2023 EJC Prelim Paper 1 Solutions

Q1

Sub $x=0$, $A+B+C=2$
$\frac{dy}{dx}=Ae^x+2Be^{2x}-2Ce^{-2x}$
Sub $x=0$, $A+2B-2C=-3$
$\frac{d^2y}{dx^2}=Ae^x+4Be^{2x}+4Ce^{-2x}$
Sub $x=0$, $A+4B+4C=11$
Solving simultaneously, $A=-1,\ B=1,\ C=2$.
So particular solution is $y=f(x)=-e^x+e^{2x}+2e^{-2x}$.

Q2	
(a)	$u_n=S_n-S_{n-1}=18-\frac{2^{n+1}}{3^{n-2}}-\left(18-\frac{2^n}{3^{n-3}}\right)=\frac{2^n}{3^{n-3}}-\frac{2^{n+1}}{3^{n-2}}=\frac{(3)2^n}{3^{n-2}}-\frac{2^{n+1}}{3^{n-2}}=\frac{2^n(3-2)}{3^{n-2}}=\frac{2^n}{3^{n-2}}$ $\frac{u_n}{u_{n-1}}=\frac{\frac{2^n}{3^{n-2}}}{\frac{2^{n-1}}{3^{n-3}}}=\frac{2}{3}$ Since $\frac{u_n}{u_{n-1}}$ is a constant independent of n, the sequence is a geometric progression.
(b)	As $n\to\infty$, $S_n=18-18\left(\frac{2}{3}\right)^n\to 18$ (Or using G.P. sum to infinity formula $S_\infty=\frac{6}{1-\frac{2}{3}}=18$) Let the common difference be d. $\frac{9}{2}\left[2(-4)+8d\right]=18$ $-8+8d=4$ $d=\frac{3}{2}$

Q3

(a) $\overrightarrow{OP}=\dfrac{\lambda\overrightarrow{OB}+(1-\lambda)\overrightarrow{OA}}{\lambda+(1-\lambda)}=\lambda\begin{pmatrix}0\\1\\-2\end{pmatrix}+(1-\lambda)\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}2-2\lambda\\1\\-1-\lambda\end{pmatrix}$

$\overrightarrow{OQ}=\dfrac{\lambda\overrightarrow{OC}+(1-\lambda)\overrightarrow{OB}}{\lambda+(1-\lambda)}=\lambda\begin{pmatrix}1\\3\\0\end{pmatrix}+(1-\lambda)\begin{pmatrix}0\\1\\-2\end{pmatrix}=\begin{pmatrix}\lambda\\1+2\lambda\\2\lambda-2\end{pmatrix}$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}3\lambda-2\\2\lambda\\3\lambda-1\end{pmatrix}$

(b) $\overrightarrow{PX}=\overrightarrow{OX}-\overrightarrow{OP}=\begin{pmatrix}2\\-1\\-5/2\end{pmatrix}-\begin{pmatrix}2-2\lambda\\1\\-1-\lambda\end{pmatrix}=\begin{pmatrix}2\lambda\\-2\\\lambda-3/2\end{pmatrix}$

Since P, Q and X are collinear, $\overrightarrow{PX}=k\overrightarrow{PQ}$ for some $k\in\mathbb{R}, k\neq 0$

$\begin{pmatrix}2\lambda\\-2\\\lambda-3/2\end{pmatrix}=k\begin{pmatrix}3\lambda-2\\2\lambda\\3\lambda-1\end{pmatrix}$

$2\lambda=k(3\lambda-2)---(1)$

$-2=2k\lambda------(2)$

$\lambda-\dfrac{3}{2}=k(3\lambda-1)--(3)$

From (2), $k=-\dfrac{1}{\lambda}$

Sub $k=-\dfrac{1}{\lambda}$ into (1), $-2\lambda^2=3\lambda-2$

$2\lambda^2+3\lambda-2=0$

$\lambda=\dfrac{1}{2}$ or $\lambda=-2$ (NA $\because 0<\lambda<1$)

Checking,

Sub $k=-\dfrac{1}{\lambda}$ into (3), $-\lambda^2+\dfrac{3}{2}\lambda=3\lambda-1$

$2\lambda^2+3\lambda-2=0$

$\lambda=\dfrac{1}{2}$ or $\lambda=-2$ (NA $\because 0<\lambda<1$)

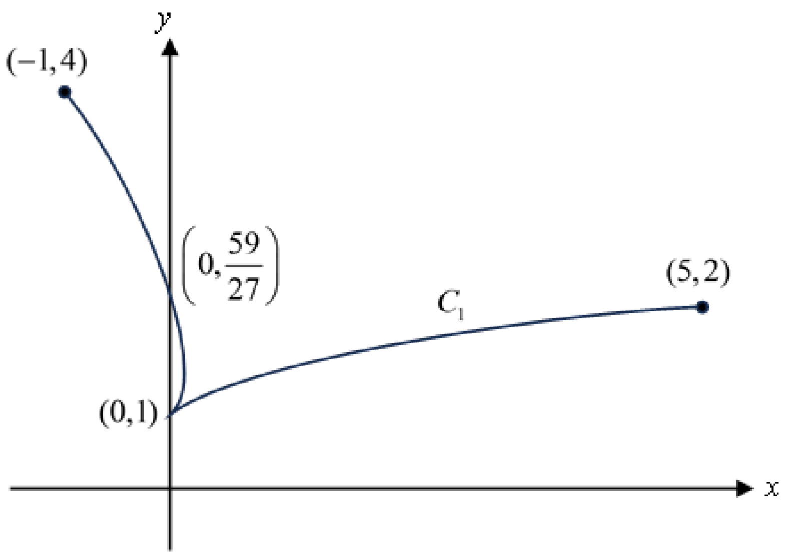
$\therefore \lambda=\dfrac{1}{2}$

Q4	
(a)	$\sum_{r=2}^{n} f(r) = \sum_{r=2}^{n} \frac{2}{r^2-1}$ $= \sum_{r=2}^{n} \frac{2}{(r-1)(r+1)}$ $= \sum_{r=2}^{n} \frac{1}{r-1} - \frac{1}{r+1}$ $= \frac{1}{1} - \frac{1}{3}$ $+ \frac{1}{2} - \frac{1}{4}$ $+ \frac{1}{3} - \frac{1}{5}$ $+ \frac{1}{4} - \frac{1}{6}$ $+ \vdots$ $+ \frac{1}{n-3} - \frac{1}{n-1}$ $+ \frac{1}{n-2} - \frac{1}{n}$ $+ \frac{1}{n-1} - \frac{1}{n+1}$ $= \frac{3}{2} - \frac{1}{n} - \frac{1}{n+1}$
(b)	$\sum_{r=11}^{3n-1} \frac{2}{r(r+2)}$ Replace r with $r-1$, $= \sum_{r-1=11}^{r-1=3n-1} \frac{2}{(r-1)(r-1+2)}$ $= \sum_{r=12}^{3n} \frac{2}{r^2-1}$ $= \left(\sum_{r=2}^{3n} \frac{2}{r^2-1}\right) - \left(\sum_{r=2}^{11} \frac{2}{r^2-1}\right)$ $= \left(\frac{3}{2} - \frac{1}{3n} - \frac{1}{3n+1}\right) - \left(\frac{3}{2} - \frac{1}{11} - \frac{1}{12}\right)$ $= \frac{23}{132} - \frac{1}{3n} - \frac{1}{3n+1}$

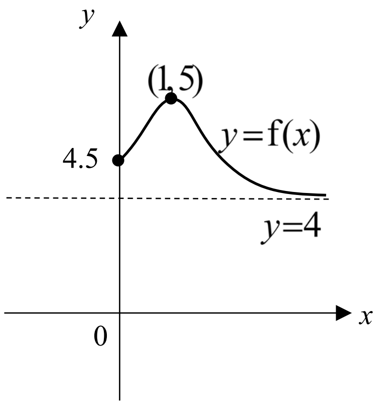
Q5
(a) $\int \sin 3x \cos x \, dx = \frac{1}{2}\int \sin 4x + \sin 2x \, dx$ $= \frac{1}{2}\left(\frac{-\cos 4x}{4} - \frac{\cos 2x}{2}\right) + c$ $= -\frac{1}{8}(\cos 4x + 2\cos 2x) + c$ **(b)** $\int e^{2x} \cos 3x \, dx$ $= \frac{1}{2}(\cos 3x)e^{2x} + \frac{3}{2}\int \sin 3x\left(e^{2x}\right) dx$ $= \frac{1}{2}(\cos 3x)e^{2x} + \frac{3}{2}\left[\left(\frac{e^{2x}}{2}\sin 3x - \frac{3}{2}\int (\cos 3x)\left(e^{2x}\right) dx\right)\right]$ $= \frac{1}{2}(\cos 3x)e^{2x} + \frac{3}{4}\sin 3x\left(e^{2x}\right) - \frac{9}{4}\int \cos 3x\left(e^{2x}\right) dx$ Moving the integral over, we have $\frac{13}{4}\int \cos 3x\left(e^{2x}\right) dx = \frac{1}{2}\cos 3x\left(e^{2x}\right) + \frac{3}{4}(\sin 3x)e^{2x} + c$ $\int \cos 3x\left(e^{2x}\right) dx = \frac{4}{13}\left(\frac{1}{2}\cos 3x\left(e^{2x}\right) + \frac{3}{4}(\sin 3x)e^{2x}\right) + c_1$ $= \frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x) + c_1$

Q6	
(a)	$y = f(x) \xrightarrow{\text{scale}} \frac{y}{3} = f(x) \xrightarrow{\text{translate}} \frac{y}{3} = f(x+4)$ So the equation of *C* is $y = 3f(x+4)$. Since we know $f(7) = -9$, substituting $x = 3$ we get $y = -27$. Hence the corresponding point is $(3, -27)$. Additionally, we are given that $f'(7) = 10$. So differentiating the equation for *C* we get $y' = 3f'(x+4)$. Again, substituting $x = 3$ we get $y' = 30$ i.e. the gradient at this point is 30.
(b)	The corresponding point is $(7, -\frac{1}{9})$. Differentiate $y = \frac{1}{f(x)}$:

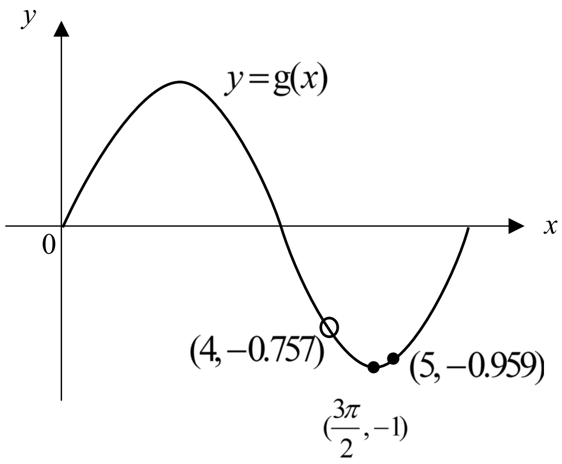
	$\frac{dy}{dx} = -\frac{f'(x)}{[f(x)]^2}$ We are given that $f(7) = -9$ and $f'(7) = 10$. So At $x = 7$, $\left.\frac{dy}{dx}\right\|_{x=7} = -\frac{(10)}{(-9)^2} = -\frac{10}{81}$

Q7	
(a)	When $x = 0$, $3t^3 + 2t^2 = 0$. $t^2(3t+2) = 0$, $t = 0$ or $t = -\frac{2}{3}$ When $t = 0$, $y = 0^2 + 1 = 1$. When $t = -\frac{2}{3}$, $y = -\left(-\frac{2}{3}\right)^3 + 2\left(-\frac{2}{3}\right)^2 + 1 = \frac{59}{27}$. When $t = -1$, $x = -1$ and $y = 4$. When $t = 1$, $x = 5$ and $y = 2$. 
(b)	The curve is a circle with centre (0, 1) and radius $\sqrt{k}$.
(c)	Substitute $x = 3t^3 + 2t^2$ and $y = -t^3 + 2t^2 + 1$ into C_2,

	$\left(3t^3+2t^2\right)^2+\left(-t^3+2t^2+1-1\right)^2=16$ $\left(9t^6+12t^5+4t^4\right)+\left(t^6-4t^5+4t^4\right)-16=0$ $10t^6+8t^5+8t^4-16=0$ Using GC, $t=0.90823$ (reject root smaller than -1 and non-real roots) Substituting, $x=3.8973$ and $y=1.9006$. The coordinates of the point of intersection are $(3.90,1.90)$.
(d)	There can be 0, 1 or 2 points of intersection.

Q8	
(a)	
(b)	Any 2 points with the same y-value, e.g. any 2 of $(0,0)$, $(\pi,0)$, $(2\pi,0)$
(c)	From graph in part (a), $R_f=(4,5]$. Since $R_f=(4,5]\subseteq[0,2\pi]=D_g$, the composite function gf exists.

(d)



Since $R_f = (4,5]$, we consider the graph of $y = g(x)$ or $y = \sin x$ restricted to $4 < x \le 5$.

When $4 < x \le 5$, $-1 \le g(x) < -0.757$, so the range of gf is $[-1, -0.757)$.

Q9

a

$$\frac{1}{\sqrt{1-a^2x^2}} = \left(1+\left(-a^2x^2\right)\right)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)\left(-a^2x^2\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-a^2x^2\right)^2 + \ldots$$

$$= 1 + \frac{a^2}{2}x^2 + \frac{3a^4}{8}x^4 + \ldots$$

The expansion is valid for

$\left|-a^2x^2\right| < 1$

$a^2x^2 < 1$ since $a^2x^2 \ge 0$

$x^2 < \frac{1}{a^2}$ since a is positive

From graph, $-\frac{1}{a} < x < \frac{1}{a}$

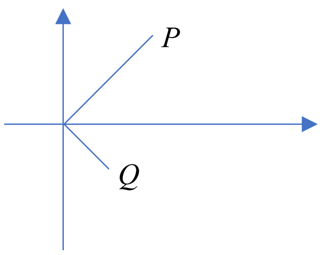
b

Since $\int -\frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$, substituting $a = 1$ into the expansion obtained in **(a)** and integrating,

$$\cos^{-1} x \approx -\int 1 + \frac{x^2}{2} + \frac{3}{8}x^4 \, dx$$

	$\cos^{-1} x \approx c - x - \frac{x^3}{6} - \frac{3}{40}x^5$ for some constant c Since $\cos^{-1} 0 = \frac{\pi}{2}$, substituting $x = 0$, $c = \frac{\pi}{2}$. $\therefore \cos^{-1} x = \frac{\pi}{2} - \frac{x^3}{6} - \frac{3}{40}x^5 + \ldots$
c	$\int_0^{\frac{1}{2}} \cos^{-1} x \, dx \approx \int_0^{\frac{1}{2}} \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3x^5}{40} \, dx = 0.65760 \ (5sf)$
d	By GC, $\int_0^{\frac{1}{2}} \cos^{-1} x \, dx = 0.65757 \ (5sf)$ Estimate in **(c)** is accurate to 4sf but not to 5sf. To improve estimate, we can include higher-order terms in the Maclaurin series expansion of $\cos^{-1} x$

Q10	
(a)	$z + w$ $= r(\cos\alpha + \mathrm{i}\sin\alpha) + r(\cos\beta + \mathrm{i}\sin\beta)$ $= r((\cos\alpha + \cos\beta) + \mathrm{i}(\sin\alpha + \sin\beta))$ $= r\left(2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \mathrm{i}\cdot 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right)$ $= 2r\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha+\beta}{2} + \mathrm{i}\sin\frac{\alpha+\beta}{2}\right)$ $\lvert z + w\rvert = 2r\cos\frac{\alpha-\beta}{2}$ $\arg(z + w) = \frac{\alpha+\beta}{2}$ <u>Alternative</u> $z + w$ $= r\mathrm{e}^{\mathrm{i}\alpha} + r\mathrm{e}^{\mathrm{i}\beta}$ $= r\mathrm{e}^{\mathrm{i}\left(\frac{\alpha+\beta}{2}\right)}\left(\mathrm{e}^{\mathrm{i}\left(\frac{\alpha-\beta}{2}\right)} + \mathrm{e}^{-\mathrm{i}\left(\frac{\alpha-\beta}{2}\right)}\right)$ $= r\mathrm{e}^{\mathrm{i}\left(\frac{\alpha+\beta}{2}\right)}\cdot 2\cos\left(\frac{\alpha-\beta}{2}\right)$ $= 2r\cos\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha+\beta}{2} + \mathrm{i}\sin\frac{\alpha+\beta}{2}\right)$ $\lvert z + w\rvert = 2r\cos\frac{\alpha-\beta}{2}$

	$\arg(z+w)=\dfrac{\alpha+\beta}{2}$
(b)	$\lvert z+w\rvert=2(2)\cos\dfrac{1}{2}\left(\dfrac{5\pi}{12}-\dfrac{\pi}{12}\right)=4\cos\dfrac{\pi}{6}=2\sqrt{3}$ $\arg(z+w)=\dfrac{1}{2}\left(\dfrac{\pi}{12}+\dfrac{5\pi}{12}\right)=\dfrac{\pi}{4}$
(c)	$\lvert v\rvert=\dfrac{\lvert z\rvert^2}{\lvert w\rvert}=2$ $\arg(v)=2\arg(z)-\arg(w)=2\times\dfrac{\pi}{12}-\dfrac{5\pi}{12}=-\dfrac{\pi}{4}$
(d)	Since $\arg(z+w)=\dfrac{\pi}{4}$ and $\arg(v)=-\dfrac{\pi}{4}$, angle POQ is a right angle. Thus area of triangle OPQ $=\dfrac{1}{2}\lvert z+w\rvert\lvert v\rvert=2\sqrt{3}$ units^2.  Alternatively, Area of triangle OPQ $=\dfrac{1}{2}(2\sqrt{3})(2)\sin\left(\dfrac{\pi}{4}-\dfrac{-\pi}{4}\right)$ $=2\sqrt{3}\sin\left(\dfrac{\pi}{2}\right)$ $=2\sqrt{3}$ units^2

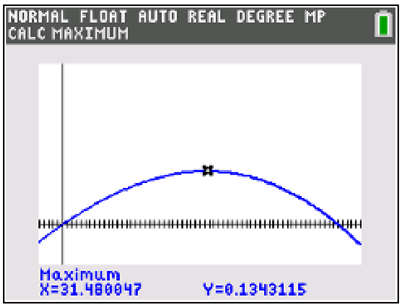
Q11	
(a)	Volume $= \pi \int_0^k \left(k + \frac{2k}{(x+1)(x+2)}\right) - \left(k - \frac{x}{3}\right) \mathrm{d}x$ $= \pi \int_0^k \frac{2k}{(x+1)(x+2)} + \frac{x}{3} \mathrm{d}x$ $= \pi \int_0^k \frac{2k}{x+1} - \frac{2k}{x+2} + \frac{x}{3} \mathrm{d}x$ $= \pi \left[2k\ln(x+1) - 2k\ln(x+2) + \frac{x^2}{6}\right]_0^k$ $= \pi \left[2k\ln(k+1) - 2k\ln(k+2) + \frac{k^2}{6} - (-2k\ln 2)\right]$ $= \pi \left[2k\ln\left(\frac{2(k+1)}{k+2}\right) + \frac{k^2}{6}\right]$ $= \pi \left[2k\ln\left(\frac{2k+2}{k+2}\right) + \frac{k^2}{6}\right]$ (shown)
(b)	$2\int_0^k \sqrt{k - \frac{x}{3}} \mathrm{d}x$ $= (-6)\int_0^k -\frac{1}{3}\sqrt{k - \frac{x}{3}} \mathrm{d}x$ $= (-6)\left[\frac{\left(k - \frac{x}{3}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^k$ $= (-4)\left[\left(\frac{2k}{3}\right)^{\frac{3}{2}} - (k)^{\frac{3}{2}}\right]$ $= 4k^{\frac{3}{2}}\left[1 - \left(\frac{2}{3}\right)^{\frac{3}{2}}\right]$
(c)	$A_{\text{in}} = \pi\left(k - \frac{0}{3}\right) = k\pi$ $A_{\text{out}} = \pi\left(k - \frac{k}{3}\right) = \frac{2k\pi}{3}$

	Thus $k\pi v_{\text{in}} = \frac{2k\pi}{3} v_{\text{out}}$ $v_{\text{in}} = \frac{2}{3} v_{\text{out}} \Rightarrow \frac{v_{\text{out}}}{v_{\text{in}}} = \frac{3}{2}$

Q12	
(a)	Using sine rule, $\frac{x}{\sin(120° - \theta)} = \frac{L}{\sin 60°}$ $x = \frac{L}{\sin 60°}\left[\sin(120° - \theta)\right]$ $= \frac{L}{\sin 60°}\left[\sin 120° \cos\theta - \cos 120° \sin\theta\right]$ $= \frac{L}{\frac{\sqrt{3}}{2}}\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right]$ $x = L\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right)$ (shown)
(b)	Let t be time in seconds. Since x increases at a constant rate, $\frac{\mathrm{d}x}{\mathrm{d}t}$ is a constant and equals $\frac{(\text{final } x) - (\text{initial } x)}{\text{time taken}}$. When $\theta = 90°$, $x = L\left(\cos 90° + \frac{1}{\sqrt{3}}\sin 90°\right) = \frac{L}{\sqrt{3}}$. When $\theta = 30°$, $x = L\left(\cos 30° + \frac{1}{\sqrt{3}}\sin 30°\right) = L\left(\frac{\sqrt{3}}{2} + \frac{1}{2\sqrt{3}}\right) = \frac{2L}{\sqrt{3}}$. So $\frac{\mathrm{d}x}{\mathrm{d}t} = \left(\frac{2L}{\sqrt{3}} - \frac{L}{\sqrt{3}}\right) \div 60 = \frac{L}{60\sqrt{3}}$. Differentiating $x = L\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right)$ with respect to θ, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = L\left(-\sin\theta + \frac{1}{\sqrt{3}}\cos\theta\right)$.

	Thus, by chain rule, $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}x}{\mathrm{d}\theta}\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$, $\frac{L}{60\sqrt{3}}=L\left(-\sin\theta+\frac{1}{\sqrt{3}}\cos\theta\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$ When $\theta=60°$, $\frac{L}{60\sqrt{3}}=L\left(-\sin 60°+\frac{1}{\sqrt{3}}\cos 60°\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $\frac{1}{60\sqrt{3}}=\left(-\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\times\frac{1}{2}\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $\frac{1}{60\sqrt{3}}=\left(\frac{-3+1}{2\sqrt{3}}\right)\times\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{1}{60}$ rad s^{-1} (or s^{-1})
(c)	For the <u>correct motion</u>, x starts at $\frac{L}{\sqrt{3}}$ when $t=0$, and increases at constant rate of $\frac{L}{60\sqrt{3}}$ (found in part (b)). Hence $x_{\text{correct}}=\frac{L}{\sqrt{3}}+\frac{L}{60\sqrt{3}}t$ For the <u>incorrect motion</u>, at time t, notice that $\theta=(90-t)°$ and hence $x_{\text{incorrect}}=L\left(\cos(90-t)°+\frac{1}{\sqrt{3}}\sin(90-t)°\right)$ $=L\left(\sin t°+\frac{1}{\sqrt{3}}\cos t°\right)$ Thus, $D=x_{\text{incorrect}}-x_{\text{correct}}$ $=L\left(\sin t°+\frac{1}{\sqrt{3}}\cos t°\right)-\left(\frac{L}{\sqrt{3}}+\frac{L}{60\sqrt{3}}t\right)$ $=L\left(\sin t°+\frac{\cos t°-1}{\sqrt{3}}-\frac{t}{60\sqrt{3}}\right)$ or equivalent
(d)	$\frac{D}{L}=\sin t°+\frac{\cos t°-1}{\sqrt{3}}-\frac{t}{60\sqrt{3}}$

From GC, plotting $Y = \sin t^\circ + \dfrac{\cos t^\circ - 1}{\sqrt{3}} - \dfrac{t}{60\sqrt{3}}$ in degree mode,



The maximum value is 0.1343.