ANDERSON SERANGOON JUNIOR COLLEGE

JC2 Preliminary Examination 2023

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **14 September 2023**

3 hours

Additional Materials: **Answer Booklets**

List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklet. If you need additional answer booklet, ask the invigilator for a continuation booklet.

Write your name and class on the cover page and on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1 Given that a is a complex number with $|a| = 3$ and $0 < \arg(a) < \frac{\pi}{4}$, shade, on an Argand diagram, the region corresponding to the complex number z for which $|z| \leq |z - 2\sqrt{3}|$ and $|z - a| \leq 3$.

Hence or otherwise, find the complex number a in cartesian form for which the maximum value of $|z|$ is $2\sqrt{3}$. [5]

2 Points F and F' are the foci of an ellipse and point P is a point on the ellipse. The line L is the tangent to the ellipse at point P. The angles θ and γ are the acute angles that FP and $F'P$ make with the tangent respectively.

Using the reflective property of ellipse, prove that $(FP)(F'P) = \dfrac{b^2}{\sin^2\theta}$ where b is the length of the semi-minor axis of the ellipse. [5]

3 **(i)** Using the substitution $u = y^{1-n}$, show that the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{y}{x^2} = \dfrac{1}{(1-n)x^3}y^n$ where $y > 0$ can be written as $\dfrac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{1-n}{x^2}u = \dfrac{1}{x^3}$. [2]

(ii) For $n = 3$, find an expression for y in terms of x. [5]

4 The curve C has polar equation given by $r = \sin\theta + \frac{1}{2}\cos 2\theta$, $0 \leq \theta \leq \pi$.

(i) Find the exact maximum value of r. [2]

(ii) Sketch the curve C. [2]

(iii) The point in the first quadrant where r is at its maximum is P. Find the area bounded by the curve C from $\theta = 0$ to P, the tangent at point P and the x-axis, leaving your answer in the form $a\sqrt{3} + b\pi + c$ where a, b, c are constants to be determined in exact form. [4]

5 Explain geometrically why $I=\int_0^1\left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x=\frac{1}{4}\pi.$ [1]

Obtain an estimate of the value of I using Simpson's rule with four intervals giving your answer to 4 decimal places. [2]

Show geometrically that

$$I+\frac{1}{2}=2\int_0^{\frac{1}{\sqrt{2}}}\left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x.$$ [3]

Applying Simpson's rule with four intervals to the integral $\int_0^{\frac{1}{\sqrt{2}}}\left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x$, obtain a second estimate of the value of I, giving your answer to 4 decimal places. [2]

Explain why the second method gives a better approximation to the value of I than the first method. [1]

6 A fishing club's waters are stocked with a particular variety of fish. Each season, breeding increases the population of fish by a **proportion** b of the population at the start of the season. Similarly, a **proportion** d of the population at the start of the season dies during the season. Also, a **number** c of fish is caught and removed from the waters each season.

(a) Write down a recurrence relation in the form $u_{n+1}=pu_n+q$, where u_n is the fish population at the start of the n^{th} season, and p and q are constants to be expressed in terms of b, c and d. [2]

(b) Solve the recurrence relation in (a) to find a general expression for u_n in terms of p, q and u_1. [3]

(c) Given that $p = 1.08$ and $q = -160$, find the size of the fish population which would remain constant from season to season. [2]

(d) The club is anxious not to have to re-stock by buying in fish each year since this is expensive. They propose to achieve this by ensuring that the initial stock is above the equilibrium population.

(i) What does the mathematical model which you have developed predict will happen under these circumstances [2]

(ii) What is actually likely to happen and why? [1]

7 **(a)** Using the substitution $v = 2u^2 + a$, show that

$$\int_{\alpha}^{\beta} u^3\sqrt{u^4 + au^2 + b}\ du = \int_{\alpha'}^{\beta'} f(v)\ dv$$

where the expressions for α', β' and $f(v)$ are to be written down clearly. [2]

(b) The parametric equations of a curve C_1 are given by

$$x = 3t^2, \quad y = t^2(2t-3) \quad \text{where } t \geq 0.$$

Find the range of values of t for which the curve C_1 lies below the x-axis. [1]

Hence show, by using a linear transformation or otherwise, that the length, r, of the curve C_1 which is below the x-axis can be expressed as

$$r = \int_{-1}^{\frac{1}{2}} 6(w+1)\sqrt{1+w^2}\ \mathrm{d}w$$ [3]

Given another curve C_2 with parametric equations

$$3x = t^3 - 3t, \quad 3y = t^3$$

Show that the area of the curved surface, s, obtained by rotating completely the arc of the curve C_2 from the points $t = 0$ to $t = -\frac{\sqrt{3}}{2}$ about the x-axis is given by

$$s = \frac{\sqrt{2}\pi r}{144}.$$ [4]

8 A pendulum consists of a ball of mass m kg tied to one end of an inextensible string with length, L cm, as shown in Figure 1 below. The string is inclined to the vertical at an angle of θ radians t seconds after the initial release of the ball. Initially, the ball is released at θ_0 radians.

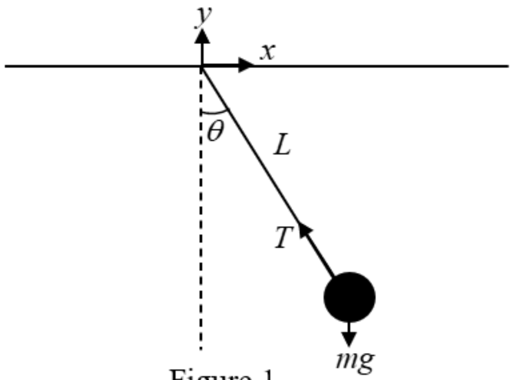

Figure 1

The horizontal and vertical displacement of the ball are defined by x cm and y cm respectively.

(i) Show that $\dfrac{d^2x}{dt^2} = L\cos\theta\dfrac{d^2\theta}{dt^2} - L\sin\theta\left(\dfrac{d\theta}{dt}\right)^2$ and

$$\frac{d^2y}{dt^2} = L\sin\theta\frac{d^2\theta}{dt^2} + L\cos\theta\left(\frac{d\theta}{dt}\right)^2.$$ [2]

The resultant horizontal force, F_x and resultant vertical force, F_y acting on the ball are given by $(-T\sin\theta)$N and $(T\cos\theta - mg)$N respectively where T is the tensile force in the string and g is the gravitational acceleration and N represents a newton which is the international unit measure for force.

Newton's 2nd law of motion states that the acceleration of a body is proportional to the resultant force acting on the body and takes place in the direction of the resultant force i.e. $\mathbf{F} = m\mathbf{a}$ where $\mathbf{a}$ is the acceleration.

(ii) By applying Newton's 2nd law of motion in the horizontal direction, we get $-T\sin\theta = m\left[L\cos\theta\dfrac{d^2\theta}{dt^2} - L\sin\theta\left(\dfrac{d\theta}{dt}\right)^2\right]$-------- (1).

Obtain another equation by applying Newton's 2nd law of motion in the vertical direction and hence show that $\dfrac{d^2\theta}{dt^2} \approx -\dfrac{g}{L}\theta$ if θ is small. [4]

For part **(iii)**, take $\dfrac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\dfrac{g}{L}\theta$.

(iii) It is given that $\theta_0 = \dfrac{\pi}{12}$ and $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = -\dfrac{\pi}{12}\sqrt{\dfrac{g}{L}}$ when $t = \sqrt{\dfrac{L}{g}}\pi$.
Find θ in terms of t, g and L and state the maximum value of θ. [5]

(iv) A 'modified' Euler method can be used to estimate the value of θ and ω where $\omega = \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$,

i.e. $\theta_2 = \theta_1 + h\left.\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{\theta=\theta_1,\ \omega=\omega_1}$ and $\omega_2 = \omega_1 + h\left.\dfrac{\mathrm{d}\omega}{\mathrm{d}t}\right|_{\theta=\theta_1,\ \omega=\omega_1}$

Given that $\omega = 0$ when $t = 0$, use the 'modified' Euler method to find an estimate for θ in terms of θ_0, g and L when $t = 2$ with a step size of 1 if θ is **not** small. [3]

9 The linear transformation $T: R^4 \to R^3$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & 5-\alpha \\ -1 & -3 & \alpha & -3 \end{pmatrix} \text{ where } \alpha \text{ is a constant.}$$

The null space of T is denoted by K_1 when $\alpha = 5$, and by K_2 when $\alpha \neq 5$.

(i) Find a basis for K_1. Find also a basis for K_2 in terms of α.
Deduce the rank of **A** for each of the cases. [8]

For the rest of the question, use $\alpha = 5$.

(ii) Find the set of vectors S consisting of all vectors $\mathbf{x}$ such that

$$\mathbf{Ax} = \lambda\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \text{ for any } \lambda \in \mathbb{R}.$$

Determine whether the set S is a vector space. [4]

(iii) Write down the basis for the range space, R of T. [1]

(iv) The set of elements of $\mathbb{R}^3$ which do not belong to the range space, R of T is denoted by U. Show that if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U$, then $3x - y + z \neq 0$. [3]

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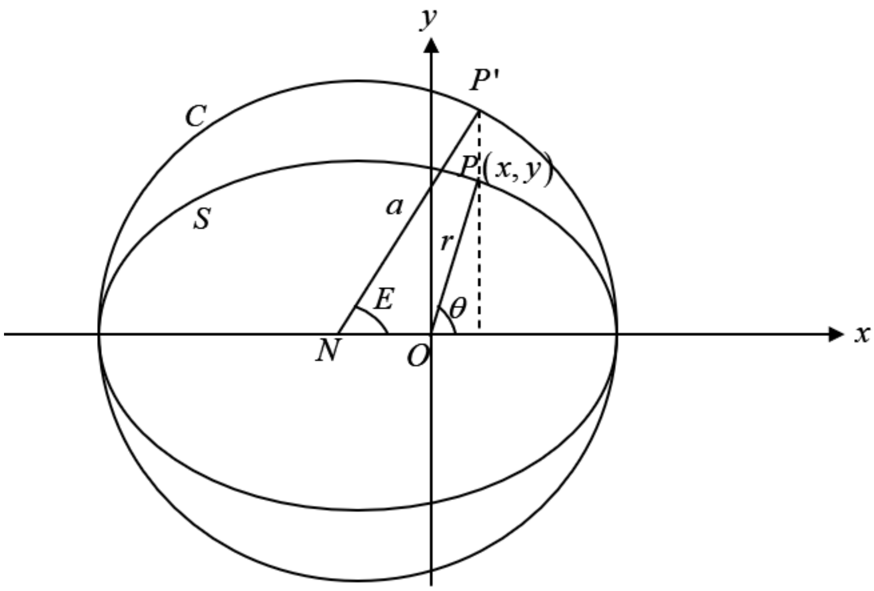

The equation of an ellipse S is given by $r=\dfrac{a\left(1-e^2\right)}{1+e\cos\theta}$, where a is a positive constant and e is the eccentricity of the ellipse. The point O, with coordinates (0,0), is a focus of the ellipse and the point P has coordinates (x, y). θ is the angle between OP and the positive x-axis. The circle C has radius a and has the same centre as the ellipse at N. The point P' is the point on C that is vertically above P.

(i) Using a single transformation or otherwise, show that $y=b\sin E$ where b is the length of the semi-minor axis of the ellipse and E is the angle between NP' and the positive x-axis. [2]

(ii) By considering the point P or otherwise, show that $r=a\left(1-e\cos E\right)$. [3]

(iii) Hence show that $\tan\dfrac{\theta}{2}=\sqrt{\dfrac{1+e}{1-e}}\tan\dfrac{E}{2}$ when $0\le\theta\le\pi$. [7]

The ellipse S where $e=\dfrac{1}{4}$ can be used to model the orbit of the planet P around the Sun at O. θ and E are known to be the true anomaly and eccentric anomaly respectively. The mean anomaly, M, is related to the eccentric anomaly by the equation $M=E-e\sin E$.

(iv) Using Newton-Raphson method with an initial estimate for E to be $\left(1+\dfrac{3}{4}e\right)M$, find an estimate for the eccentric anomaly when $M=\dfrac{4\pi}{19}$, correct to 4 decimal places and hence find an estimate for the true anomaly. [4]

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