delta t}$
- **Average velocity** is defined as the total displacement over the total time taken.
- Average velocity = $\dfrac{\text{change of displacement}}{\text{time taken}} = \dfrac{\Delta s}{\Delta t}$
- **Instantaneous speed** is defined as the rate of change of distance (with time).
- Instantaneous speed= instantaneous rate of change of distance travelled = $\dfrac{dD}{dt}$
- **Instantaneous velocity** is defined as the rate of change of displacement (with time).
- Instantaneous velocity = instantaneous rate of change of displacement = $\dfrac{ds}{dt}$

Example 3:
In the previous example, the boy took 100 s to complete the whole journey ABCD. What is his average velocity over the whole journey?

A car starts moving from rest along a straight path without reversing, and its distance from its starting point at various times are shown in the table below.

time / s	0	1.0	2.0	3.0	4.0	5.0
distance travelled / m	0	5.0	25	68	147	210

Estimate its instantaneous speed at 3.5 s from the start.

Answer:
Instantaneous speed (estimated) at 3.5 s

$= \dfrac{\text{Change of displacement between 3 s and 4 s}}{\textit{time}\text{ taken}}$

=

=

3. DISPLACEMENT-TIME GRAPH

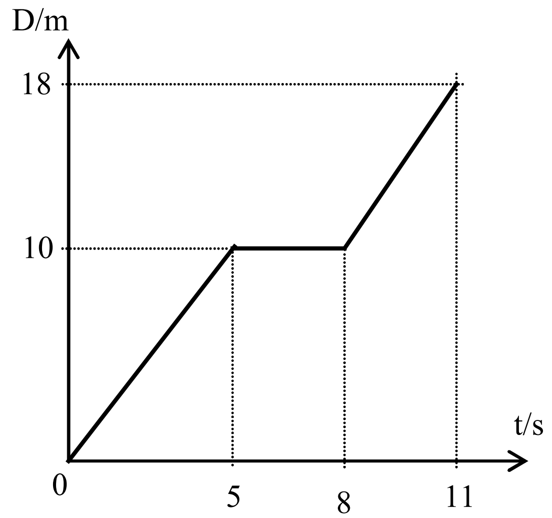
- Graphical depictions provide a comprehensive account of how objects move.
- The two primary aspects that receive attention in graphical interpretation are the **area** and the **gradient**.
- When graphing a dependent variable y over time (with time as the horizontal axis), the gradient signifies the rate at which y changes.

The following table summarises the graphs you will learn:

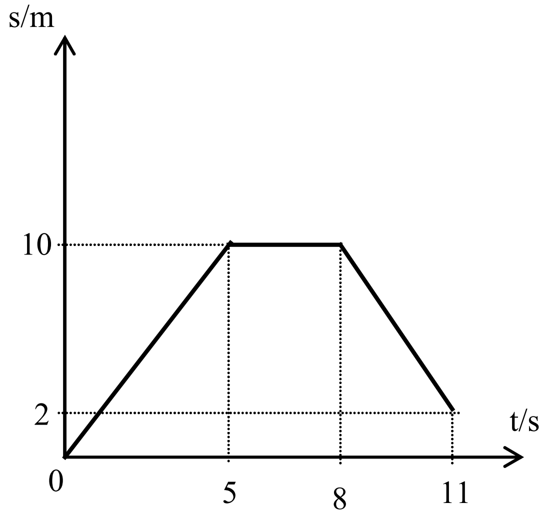
Type of graph	Gradient represents	Area represents
Displacement – time	Velocity	Not Applicable
Velocity – time	Acceleration	Change in Displacement
Acceleration-time	Not Applicable	Change in Velocity

Example 4:
A body moves at constant speed along a straight-line path, and travels 10 m in a time of 5 s. It then stays at rest for 3 s. After that, it reverses its direction of travel and travels a distance of 8 m in 3 s. Draw a distance-time graph, as well as a displacement-time graph, to illustrate the motion of the body.

Answer:



distance-time graph



displacement-time graph

3.1 Gradient of a displacement-time graph (and distance-time graph)

Gradient of a distance-time graph = $\frac{dD}{dt}$ = speed

Gradient of a displacement-time graph = $\frac{\mathrm{ds}}{\mathrm{dt}}$ = velocity

Example 5:
In the previous example, calculate the speeds for the three stages of the motion of the body, and draw a speed-time graph for the motion. Similarly, calculate the velocities for the three stages, and draw a velocity-time graph for the motion.

Answer:

From distance-time graph,

During the first 5 s, speed = $\dfrac{\Delta D}{\Delta t}$ =

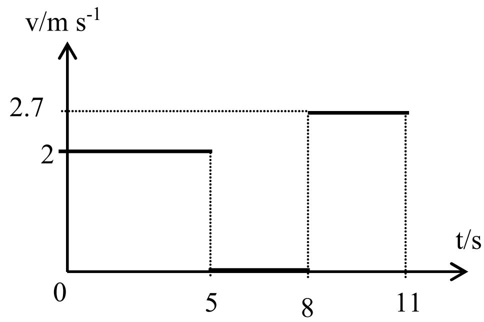
From t = 5 s to t = 8 s, speed =

From t = 8 s to t = 11 s, speed =

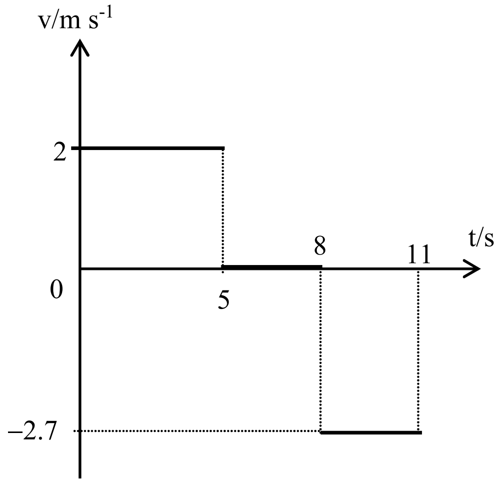
From displacement-time graph,

For the first 8 s, velocity = speed, since there is no change in direction of travel.

From t = 8 s to t = 11 s, velocity = $\dfrac{2-10}{11-8}$ = –2.667 ≈ –2.7 m s^{-1}



speed-time graph



velocity-time graph

Example 6:

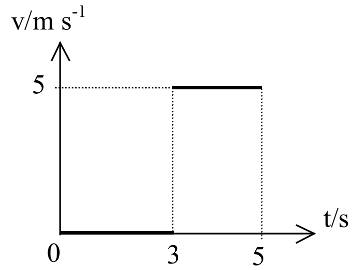
Sketch the velocity-time graph and deduce the corresponding displacement-time graph for each of the following situations. Assume that the body moves along a straight-line path.

(a) A body stays at rest for 3.0 s, then starts moving at a constant velocity of 5.0 m s^{-1} for a further 2.0 s.

(b) A body's velocity starts to increase at a uniform rate from rest. It reaches a maximum velocity after 4.0 s before decelerating uniformly to a complete stop in another 4.0 s. At no time does its velocity stay constant.

Answer:

(a)



From t = 3 s to t = 5 s, time taken = 2 s

∴ change of displacement = velocity × time taken

= 5 × 2

= 10 m

(b)

At t = 0, velocity v = 0 $\Rightarrow \dfrac{ds}{dt} = 0$

$\Rightarrow$ At t = 0 s, gradient of s-t graph = 0.

Similarly,

at t = 8 s, v = 0 $\Rightarrow \dfrac{ds}{dt} = 0$

$\Rightarrow$ At t = 8 s, gradient of s-t graph = 0.

From t = 0 to t = 4 s,
v increases at a constant rate

$\Rightarrow \dfrac{ds}{dt}$ increases at a constant rate.

$\Rightarrow$ gradient of s-t graph increases uniformly.

At t = 4 s, v is maximum

$\Rightarrow \dfrac{ds}{dt}$ = gradient of s-t graph is maximum at t = 4 s.

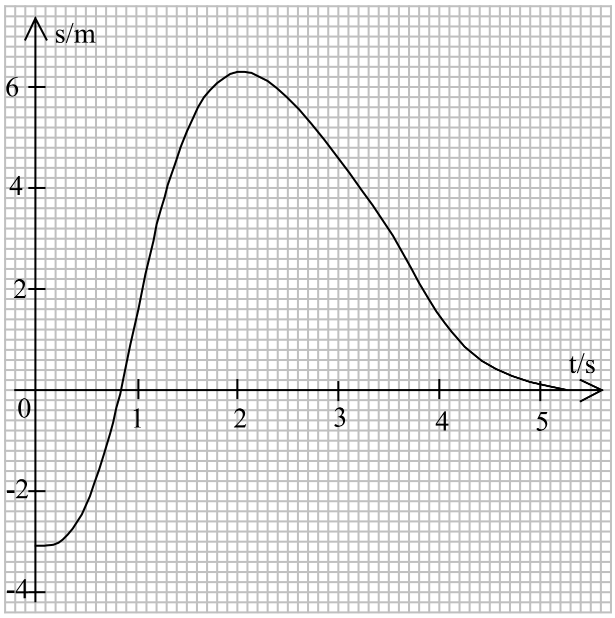
From t = 4 s to t = 8 s,
v decreases at a constant rate

$\Rightarrow \dfrac{ds}{dt}$ decreases at a constant rate.

$\Rightarrow$ gradient of s-t graph decreases uniformly to zero

Example 7

The graph below shows the variation with time of the displacement of an object from a certain point A. The object's motion is restricted to a straight line path. Study it and answer the questions below.



(a) State the times at which the object was momentarily at rest, and mark them on the curve.
(b) State the time intervals during which the object was moving towards point A.
(c) Estimate the time interval during which the velocity of the object was maximum and determine this maximum velocity.
(d) Determine the velocity of the object at t = 3.8 s.
(e) State a time at which the object was speeding up.

Answer:

(a) The object was at rest at times

(b)

(c) Velocity = gradient of displacement-time graph
When velocity is maximum, the gradient is maximum
$\therefore$ Time interval of maximum velocity is

Maximum velocity = maximum value of gradient

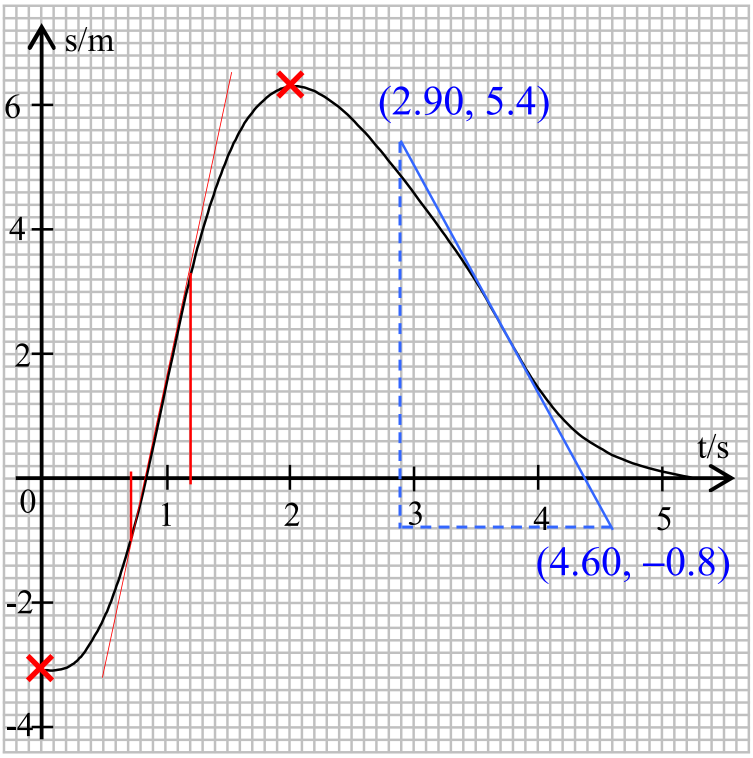
=

=

(d) Velocity at 3.8 s = gradient of the curve at 3.8 s

=

=

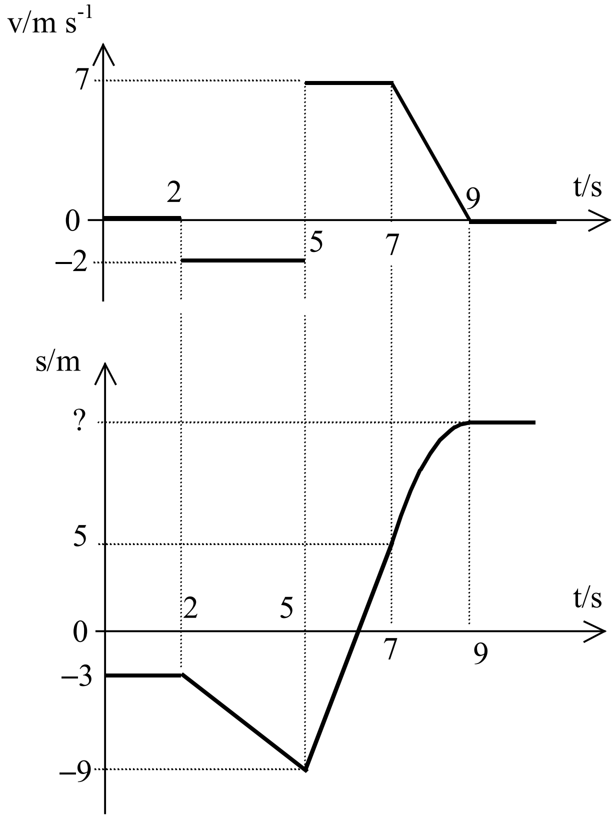


(e) Object speeds up
$\Rightarrow$ magnitude of velocity increases with time
$\Rightarrow$ magnitude of gradient of *s-t* graph increases with time (graph becomes steeper)
Hence, the object is speeding up at approximately

Example 8:

Study the following velocity-time graph of a body moving along a straight line, and draw the corresponding displacement-time graph. It is known that the body had an initial displacement of –3 m as measured from an arbitrary point O.

Answer:



From t = 2 s to t = 5 s,

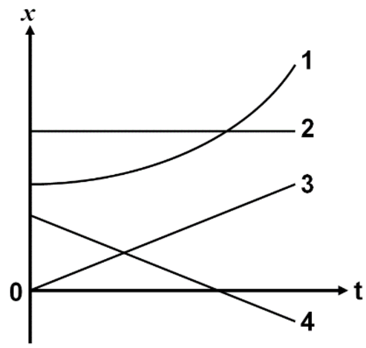
$v = -2$ m s^{-1}
$\Rightarrow$ Δs = velocity × time interval
= –2 × 3
= –6 m

$\Rightarrow$ new displacement at 5 s
= initial displacement + change in displacement
= (–3) + (–6)
= –9 m

What is the final displacement of the body?

Test yourself!

The figure on the right shows graphs of an object's position along the x-axis versus time for four separate trials. Which of the following statements is or are correct?



A. The velocity is greater in trial 2 than in trial 3.
B. The velocity is not constant during trial 1.
C. During trial 4, the object changes direction as it passes through the point x = 0.
D. The velocity is constant during trials 2, 3, and 4.

4. ACCELERATION

Acceleration is defined as **the rate of change of velocity** or as the change of velocity **per** unit time.

i.e. Acceleration = $\dfrac{\text{change in velocity}}{\text{time taken}} = \dfrac{dv}{dt}$

Average acceleration is defined as the total change in velocity over the total time interval.

Average acceleration = $\dfrac{\text{overall change in velocity}}{\text{total time taken}}$

Acceleration is a vector quantity.

Note: Occasionally we come across a term ***deceleration*** which refers to objects which are slowing down (magnitude of velocity decreasing), regardless of the direction of motion. This term is different from ***negative acceleration*** which tells us the direction of acceleration with respect to a sign convention (and can result in speeding up or slowing down).

<u>Example 9:</u>
A car was moving initially moving at 1.0 m s^{-1} towards the north. After 5.0 s, its speed has increased to 3.5 m s^{-1}, while its direction of travel has not changed. What is the magnitude and direction of its average acceleration over this 5.0 s period?

<u>Answer:</u>

Acceleration = $\dfrac{\text{final velocity - initial velocity}}{\text{time taken}}$

=

=

<u>Example 10:</u>
A car was initially moving at 3.0 m s^{-1} towards the north. After 10 s, its velocity has changed to 6.0 m s^{-1} towards the south. What was its acceleration over this 10 s period?

<u>Answer:</u>
Taking "north" as the +ve direction.

Acceleration = $\dfrac{\Delta v}{\Delta t}$

=

=

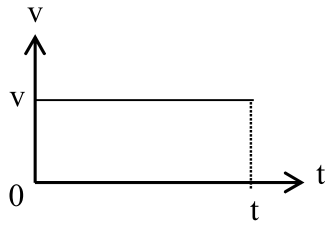
- Hence, the acceleration of the car was **0.90 m s^{-2} towards the south**.
- The car slows down to a stop, reverses direction, then accelerates to 6.0 m s^{-1} (toward the south).

5. VELOCITY-TIME GRAPH

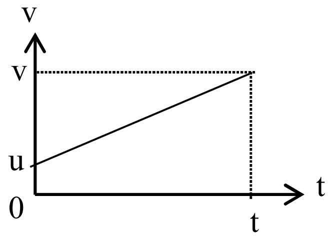
Gradient of a velocity-time (v-t) graph = $\dfrac{dv}{dt}$ = *instantaneous* acceleration

5.1 Area under a Velocity-Time Graph

Some examples:



Case 1 (zero acceleration):
A body moves with constant velocity v for a time t.
Change of displacement, s = velocity × time
$= vt$
= Area under v-t graph.

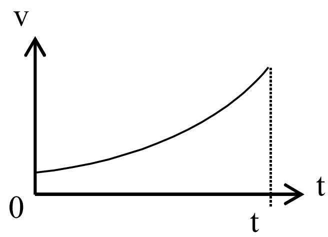


Case 2 (uniform acceleration):
A body accelerates uniformly from initial velocity u, attaining final velocity v in time t.

Average velocity = $\dfrac{1}{2}(u+v)$

s = average velocity × time = $\dfrac{1}{2}(u+v)\,t$

= Area (of trapezium) under v-t graph

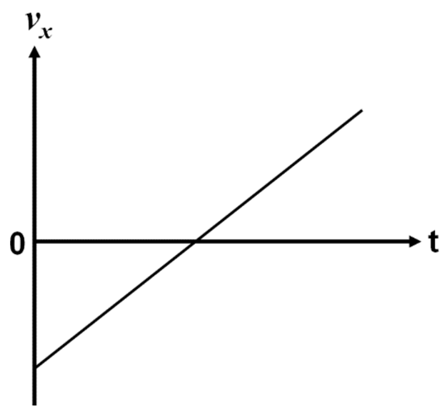


Case 3 (non-uniform acceleration):
s = estimated area under v-t curve
using the "trapezium rule"
(divide the area under the curve into small trapeziums)

Test yourself!

The figure on the right shows a graph of velocity versus time for a moving object. The object moves on an east-west axis, with west chosen as the positive direction. Which of the following verbal descriptions matches the graph?

A. The object speeds up continuously while moving westward.
B. The object moves toward the origin while slowing, then moves away from the origin while speeding up.
C. The object moves west while slowing to a stop, then moves east while speeding up.
D. The object moves east while slowing to a stop, then moves west while speeding up.

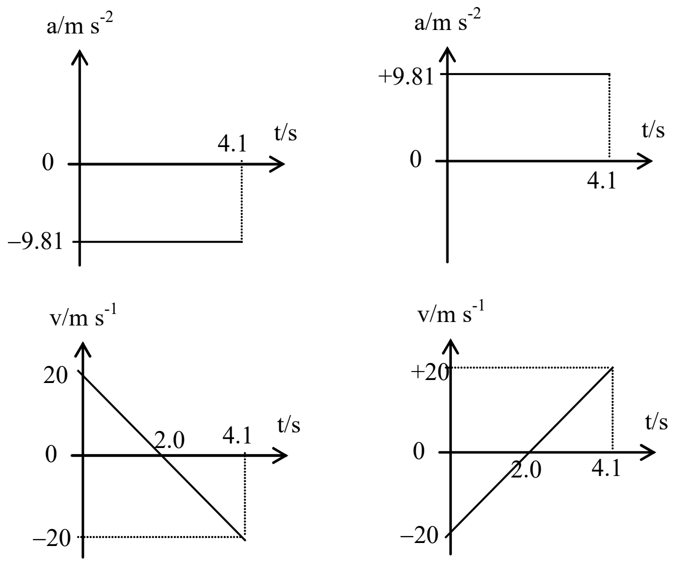


6. FREE FALL IN A GRAVITATIONAL FIELD WITHOUT AIR RESISTANCE

Example 11:
A ball is thrown vertically upwards, leaving the hand with a speed of 20 m s^{-1}. It reaches its greatest height before falling back down to its starting location. Ignoring air resistance, sketch both the acceleration-time and velocity-time graphs for the whole motion of the ball. State your chosen sign convention.

Answer:

Alternative graphs:



The **vertically upwards** direction is chosen to be the positive direction.

The **vertically downwards** direction is chosen to be the positive direction.

Choosing ↑ to be the +ve direction, consider the motion from start to greatest height:

Acceleration = $\dfrac{\Delta v}{\Delta t}$ = $\dfrac{v_f - v_i}{t}$

⇒ time taken to reach maximum height, t = 2.039 ≈ 2.0 s.
Similarly, the time taken to fall back to the starting level is also 2.0 s.

Example 12:
A rubber ball is allowed to fall from rest. It hits the ground 2.0 s later. The impact of the ball with the ground lasts for 0.10 s, after which the ball bounces back up to its original height. Assuming that the speed of the ball is the same before and after the collision (i.e., the collision is totally elastic), sketch the velocity-time graph and acceleration-time graph for the ball's motion until it reaches its greatest height after this one bounce.

Answer:

acceleration, $a = \frac{\Delta v}{\Delta t}$ = +9.81 m s^{-2} (constant)

- take the vertically downwards direction as positive.

Time taken, Δt = 2.0 s

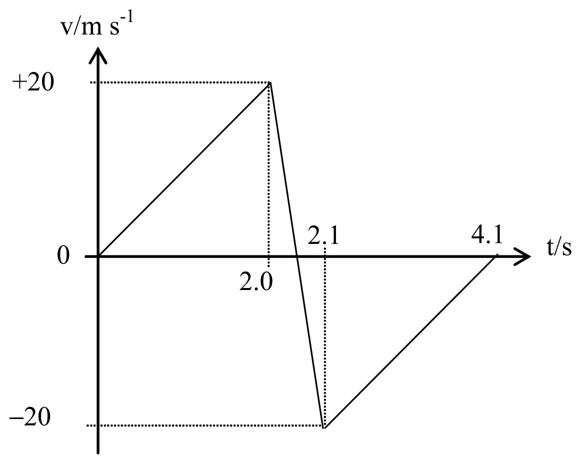
$\Rightarrow \quad \frac{\Delta v}{2.0} = +9.81$

$\Rightarrow$ change in velocity, Δv = +19.62 ≈ +20 m s^{-1}

Since the initial velocity, u = 0,
final velocity before impact = $u + \Delta v$ = 0 + 20 = +20 m s^{-1}

Given: duration of impact = 0.10 s.
Given: speed after impact = speed before impact.

∴ velocity immediately after impact = –20 m s^{-1} (i.e. only direction has changed)



acceleration during impact with the ground = $\frac{\Delta v}{\Delta t}$ =

Thus, the acceleration-time graph is as follows:

Test yourself!

In a baseball game, a batter hits a pitched ball that goes straight up in the air above the home plate. At the top of its motion, the ball is momentarily stationary. At this point,
A. the acceleration is zero because the ball is stationary at that instant.
B. the acceleration is zero because the motion is changing from slowing down to speeding up.
C. the acceleration is zero because, at that instant, the force from the impact with the bat is balanced by the gravitational pull of the earth.
D. the acceleration is 9.81 m s^{-2} downward.

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7. DERIVING THE EQUATIONS OF KINEMATICS

Note: only applies to uniform motion (*a* = constant)

Acceleration $a = \dfrac{\Delta v}{\Delta t} = \dfrac{v - u}{t}$

$$\boxed{v = u + at} \qquad (1)$$

$$\text{average velocity} = \frac{\text{change of displacement}}{\text{time taken}}$$

⇒ **change** of displacement, *s* = average velocity × time taken

Average velocity over the time interval $t = \frac{1}{2}(u+v)$.

$$\Rightarrow \quad \boxed{s = \frac{1}{2}(u+v)t} \qquad (2)$$

$$\therefore s = \frac{1}{2}(u + u + at)t$$

$$\boxed{s = ut + \frac{1}{2}at^2} \qquad (3)$$

From (1), we get: $t = \dfrac{v-u}{a}$

Substituting into (3):

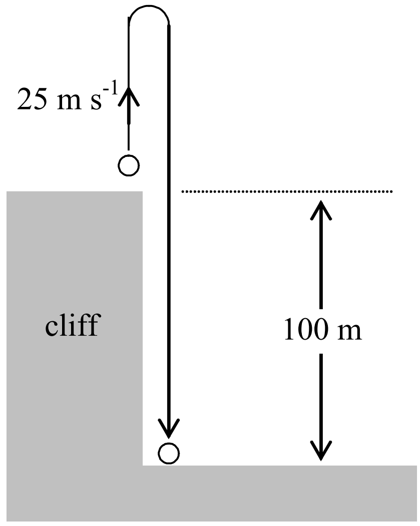
$$s = u\left(\frac{v-u}{a}\right) + \frac{1}{2}a\left(\frac{v-u}{a}\right)^2$$

$$\boxed{v^2 = u^2 + 2as} \qquad (4)$$

These 4 equations can only be used when the **acceleration is constant**.

Example 13:
A ball is thrown vertically upwards from a cliff of height 100 m with a starting speed of 25 m s^{-1}.

(a) Calculate the velocity of the ball at 2.0 s and at 6.0 s.
(b) Determine the time at which the ball reaches its greatest height.
(c) Find the greatest height above the cliff reached by the ball.
(d) Determine the total time taken for the ball to reach the base of the cliff.
(e) Calculate the speed with which the ball hits the base of the cliff.



Answer:
(a) $v = u + at$
Take the vertically-upward direction as positive.

At t = 2.0 s,

Thus, the velocity of the ball at 2.0 s is **5.4 m s^{-1} vertically upwards**.

At t = 6.0 s,

Thus, the velocity of the ball at 6.0 s is **34 m s^{-1} vertically downwards**.

(b) At its greatest height, velocity of ball = 0 (momentarily at rest).
$v = u + at$

(taking vertically upwards as positive)

(c) $v^2 = u^2 + 2as$

∴ the displacement of the ball at its greatest height = 32 m *above* the cliff-top,

(d) Consider the motion of the ball from its launch to its impact with the base:

$$s = ut + \frac{1}{2}at^2$$

(taking vertically upwards as positive)

Solving this quadratic equation:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-(-50) \pm \sqrt{(-50)^2 - 4(9.81)(-200)}}{2(9.81)}$$

= 7.7 s or –2.6 s

The negative "answer" is discarded because it does not make sense in the context of this question.

∴ The total time taken for ball to reach the cliff base = **7.7 s**

(e) $v^2 = u^2 + 2as$

(taking vertically upwards as positive)

Hence, the <u>speed</u> of impact of the ball with the base of the cliff = **51 m s^{-1}**

(From the context of the question, we know that the ball hits the ground with a downward velocity, so the <u>velocity</u> is actually –51 m s^{-1})

<u>Test yourself!</u>

Suppose you throw two identical stones from the top of a tall cliff. You throw one directly upward with a speed of 10 m s^{-1} and the other directly downward with the same speed. When they hit the ground below, how do their speeds compare? (Neglect air resistance in this question.)

A. The one thrown upward is travelling faster.
B. The one thrown downward is travelling faster.
C. Both are travelling at the same speed.

8. FREE FALL IN A GRAVITATIONAL FIELD WITH AIR RESISTANCE

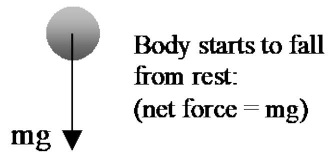
Without air resistance: downward acceleration (due to gravity) = 9.81 m s^{-2}

With air resistance present:

- A falling object will still accelerate downwards.
- But this acceleration will not remain steady at 9.81 m s^{-2}.
- The acceleration will, instead, decrease until it eventually reaches zero.
- At this point the object is no longer accelerating and falls at constant velocity (called terminal velocity).

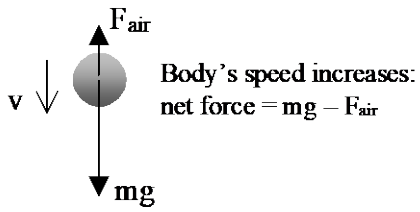
When a body first starts to fall from rest:

- Its initial velocity is zero.
- Since it is not moving, it does not experience any air resistance.
- Hence, the only force acting on the body is the force of gravity (its weight).
- Acceleration, $a = g = 9.81$ m s^{-2} downwards.



As the body gains velocity in the downwards direction:

- It experiences increasing air resistance against its motion, in the upwards direction.
- Hence, the net force it experiences is $mg - F_{air}$
- This net force is smaller in magnitude than mg alone.



Thus, applying $F = ma$, we get:

$$mg - F_{air} = ma$$

$$\Rightarrow a = g - \frac{F_{air}}{m}$$

which is less than the acceleration when the body was at rest at the start.

Eventually, the velocity of the body is so large that the air resistance it experiences is equal to the body's weight:

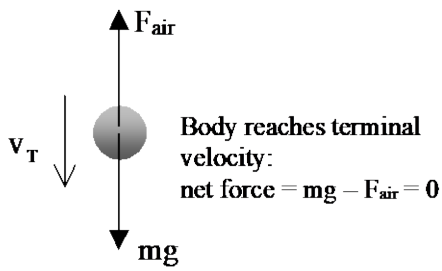
Using $F = ma$,

$mg - F_{air} = ma$

$0 = ma$, since weight equals air resistance.

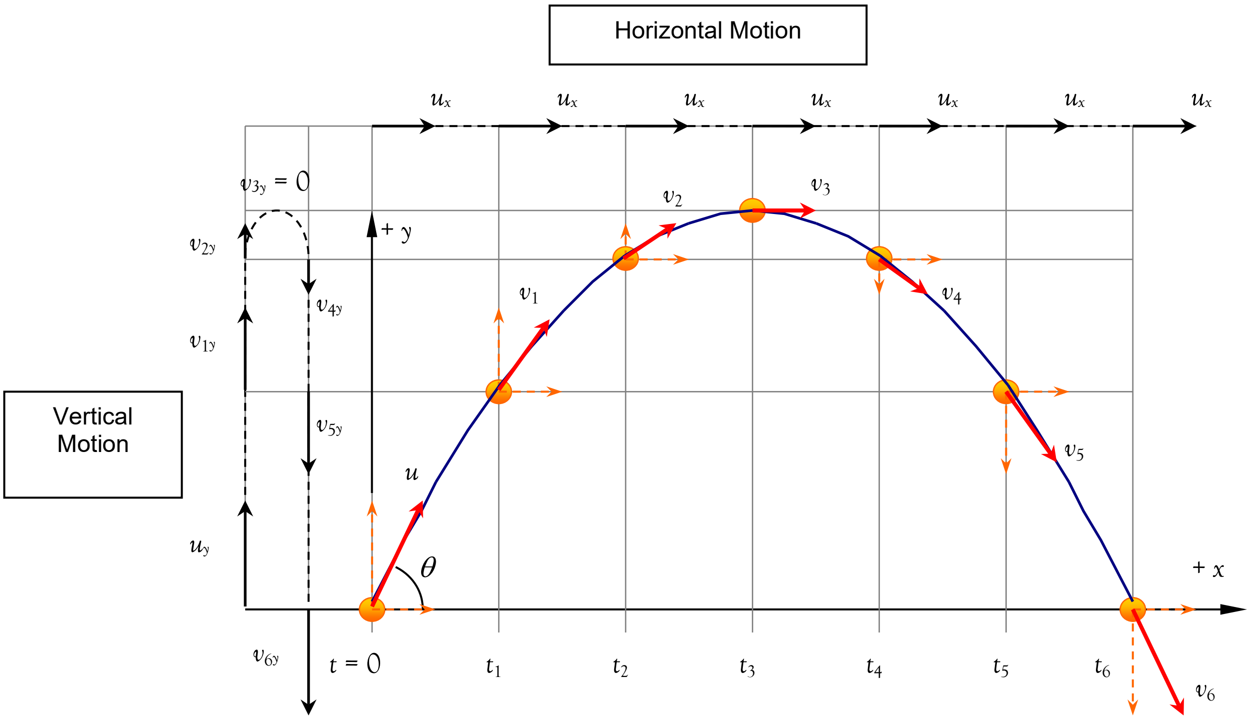
$\Rightarrow a = 0$

$\Rightarrow$ body falls at constant velocity (called terminal velocity) from this point onwards.

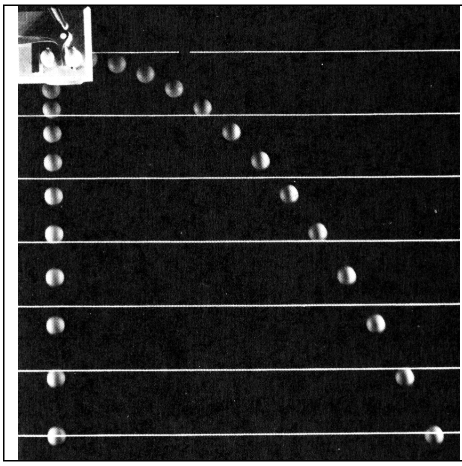


9. PROJECTILE MOTION

- When an object is thrown into the air at an angle to the ground, it will follow a **parabolic** path.
- So long as the object is not in contact with anything and air resistance is negligible, the **only force** acting on the object is the force of gravity acting **vertically downwards** (i.e. its weight).
- By Newton's Second Law (F = ma), this means that the **only acceleration** the object experiences is the acceleration due to gravity, about 9.81 m s^{-2} **vertically downwards**.
- Horizontally, the object's velocity is constant because horizontally, the object experiences no force.



In summary, projectile motion is simply a combination of two motions:
- **horizontal** motion at constant velocity, combined with
- **vertical** free motion under gravity (constant downward acceleration of 9.81 m s^{-2}). The vertical projection of the object behaves like a stone thrown vertically upwards.

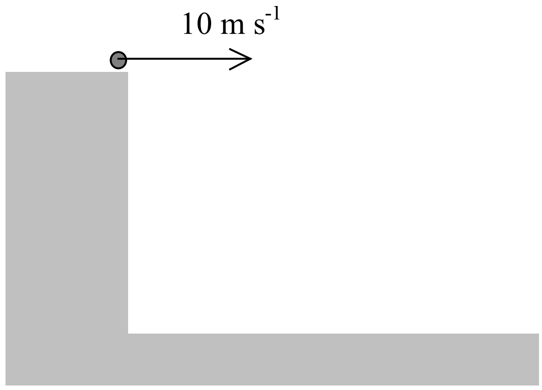


Ball 1 is released from rest at the same instant that Ball 2 is fired to the right with an initial horizontal velocity. Successive images are separated by equal time intervals.
Note the **<u>independence of horizontal and vertical motion</u>**: at any given time, both balls have the same *y* position, *y* velocity, and *y* acceleration, despite having different *x* positions and *x* velocities.

Example 14:

A stone is launched horizontally with a speed of 10 m s^{-1} from the edge of a cliff.

(a) State the initial horizontal and vertical velocities.
(b) Find the horizontal and vertical velocities of the stone after 1.0 s.
(c) Draw vector diagrams to find the effective velocities of the stone at 1.0 s.
(d) The stone hits the ground 100 m from the base of the cliff. How high is the cliff?
(e) What is the velocity of impact?



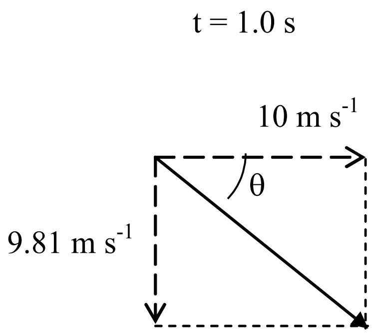
Answer:

(a) Initial horizontal velocity =
Initial vertical velocity =

(b) Horizontal velocity stays constant, because no horizontal force acts on stone.
$\therefore$ horizontal velocity after 1.0 s =

Vertical velocity after 1.0 s,

(c)



At t = 1.0 s:

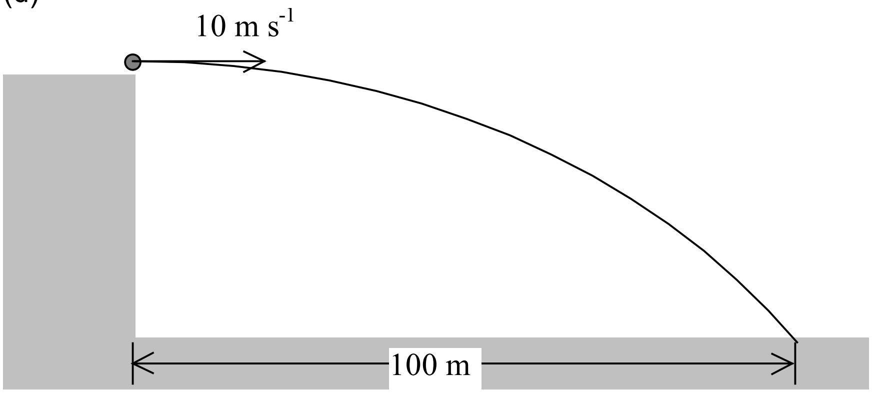
Speed =
=

tan θ =

$\Rightarrow \theta$ =

$\therefore$ velocity of stone = **14 m s^{-1} at an angle of 44° below the horizontal**.

(d)

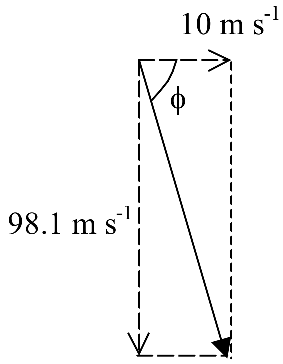


<u>Consider horizontal motion:</u>

<u>Consider vertical motion from start to impact:</u>
Initial vertical velocity = 0 m s^{-1}.
Vertical acceleration = +9.81 m s^{-1} downwards.
Time taken = 10 s.

Thus, the height of the cliff = **4.9 × 10^2 m**

(e) Horizontal speed on impact, v_x = (constant).



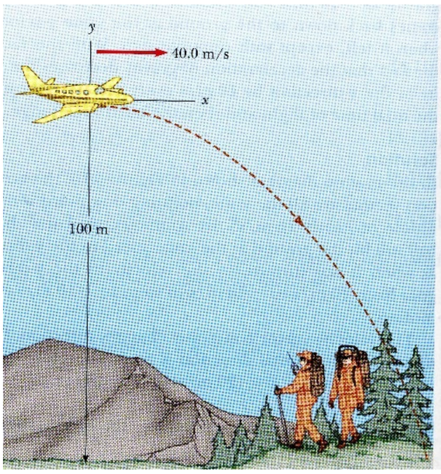
Thus, the stone hits the ground with a velocity of **99 m s^{-1} at an angle of 84^o below the horizontal.**

Example 15:

A rescue plane drops a package of emergency rations to some stranded hikers as shown. The plane is travelling horizontally at 40.0 m s^{-1} at a height of 100 m above ground.

(a) Where does the package strike the ground relative to the point at which it was released?

(b) What are the horizontal and vertical components of the velocity of the package just before it hits the ground?



Answer:

The package is initially moving horizontally (no vertical velocity at all), so at the moment in time when it is released from the plane, u_x = 40.0 m s^{-1} and u_y = 0 m s^{-1}

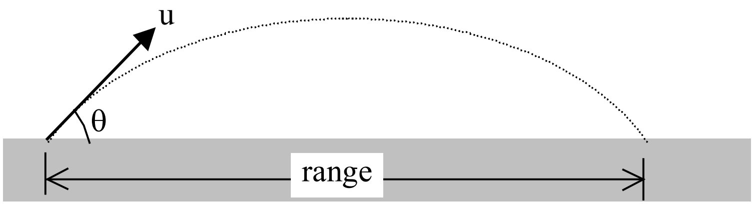
(a) We know that the package falls a vertical distance s_y = –100 m

So, the package strikes the ground 181 m ahead of the point of release.

(b) From (a), we found that the package took 4.52 s to hit the ground.
Its horizontal velocity, $v_x = u_x =$ *40.0* m s^{-1} (to the right)

Example 16: Range

If an object is thrown with a starting speed u at an angle θ above the horizontal on flat ground, the horizontal distance travelled by an object before it hits the ground is called the **range**. Derive an expression for the range in terms of u, θ and g (acceleration due to gravity). What is the value of angle θ for maximum range to be achieved?



Answer:

Vertically:

Initial vertical velocity, $u_y = +u \sin\theta$, taking the vertically-upwards direction as positive.

Vertical acceleration = 9.81 m s^{-2} downwards
= –9.81 m s^{-2}
= $-g$

To find the object's **time of flight**, we can apply $s_y = u_y t + \frac{1}{2} a_y t^2$ and substitute $s_y = 0$ (because the net vertical displacement is zero when the object returns to the ground).

$0 = (+u \sin\theta)t + \frac{1}{2}(-g)t^2$

$\Rightarrow t = 0$ (trivial) or $t = \dfrac{2u\sin\theta}{g}$

We use $t = \dfrac{2u\sin\theta}{g}$ to find the range.

$\therefore$ range = $u_x \times$ time

$= (u\cos\theta)\dfrac{2u\sin\theta}{g} = \dfrac{u^2(2\sin\theta\cos\theta)}{g}$

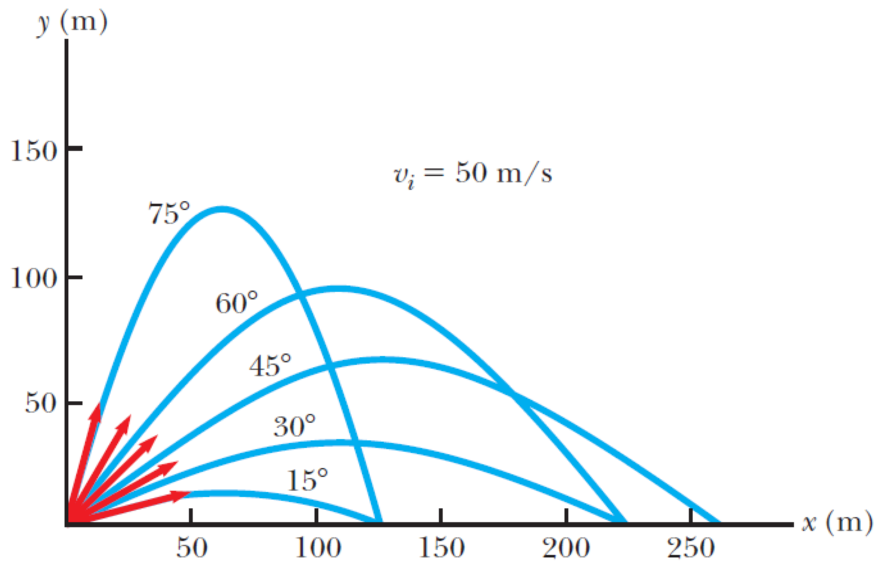
$\therefore$ $\boxed{\text{range} = \dfrac{u^2\sin 2\theta}{g} \quad \text{(no need to memorise)}}$

For **maximum range**, $\sin 2\theta = 1$

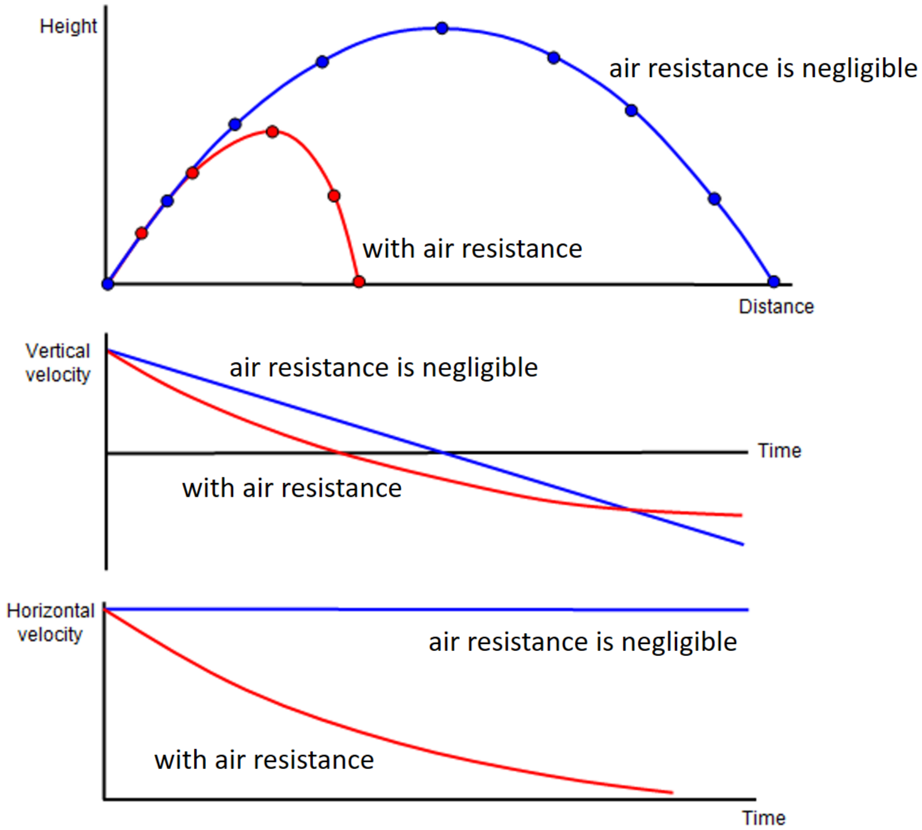
$\therefore$ **θ = 45°**

So, the maximum range = $\dfrac{u^2}{g}$

Notice that it is possible to obtain the same horizontal range with two different launching angles (refer to the figure below).



Notice that when air resistance is negligible, the parabolic trajectory is symmetrical about its highest point. When there is significant air resistance (e.g. when travelling at high speeds), the trajectory is no longer symmetrical (refer to the figures below).



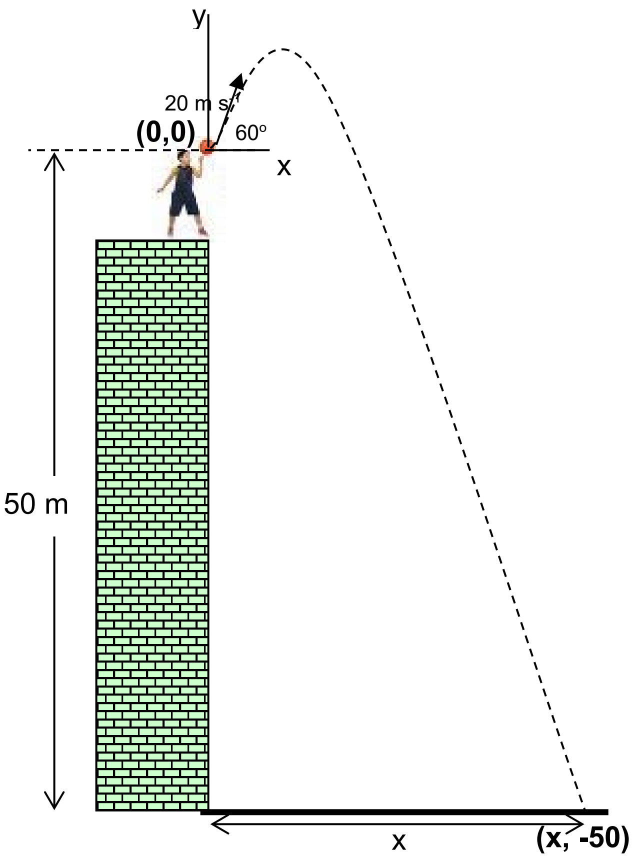
Example 17:

A basketball is thrown upward from the top of a 50 m-tall building at an angle of 60° to the horizontal with a speed of 20 m s^{-1}.

(a) How long does it take for the basketball to hit the ground?

(b) Find the basketball's speed upon impact.

(c) Find the horizontal range of the basketball.



Answer:

(a) When the ball hits the ground,

$s_y = -50$ m

The ball hits the ground after 5.42 s.

(b) Horizontal speed upon impact, $v_x = u_x =$

Resultant speed upon impact, $v = \sqrt{v_x^2 + v_y^2} =$

=

(c) From (a), the time of flight is 5.42 s.

So the range is

Example 18:

Derive an expression relating the horizontal distance travelled and the vertical displacement of an object undergoing projectile motion. This expression represents the path followed by the object. This "path" is called the **trajectory** of the object.

Answer:

Let the initial speed of the object be u.
Let its initial angle from the horizontal be θ.

Hence, at time t,
horizontal distance travelled, $x = u_x \times t$

$$\Rightarrow x = (u \cos \theta)\, t \quad \text{---(1)}$$

Vertical distance travelled in time t, $y = u_y t + \frac{1}{2} a_y t^2$

$$\Rightarrow y = (+u \sin \theta)t + \frac{1}{2}(-g)t^2$$

Using the same sign convention as the previous question,

$\therefore\ y = (u \sin \theta)t - \frac{1}{2} g t^2 \quad$ ---(2)

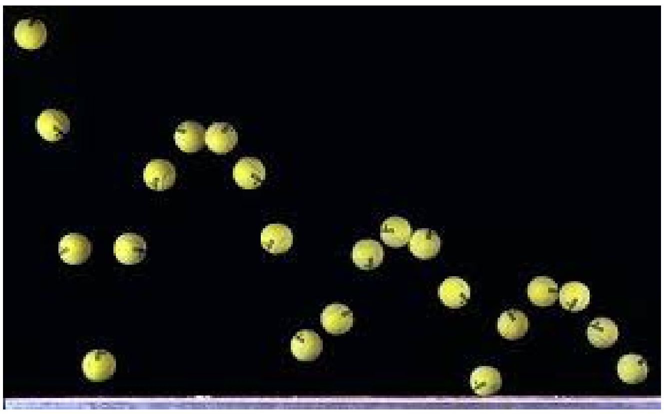
From (1), $t = \dfrac{x}{u \cos \theta}$

Substitute this into (2):

$$y = (u \sin \theta)\frac{x}{u \cos \theta} - \frac{1}{2} g \left(\frac{x}{u \cos \theta}\right)^2$$

$$y = x \tan \theta - \frac{g}{2u^2 \cos^2 \theta} x^2$$

Don't worry about the details of this equation. This is the equation of a **parabola**. With projectile motion (neglecting air resistance), the trajectory is always a parabola.



This is a strobe photo of a bouncing ball. The individual images are separated by equal time intervals.

Notice that the ball follows a parabolic trajectory after each bounce.

The ball rises a little less after each bounce. Why?

Test yourself!

If you throw a stone horizontally out over the surface of a lake, its time of flight (i.e. its time in the air before it hits the water) is determined only by
A. the height from which you throw it.
B. the height from which you throw it and its initial speed.
C. its initial speed and the horizontal distance to the point where it splashes down.

Two stones are launched from the top of a tall building. One stone is thrown in a direction 20° above the horizontal with a speed of 10 m s^{-1}; the other is thrown in a direction 20° below the horizontal with the same speed. How do their speeds compare just before they hit the ground below? (Neglect air resistance in this question.)
A. The one thrown upward is travelling faster.
B. The one thrown downward is travelling faster.
C. Both are travelling at the same speed.

Reference:

College Physics (8th Edition), Young and Geller, Pearson International Edition, Pg 29 – 58, 75 – 85.

Computer simulations

1. Applet on direction of velocity and acceleration:
   http://phet.colorado.edu/en/simulation/moving-man

2. Projectile motion:
   https://phet.colorado.edu/en/simulation/projectile-motion

SUMMARY

Term	Definition	Formula or concept
Distance, *d*	**Distance** is the total length of the path travelled by a body regardless of direction travelled.	Algebraic sum of the area under a *v-t graph* (no negative areas) = total distance travelled
Displacement, *s*	**Displacement**: the distance travelled in a straight line in a specified direction, from some reference point.	Vector sum of the area under a *v-t* graph (positive and negative areas) = change in displacement, Δs
Speed,	**Speed** is defined as the **rate of change of distance travelled** or as the distance travelled **per** unit time.	Average speed = $\dfrac{\text{Total distance travelled}}{\text{Total time taken}}$
Velocity, *v*	**Velocity** is defined as the **rate of change of displacement** or as the change of displacement **per** unit time.	area under an a-t graph = change in velocity, Δv gradient of s-t graph, $\dfrac{ds}{dt}$ = instantaneous velocity, *v* Average velocity = $\dfrac{\Delta s}{\text{Total time taken}}$
Acceleration, *a*	**Acceleration** is defined as **the rate of change of velocity** or as the change of velocity **per** unit time.	*a*=gradient of the *v-t* graph, $a=\dfrac{dv}{dt}$
Equations of uniform motion (*a* = constant)	Must rectilinear (in 1-direction)	$s=ut+\frac{1}{2}at^2$ $v=u+at$ $v^2=u^2+2as$ $s=\frac{1}{2}(u+v)t$
Projectile motion	Vertical direction	$s_y=u_yt+\frac{1}{2}gt^2$ $v_y=u_y+gt$ $v_y{}^2=u_y{}^2+2gs_y$ $s_y=\frac{1}{2}(u_y+v_y)t$ (Remember to check sign convention of *g*!)
	Horizontal direction (without air resistance)	$s_x=u_xt$ $v_x=u_x=\text{constant}$