NUMERICAL METHODS [FM]
TRAPEZIUM AND SIMPSON'S RULE

1 The function f is given by $f(x)=x(4-x)(8-x)^2$ for $x\in\mathbb{R}$.

(i) Use the trapezium rule with 4 equal intervals to estimate $\int_0^4 f(x)\,dx$. [3]

(ii) Explain why the trapezium rule underestimates the value of $\int_0^4 f(x)\,dx$. [1]

(iii) Use the Simpson's rule with 4 equal intervals to estimate $\int_0^4 f(x)\,dx$. [3]

(iv) Explain why the Simpson's rule gives a better estimate than the trapezium rule. [1]

(v) Use the trapezium rule with the same interval width as **(i)** to estimate $\int_0^8 f(x)\,dx$. [2]

[EJC/FM/2019/P2/Q3]

2 Let f be a function such that $f(-x)=-f(x)$ for $0\le x\le a$ where a is a positive constant.

The integral I is defined by $I=\int_0^{2a}\left[f(x-a)+b\right]dx$ where b is a constant.

(i) Use Trapezium rule with 3 ordinates, estimate I in terms of a and b. [3]

(ii) Use your answer in **(i)** to estimate the value of $\int_0^{\frac{\pi}{6}}\sqrt{3}+\tan\left(x-\frac{\pi}{12}\right)dx$. [1]

(iii) Show that the estimate in **(ii)** is the exact value of $\int_0^{\frac{\pi}{6}}\sqrt{3}+\tan\left(x-\frac{\pi}{12}\right)dx$. [2]

[TJC/FM/2019/MYE/Q3]

3 **(i)** Using the Simpson's rule with 5 ordinates, find an approximation to

$$\int_0^2 \sqrt[3]{x^4-2x^3+3x}\ dx,$$

leaving your answer to 3 decimal places. [3]

(ii) Hence, without the use of a calculator, find an approximation to $\int_{-2}^2 \sqrt[3]{x^4-2|x|^3+3|x|}\ dx$, leaving your answer to 3 decimal places. Justify your answer.

[2]

[NJC/FM/2018/MYE/Q2]

4 **(a)** The arc of the curve $y=-\ln(\cos x)$, from the point where $x=0$ to the point where $x=\dfrac{\pi}{6}$, is denoted by C. Show that S, the area of the surface generated when C is rotated through one revolution about the x-axis, is $2\pi\int_0^{\frac{\pi}{6}} \sec x \ln(\sec x)\,\mathrm{d}x$. [3]

Use the trapezium rule with 5 ordinates to find an approximation for the integral $\int_0^{\frac{\pi}{6}} \sec x \ln(\sec x)\,\mathrm{d}x$, giving your answer correct to 4 decimal places. Explain, with the aid of a sketch, whether the estimate is an over-estimate or an under-estimate. [5]

(b) The region R is bounded by the curves $x=\dfrac{y^4}{4}-\dfrac{y^2}{2}$ and $x=\dfrac{y^2}{2}$, from the point where $y=0$ to the point where $y=2$. Sketch the two curves in a single diagram, indicating clearly the region R. Find the exact volume of the solid formed when R is rotated completely about the line $y=-\dfrac{5}{8}$, giving your answer in the form $k\pi$, where k is a constant to be determined. [6]

[RI/FM/2019/MCT/Q6]

5 **(i)** By using the trapezium rule, with 6 ordinates, find an approximate value for $\int_0^{\pi} x^2 \sin\left(\dfrac{x}{2}\right)\mathrm{d}x$, giving your answer correct to 3 decimal places. Explain, with the aid of a sketch, why the value obtained is an overestimate. [3]

(ii) Explain the feasibility in applying Simpson's rule to the ordinates used in part **(i)**. [1]

(iii) Use Simpson's rule, with 4 intervals of equal width, to find an estimate for the integral in part **(i)**, leaving your answer correct to 3 decimal places. [2]

(iv) Explain why Simpson's rule usually gives a more accurate answer than the trapezium rule. [1]

[HCI et al/FM/2018/P1/Q7b]

6 By considering an appropriate area under the curve $y=\dfrac{1}{x^2}$ for $x>0$, use the trapezium rule with strips of unit width to show that

$$\frac{1}{2}-\frac{1}{n}-\frac{1}{2n^2}<\sum_{r=2}^{n-1}\frac{1}{r^2}.$$ [3]

Prove that the area of trapezium bounded by the x-axis, the lines $x=r\pm\frac{1}{2}$, and the tangent to the curve $y=\dfrac{1}{x^2}$ at the point $\left(r,\dfrac{1}{r^2}\right)$ is $\dfrac{1}{r^2}$. [3]

By considering an appropriate area under the curve $y=\dfrac{1}{x^2}$ for $x>0$, show that

$$\sum_{r=2}^{n-1}\frac{1}{r^2} < \frac{2}{3}-\frac{2}{2n-1}.$$ [3]

Hence, without using a graphic calculator, show that

$$1.49004<1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{100^2}<1.65672.$$ [2]

[ACJC/FM/2018/P2/Q5]

7 Let $I=\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$. An estimate of I denoted by I_n is obtained by using the trapezium rule with n trapeziums each of equal width.

(i) Find I_3. [2]

(ii) It is given that $I \approx I_n + k\left(\frac{1}{n^2}\right)$, where k is a constant independent of n. By taking $I_4 = 0.7828$ and using your value of I_3, find a better approximation to I, leaving your answer to four decimal places. [3]

Hence, find an approximation for π, giving your answer to four decimal places. [2]

[TJC/FM/2019/P1/Q2]

8 The ***mid-point rule*** using one strip to approximate the integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is given by

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx (b-a)\,\mathrm{f}\left(\frac{a+b}{2}\right).$$

The integral $I=\int_0^2 \mathrm{f}(x)\,\mathrm{d}x$, for a given $\mathrm{f}(x)$, is being evaluated numerically by the mid-point rule and the trapezium rule. The following estimates to 6 decimal places have been obtained.

Number of strips	Mid-point rule	Trapezium rule
1	3.464102	3.650282
2	3.510411	I_T

(i) Show that $I_T = 3.557192$. [3]

(ii) Use Simpson's rule with two strips to determine an estimate of I. [2]

(iii) Give a value of I to a degree of accuracy that appears justified. [1]

[VJC/FM/2019/P1/Q2]

9 **(i)** Find the exact value of $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$. [2]

(ii) Use the trapezium rule with 7 ordinates to find an estimate for $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$, leaving your answer correct to 8 decimal places. [3]

(iii) Use Simpson's rule, with 6 intervals of equal width, to find an estimate for $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$, leaving your answer correct to 8 decimal places. [2]

The Simpson's $\frac{3}{8}$ rule is another numerical integration method that is based on a cubic interpolation rather than a quadratic interpolation. This method can be used when the number of intervals are multiples of 3. The Simpson's $\frac{3}{8}$ rule states that where $h = \frac{b-a}{n}$,

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{3h}{8}\left(\mathrm{f}(x_0) + 3\sum_{i=1}^{\frac{n}{3}} \mathrm{f}(x_{3i-2}) + 3\sum_{i=1}^{\frac{n}{3}} \mathrm{f}(x_{3i-1}) + 2\sum_{i=1}^{\frac{n}{3}-1} \mathrm{f}(x_{3i}) + \mathrm{f}(x_n) \right)$$

(iv) Use Simpson's $\frac{3}{8}$ rule, with 6 intervals of equal width, to find an estimate for $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$, leaving your answer correct to 8 decimal places. [3]

(v) Compare how well the above three methods provide estimates for $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$. Give an advantage of the Simpson's $\frac{3}{8}$ rule as compared to the Simpson's rule. [3]

[HCI/FM/2019/MCT/Q7]

10 **(a)** Using Simpson's rule with 3 ordinates, obtain a value for $\int_{x_0-h}^{x_0+h} x^4\,\mathrm{d}x$, where h is a positive constant, and show that the value you obtain is in error by kh^5, where k is a positive constant to be determined exactly. [5]

(b) The differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy - y^2$$

is to be solved numerically using the second order *Taylor* series

$$y(x_0+h) \approx y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0).$$

Given that $y(0)=1$ and taking $h=0.2$, calculate successively the value of $y(0.2)$ and $y(0.4)$. [4]

[NJC/FM/2019/P1/Q6]

11 Explain why Simpson's rule usually gives a better approximation than trapezium rule does. [2]

A curve has equation $y = \mathrm{f}(x) = \sin x, 0 \le x \le \pi$.

(i) Write down a definite integral for the length of the curve. [1]

(ii) By using a calculator, evaluate, correct to 10 significant figures, the integral in part **(i)** by trapezium rule and by Simpson's rule, each with 9 ordinates. [6]

[2016 CJC/Promo/8]

<u>Answers</u>

1 (i) 366 (iii) 392 (v) 252

2 (i) $I = 2ab$ (ii) $\dfrac{\sqrt{3}\pi}{6}$

3 (i) 2.394 (ii) omitted

4 (a) 0.0279 (b) 4π

5 (i) 9.341 (iii) 9.116

7 (i) $I_3 = \dfrac{203}{260}$ (≈ 0.781) (ii) $I \approx 0.7854$ (iii) $\pi \approx 3.1416$

8 (ii) $I \approx 3.53$ (3sf) (iii) omitted

9 **(i)** $\dfrac{\pi}{4}$ **(ii)** 0.78424077 **(iii)** 0.78539795 (iv) 0.78539586

10 (a) $2{x_0}^4 h + 4{x_0}^2 h^3 + \dfrac{2}{5}h^5$; $k = \dfrac{4}{15}$ (b) $y(0.2) \approx 0.88$; $y(0.4) \approx 0.842$

11 **(i)** $\int_0^{\pi} \sqrt{1 + (\cos x)^2}\,\mathrm{d}x$

(ii) Trapezium rule: 3.820197715; Simpson's rule: 3.820282406