2023 NYJC J1 H2 Mathematics End-of-Year Exam 9758/1

1 (a) Sketch, on the same axes, the graphs of $y=5-\frac{1}{x-1}$ and $y=\frac{1}{(x-5)^2}$. [2]

(b) Solve the inequality $5-\frac{1}{x-1}<\frac{1}{(x-5)^2}$. [3]

2 (a) Find $\mathrm{f}'(x)$ where $\mathrm{f}(x)=\ln\left(x+\sqrt{1+x^2}\right)$, giving your answer in its simplest form. [3]

(b) Find $\mathrm{g}'(x)$ where $\mathrm{g}(x)=\tan^{-1}\left(\frac{2+x}{1-2x}\right)$, giving your answer in its simplest form. [3]

3 With reference to the origin O, the points A and B have position vectors **a** and **b** respectively.

(a) A point C has position vector **a** + **b**. State the geometric shape formed by O, A, B and C. [1]

Point M lies on AC, between A and C, such that $AM : MC = 1 : 2$.

(b) Show that the area of triangle OAM can be written as $\alpha|\mathbf{a}\times\mathbf{b}|$, where α is a constant to be found. [2]

It is given that $|\mathbf{b}| = 5$ and the shortest distance from A to OB is $\sqrt{3}$.

(c) Using your result from part **(b)**, find the exact value of the area of triangle OAM. [2]

(d) Point N lies on OM, between O and M, such that $ON : NM = 3 : 1$. Show that A, N and B are collinear and hence find the ratio $AN : NB$. [3]

4 (a) Describe fully a sequence of transformations which transforms the curve $y^2=x$ onto the curve $\frac{y^2}{4}=1-x$. [2]

(b) A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x)=\begin{cases}2\sqrt{1-x} & \text{for } 0\le x<1,\\ 0 & \text{otherwise.}\end{cases}$$

(i) Sketch the curve of $y = \mathrm{f}(x)$ for $-2\le x\le 2$, stating the coordinates of the end points and the coordinates of any points where the curve crosses the axes. [3]

(ii) Sketch the curve with equation $y=|\mathrm{f}(x)-1|$ for $-2\le x\le 2$, stating the coordinates of the end points and the coordinates of any points where the curve crosses the axes. [3]

5 The function g is defined by

$$\mathrm{g}: x\to e^{\left(x-\frac{1}{2}\right)^2},\ x\in\ ,\ -1<x<a.$$

(a) State the greatest value of a for which the function g^{-1} exists. [1]

For the rest of the question, let $a = 0$.

(b) Sketch the graphs of $y = \mathrm{g}(x)$, $y = \mathrm{g}^{-1}(x)$ and $y = \mathrm{g}^{-1}\mathrm{g}(x)$ on the same diagram, showing clearly the relationship between them. You should also state the coordinates of the end points of the graphs. [3]

The function h is defined by

$$\mathrm{h}: x \rightarrow 1+\ln x, x \in \quad, x>0.$$

(c) (i) Find hg(x) and state its domain. [2]

(ii) Hence, or otherwise, find the exact value of $(\mathrm{hg})^{-1}\left(\frac{3}{2}\right)$. [2]

6 Let $\mathrm{f}(r)=\ln(r)$, where r is a positive integer and $r \geq 2$.

(a) Show that $\mathrm{f}(r-1)-2\mathrm{f}(r)+\mathrm{f}(r+1)=\ln\left(1-\frac{1}{r^2}\right)$. [1]

(b) Show that $\sum_{r=2}^{N} \ln\left(1-\frac{1}{r^2}\right)=\ln\left(\frac{N+1}{2N}\right)$. [3]

(c) Explain why the series $\sum_{r=2}^{\infty} \ln\left(1-\frac{1}{r^2}\right)$ converges and state the exact value of the sum to infinity of this series. [2]

(d) Find an expression for the series

$$\ln\left(1-\frac{1}{(2k+1)^2}\right)+\ln\left(1-\frac{1}{(2k+2)^2}\right)+\ln\left(1-\frac{1}{(2k+3)^2}\right)+\ldots,$$

giving your answer as a single logarithmic function in terms of k. [3]

7 The parametric equations of a curve are $x=\frac{1}{2}\mathrm{e}^t-2\mathrm{e}^{-t}$ and $y=\mathrm{e}^t+2\mathrm{e}^{-t}$.

(a) Using calculus, find the exact gradient of the normal to the curve at the point where $t=\ln 4$. [3]

(b) Show that the equation of the normal is $5x+7y=39$. [2]

(c) Find the exact value of t where the normal meets the curve again. [4]

8 It is given that $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=\tan(\alpha+\beta x)$, for constants α and β.

(a) Show that $\mathrm{f}'(x)=\beta(1+y^2)$. Hence find $\mathrm{f}''(x)$ and $\mathrm{f}'''(x)$ in terms of β and y. [5]

(b) In the case where $\alpha=\frac{\pi}{4}$, use your results from part (a) to find the Maclaurin series for f(x) in terms of β, up to and including the term in x^3. [3]

(c) Using your result from part (b), state the equation of the tangent to the curve $y=\tan\left(\frac{\pi}{4}+3x\right)$ at $x=0$. [1]

(d) Explain why a Maclaurin series for $y=\cot 3x$ cannot be found. [1]

9 Referred to the origin O, the points A, B and C have position vectors $3\mathbf{i}-\mathbf{j}$, $5\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ and $2\mathbf{i}+\alpha\mathbf{j}+\sqrt{2}\mathbf{k}$ respectively, where α is a non-zero constant.

(a) Given that the acute angle between the x-axis and the line passing through A and C is 60°, find the value of α. [3]

The line l passes through A and B.

(b) A point Q lies on l such that AQ is 3 units. Find the two possible position vectors of Q. [4]

The plane π containing A has equation $x + 2z = 3$.

(c) Find the position vector of the point F on π that is closest to B. [3]

(d) Hence find a vector equation of the line of reflection of l in π. [3]

10 A radio game show called the Inverted Tower of Ionah is broadcasted live everyday. Each day, a participant is invited to play the game. The participant has to stack up discs of the same thickness but increasing cross-sectional area to build a tower. Each level of the tower consists of one disc. The base of the tower is a disc with base area 6 cm^2. For each successive level, a disc with base area 8 cm^2 larger than the preceding disc is placed on the tower. The game will stop if the tower collapses and the participant loses the game.

Alan is a participant who participated in the game on Day 1.

(a) Alan claims that one of the discs he used has base area 140 cm^2. Explain if this is possible. [2]

(b) Alan's completed tower consists of 22 discs with a total volume of 4950 cm^3. Assuming that the discs have uniform cross-sectional areas with the same thickness of h cm, find the value of h. [2]

The participant wins the prize money for the day if he is able to build a tower with 30 levels within 2 minutes. Ben will participate in the game on Day 2. He wants to win the game and decides to practise before the game. He uses 3 minutes to complete the tower in his first attempt and the time for each subsequent attempt is 5% less than the time for the previous attempt.

(c) Find the minimum number of attempts he needs to build the tower within 2 minutes. [2]

On Day 1 of the radio game show, the prize money is \$200. For each day of the game, if the prize money is not won, the radio station will donate 20% of the prize money for that day to charity before topping up the balance prize money by \$150 for the next day. The prize money is increased in the same way every day until it is won.

(d) If there are no winners since Day 1, show that the prize money at the start of Day n is $\$\left[750-550(0.8)^{n-1}\right]$. [3]

If the prize money is won, it will start from \$200 again on the next day. It is now given that the first and second winner of the game won the prize money on Day k and Day 24 respectively.

(e) By using part **(d)**, find the set of possible values of k such that the first winner won a prize money of at least \$100 more than the second winner. [3]

11 **(a)**

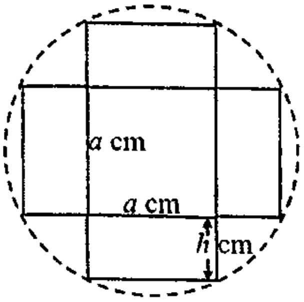


Figure 1

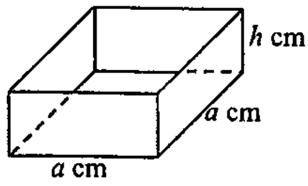


Figure 2

Jane bought a piece of circular cardboard of negligible thickness and radius 20 cm to make an open box with a square base. **Figure 1** shows the net of a square-based box that she cuts from the circular cardboard. The net consists of a square of side a cm and four identical rectangles, each with sides a cm by h cm. The net is folded to form a box which has a square base of side a cm and vertical height h cm, as shown in **Figure 2**.

(i) Show that the volume of the box is $\left(a^2\sqrt{400-0.25a^2}-0.5a^3\right)\text{cm}^3$. [2]

(ii) By differentiation, find the values of a and h which give a maximum volume of the box, giving your answers correct to 1 decimal place. You do not need to show that these values give a maximum volume. [5]

(b) Jane later bought an open metal cone to make a large candle. The inverted cone has height 16 cm and the circular opening at the top has radius 10 cm as shown in **Figure 3**. Melted wax is poured at a constant rate of 5 cm^3 per second into the inverted cone. At time t seconds after the start, the depth of the melted wax is x cm. Find the rate of increase of the depth of the melted wax when the depth of the melted wax is 12 cm. [5]

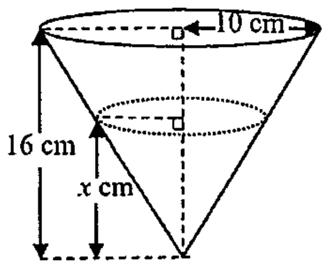
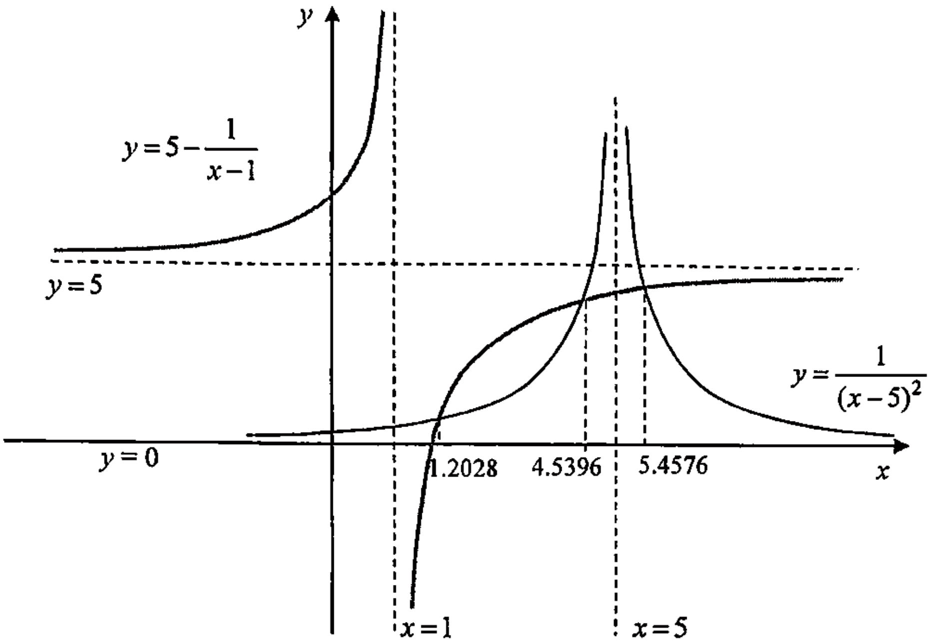


Figure 3

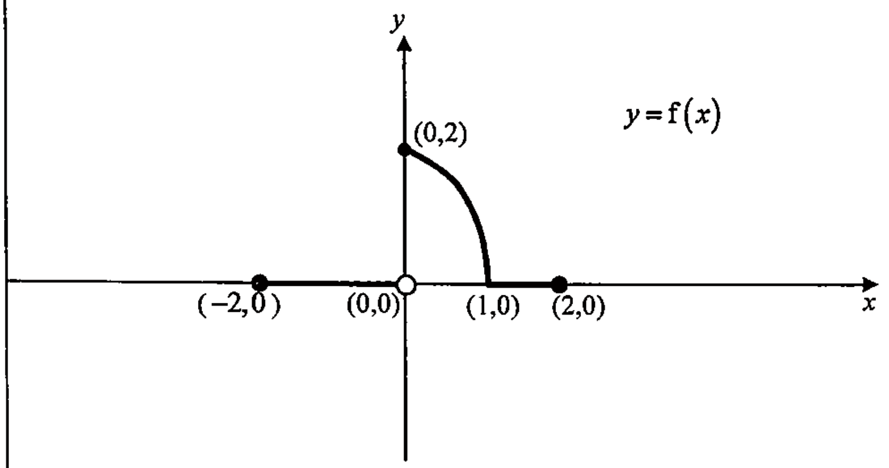
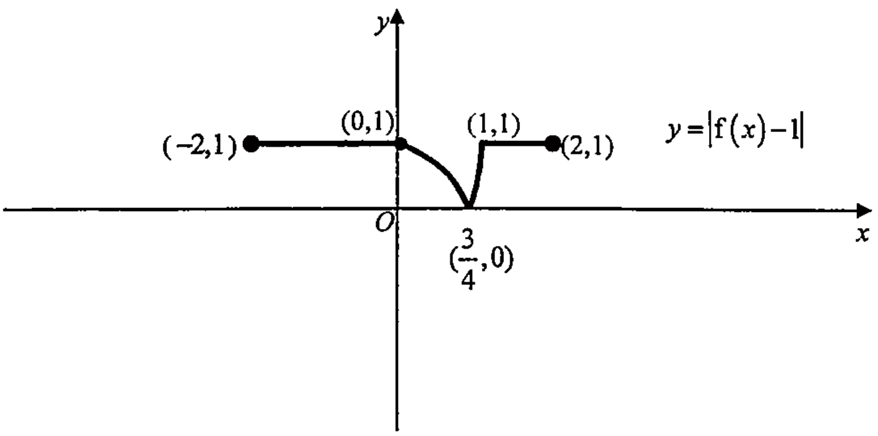
[The volume of a cone of base radius r and height h is given by $V=\frac{1}{3}\pi r^2 h$.]

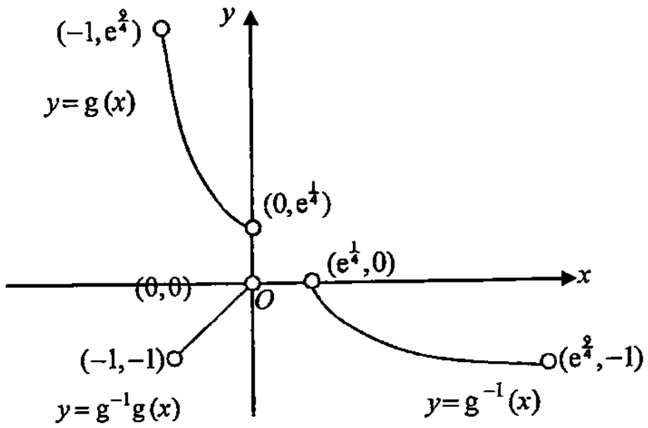
Q1	**Suggested Answers**
(a)	
(b)	Using GC, the graphs in **(a)** intersect at $x=1.2028,\ 4.5396,\ 5.4576$ For $5-\frac{1}{x-1}<\frac{1}{(x-5)^2}$ $1<x<1.20$ or $4.54<x<5.46,\ x\neq 5$ Alternatively, $1<x<1.20$ or $4.54<x<5$ or $5<x<5.46$

Q2	Suggested Answers
(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x+\sqrt{1+x^2}\right)\right)$ $=\frac{1}{x+\sqrt{1+x^2}}\times\left(1+\frac{1}{2}\times\frac{2x}{\sqrt{1+x^2}}\right)$ $=\frac{1}{x+\sqrt{1+x^2}}\times\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$ $=\frac{1}{\sqrt{1+x^2}}$
(b)	$\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}\left(\frac{2+x}{1-2x}\right)=\frac{1}{1+(\frac{2+x}{1-2x})^2}\cdot\frac{(1-2x)-(2+x)(-2)}{(1-2x)^2}$ $=\frac{1-2x+4+2x}{(1-2x)^2+(2+x)^2}$ $=\frac{5}{5x^2+5}=\frac{1}{x^2+1}$

Q3	Suggested Answers
(a)	O, A, C and B forms a parallelogram. Or O, A, C and B are vertices of a parallelogram.
(b)	$\overrightarrow{OC} = \mathbf{a}+\mathbf{b}$ $\overrightarrow{OM} = \frac{1}{3}\left(2\overrightarrow{OA}+\overrightarrow{OC}\right) = \frac{1}{3}(2\mathbf{a}+\mathbf{a}+\mathbf{b}) = \frac{1}{3}(3\mathbf{a}+\mathbf{b}) = \mathbf{a}+\frac{\mathbf{b}}{3}$ Area of triangle OAM $= \frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OM}\right\| = \frac{1}{2}\left\|\mathbf{a}\times\left(\mathbf{a}+\frac{\mathbf{b}}{3}\right)\right\|$ $= \frac{1}{2}\left\|\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\frac{\mathbf{b}}{3}\right\| = \frac{1}{2}\left\|0+\mathbf{a}\times\frac{\mathbf{b}}{3}\right\|$ $= \frac{1}{6}\|\mathbf{a}\times\mathbf{b}\|$
(b)	**Alternative method:** Area of triangle OAC $= \frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\| = \frac{1}{2}\|\mathbf{a}\times(\mathbf{a}+\mathbf{b})\|$ $= \frac{1}{2}\|\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\| = \frac{1}{2}\|0+\mathbf{a}\times\mathbf{b}\|$ $= \frac{1}{2}\|\mathbf{a}\times\mathbf{b}\|$ Area of triangle OAM $= \frac{1}{3}\times$ area of triangle OAC $= \frac{1}{6}\|\mathbf{a}\times\mathbf{b}\|$
(c)	The shortest distance from A to $OB = \|\mathbf{a}\times\hat{\mathbf{b}}\| = \left\|\mathbf{a}\times\frac{\mathbf{b}}{\|\mathbf{b}\|}\right\| = \sqrt{3}$ $\|\mathbf{a}\times\mathbf{b}\| = 5\sqrt{3}$ Area of triangle OAM $= \frac{1}{6}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{6}\times 5\sqrt{3} = \frac{5\sqrt{3}}{6}$
(c)	**Alternative method 1:** Area of triangle OAM $= \frac{1}{6}\|\mathbf{a}\times\mathbf{b}\| = \frac{1}{6}\times$ area of parallelogram $OACB$ $= \frac{1}{6}\times$ base $OB\times$ height of parallelogram $= \frac{1}{6}\times 5\times\sqrt{3} = \frac{5\sqrt{3}}{6}$ **Alternative method 2:**

	Let θ be the angle between **a** and **b** Area of triangle OAM $=\frac{1}{6}\lvert\mathbf{a}\times\mathbf{b}\rvert$ OR $=\frac{1}{6}\times(2\times\text{area of }\square AOB)$ $=\frac{1}{6}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $\quad=\frac{1}{6}\times(2\times\frac{1}{2}\times5\times\sqrt{3})$ $=\frac{1}{6}\times\lvert\mathbf{a}\rvert\times5\times\frac{\sqrt{3}}{\lvert\mathbf{a}\rvert}=\frac{5\sqrt{3}}{6}$ $\quad=\frac{5\sqrt{3}}{6}$
(d)	Since $ON:NM=3:1$, $\overrightarrow{ON}=\frac{3}{4}\overrightarrow{OM}=\frac{3}{4}\left(\mathbf{a}+\frac{\mathbf{b}}{3}\right)=\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}$ $\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}$ $=\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}-\mathbf{a}=\frac{1}{4}(\mathbf{b}-\mathbf{a})=\frac{1}{4}\overrightarrow{AB}$ Since $\overrightarrow{AN}$ is parallel to $\overrightarrow{AB}$ and A is a common point, A, N and B are collinear. $AN:NB=1:3$

Q4	Suggested Answers
(a)	(1) Translate the curve by −1 unit in the direction of x-axis (2) Reflect the curve in the y-axis (3) Scale the curve by factor 2 parallel to the y-axis (1) Reflect the curve in the y-axis (2) Translate the curve by 1 unit in the direction of x-axis (3) Scale the curve by factor 2 parallel to the y-axis (1) Translate the curve by −4 units in the direction of x-axis (2) Reflect the curve in the y-axis (3) Scale the curve by factor 1/4 parallel to the x-axis
(b)(i)	
(b)(ii)	

Q5	Suggested Answers
(a)	Greatest value of $a = ½$ <u>Explanation (for students)</u> Since $\mathrm{g}(x)=\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}$ is an even function that is symmetrical about $x=\dfrac{1}{2}$, for g to be a 1-1 function, the largest $\mathrm{D}_g=(-1,\tfrac{1}{2})$. Alternatively, use GC to find the minimum point of $y=\mathrm{g}(x)$.
(b)	Given $a=0$. So, 
(c)(i)	$\mathrm{hg}(x)=\mathrm{h}\left(\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}\right)=1+\ln\left(\mathrm{e}^{\left(x-\frac{1}{2}\right)^2}\right)=1+\left(x-\dfrac{1}{2}\right)^2$ $\mathrm{D}_{\mathrm{hg}}=(-1,0)$
(c)(ii)	"Hence" method: Let $x=(\mathrm{hg})^{-1}\left(\tfrac{3}{2}\right)$ $(\mathrm{hg})(x)=\tfrac{3}{2},\quad -1<x<0$ $1+\left(x-\dfrac{1}{2}\right)^2=\dfrac{3}{2}$ $\left(x-\dfrac{1}{2}\right)^2=\dfrac{1}{2}$ $x=\dfrac{1}{2}\pm\dfrac{1}{\sqrt{2}}$ (reject $x=\dfrac{1}{2}+\dfrac{1}{\sqrt{2}}$ since $-1<x<0$) $\therefore x=(\mathrm{hg})^{-1}\left(\tfrac{3}{2}\right)=\dfrac{1}{2}-\dfrac{1}{\sqrt{2}}=\dfrac{1-\sqrt{2}}{2}$
(c)(ii)	"Otherwise" method:

	Let $y = \text{hg}(x) = 1 + \left(x - \frac{1}{2}\right)^2, -1 < x < 0$ $\left(x - \frac{1}{2}\right)^2 = y - 1$ $x = \frac{1}{2} \pm \sqrt{y-1}$ (Reject $x = \frac{1}{2} + \sqrt{y-1}$ since $-1 < x < 0$) $\Rightarrow x = (\text{hg})^{-1}(y) = \frac{1}{2} - \sqrt{y-1}$ $\Rightarrow (\text{hg})^{-1}(x) = \frac{1}{2} - \sqrt{x-1}$ $\therefore (\text{hg})^{-1}\left(\frac{3}{2}\right) = \frac{1}{2} - \sqrt{\frac{3}{2}-1} = \frac{1}{2} - \frac{1}{\sqrt{2}} = \frac{1-\sqrt{2}}{2}$

6	Suggested Answers
(a)	$\mathrm{f}(r-1)-2\mathrm{f}(r)+\mathrm{f}(r+1)=\ln(r-1)-2\ln(r)+\ln(r+1)$ $=\ln\left(\dfrac{(r-1)(r+1)}{r^2}\right)$ $=\ln\left(\dfrac{r^2-1}{r^2}\right)$ $=\ln\left(1-\dfrac{1}{r^2}\right)$
(b)	$\sum_{r=2}^{N}\ln\left(1-\dfrac{1}{r^2}\right)=\sum_{r=2}^{N}\left[\mathrm{f}(r-1)-2\mathrm{f}(r)+\mathrm{f}(r+1)\right]$ $=\mathrm{f}(1)-2\mathrm{f}(2)+\mathrm{f}(3)$ $+\mathrm{f}(2)-2\mathrm{f}(3)+\mathrm{f}(4)$ $+\mathrm{f}(3)-2\mathrm{f}(4)+\mathrm{f}(5)$ $+\ldots$ $+\mathrm{f}(N-3)-2\mathrm{f}(N-2)+\mathrm{f}(N-1)$ $+\mathrm{f}(N-2)-2\mathrm{f}(N-1)+\mathrm{f}(N)$ $+\mathrm{f}(N-1)-2\mathrm{f}(N)+\mathrm{f}(N+1)$ $=\mathrm{f}(1)-2\mathrm{f}(2)+\mathrm{f}(2)+\mathrm{f}(N)-2\mathrm{f}(N)+\mathrm{f}(N+1)$ $=\ln 1-\ln 2-\ln N+\ln(N+1)$ $=\ln\left(\dfrac{N+1}{2N}\right)$
(c)	$\dfrac{N+1}{2N}=\dfrac{1}{2}+\dfrac{1}{2N}$ As $N\to\infty$, $\dfrac{1}{2N}\to 0$ thus $\dfrac{1}{2}+\dfrac{1}{2N}\to\dfrac{1}{2}$. Hence $\sum_{r=2}^{N}\ln\left(1-\dfrac{1}{r^2}\right)\to\ln\left(\dfrac{1}{2}\right)$ which is a constant/finite value. Hence the series converges. $\sum_{r=2}^{\infty}\ln\left(1-\dfrac{1}{r^2}\right)=\ln\left(\dfrac{1}{2}\right)$
(d)	**Method 1:**

	$\ln\left(1-\frac{1}{(2k+1)^2}\right)+\ln\left(1-\frac{1}{(2k+2)^2}\right)+\ln\left(1-\frac{1}{(2k+3)^2}\right)+\ldots$ $=\sum_{r=2k+1}^{\infty}\ln\left(1-\frac{1}{r^2}\right)$ $=\sum_{r=2}^{\infty}\ln\left(1-\frac{1}{r^2}\right)-\sum_{r=2}^{2k}\ln\left(1-\frac{1}{r^2}\right)$ $=\ln\left(\frac{1}{2}\right)-\ln\left(\frac{2k+1}{2(2k)}\right)$ $=\ln\left(\frac{4k}{2(2k+1)}\right)$ $=\ln\left(\frac{2k}{2k+1}\right)$ **Method 2 (Change of index):** $\ln\left(1-\frac{1}{(2k+1)^2}\right)+\ln\left(1-\frac{1}{(2k+2)^2}\right)+\ln\left(1-\frac{1}{(2k+3)^2}\right)+\ldots$ $=\sum_{r=1}^{\infty}\ln\left(1-\frac{1}{(2k+r)^2}\right)$ Replace r with $r-2k$ $=\sum_{r-2k=1}^{\infty}\ln\left(1-\frac{1}{(2k+r-2k)^2}\right)$ $=\sum_{r=2k+1}^{\infty}\ln\left(1-\frac{1}{r^2}\right)$ $=\sum_{r=2}^{\infty}\ln\left(1-\frac{1}{r^2}\right)-\sum_{r=2}^{2k}\ln\left(1-\frac{1}{r^2}\right)$ $=\ln\left(\frac{1}{2}\right)-\ln\left(\frac{2k+1}{2(2k)}\right)$ $=\ln\left(\frac{4k}{2(2k+1)}\right)$ $=\ln\left(\frac{2k}{2k+1}\right)$

Q7	Suggested Answers
(a)	$\frac{dx}{dt}=\frac{1}{2}e^{t}+2e^{-t}$ $\frac{dy}{dt}=e^{t}-2e^{-t}$ $\frac{dy}{dx}=\frac{e^{t}-2e^{-t}}{\frac{1}{2}e^{t}+2e^{-t}}$ At $t=\ln 4$, $\frac{dy}{dx}=\frac{e^{\ln 4}-2e^{-\ln 4}}{\frac{1}{2}e^{\ln 4}+2e^{-\ln 4}}=\frac{7}{5}$ Gradient of normal at $t=\ln 4$ is $-\frac{5}{7}$
(b)	At $t=\ln 4$, $x=\frac{3}{2}$, $y=\frac{9}{2}$. Equation of normal is $y-\frac{9}{2}=-\frac{5}{7}\left(x-\frac{3}{2}\right)$ $14y-63=-10x+15$ $5x+7y=39$ (shown)
(c)	Sub $x=\frac{1}{2}e^{t}-2e^{-t}$ and $y=e^{t}+2e^{-t}$ into $5x+7y=39$ $5\left(\frac{1}{2}e^{t}-2e^{-t}\right)+7\left(e^{t}+2e^{-t}\right)=39$ $19e^{t}+8e^{-t}-78=0$ Let $a=e^{t}$ $19a+\frac{8}{a}-78=0$ $19a^{2}-78a+8=0$ $(a-4)(19a-2)=0$ $a=4$ or $a=\frac{2}{19}$ $t=\ln 4$ or $t=\ln\frac{2}{19}$ (reject)

Q8	Suggested Answers
(a)	**Method 1:** $\mathrm{f}(x)=\tan(\alpha+\beta x)$ $\mathrm{f}'(x)=\beta\sec^2(\alpha+\beta x)$ $=\beta\left[1+\tan^2(\alpha+\beta x)\right]$ $=\beta\left(1+y^2\right)$ (shown) **Method 2:** $y=\tan(\alpha+\beta x)$ $\tan^{-1}y=\alpha+\beta x$ $\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\beta$ $\mathrm{f}'(x)=\frac{\mathrm{d}y}{\mathrm{d}x}=\beta\left(1+y^2\right)$ (shown) $\mathrm{f}''(x)=\beta(2y)\frac{\mathrm{d}y}{\mathrm{d}x}$ ------(1) $=2\beta y\left[\beta\left(1+y^2\right)\right]$ $=2\beta^2y\left(1+y^2\right)$------(2) or $\mathrm{f}''(x)=2\beta^2\left(y+y^3\right)$----(3) Differentiate (1) w.r.t. x, $\mathrm{f}'''(x)=2\beta\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2\beta y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ $=2\beta\left(\beta^2\right)\left(1+y^2\right)^2+2\beta y\left(2\beta^2y\right)\left(1+y^2\right)$ $=2\beta^3\left(1+y^2\right)\left(1+y^2+2y^2\right)$ $=2\beta^3\left(1+y^2\right)\left(1+3y^2\right)$ or differentiate (2) w.r.t. x, $\mathrm{f}'''(x)=\left(2\beta^2y\right)\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)+2\beta^2\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+y^2\right)$ $=2\beta^2\frac{\mathrm{d}y}{\mathrm{d}x}\left(2y^2+1+y^2\right)$ $=2\beta^2\left[\beta\left(1+y^2\right)\right]\left(2y^2+1+y^2\right)$ $=2\beta^3\left(3y^2+1\right)\left(1+y^2\right)$ or differentiate (3) w.r.t. x,

	$f'''(x)=(2\beta^2)\left(\frac{dy}{dx}+3y^2\frac{dy}{dx}\right)$ $=2\beta^2\frac{dy}{dx}(1+3y^2)$ $=2\beta^2\left[\beta(1+y^2)\right](1+3y^2)$ $=2\beta^3(1+3y^2)(1+y^2)$
(b)	For $\alpha=\frac{\pi}{4}$, When $x=0$, $f(0)=\tan(\alpha)=\tan\left(\frac{\pi}{4}\right)=1$ $f'(0)=\beta\left(1+(1)^2\right)=2\beta$, $f''(0)=2\beta^2(1)\left(1+(1)^2\right)=4\beta^2$, $f'''(0)=2\beta^3\left(1+3(1)^2\right)\left(1+(1)^2\right)=16\beta^3$ Hence the Maclaurin series for f(x) is $f(x)=1+2\beta x+\frac{4\beta^2}{2!}x^2+\frac{16\beta^3}{3!}x^3+\cdots$, i.e. $f(x)=1+2\beta x+2\beta^2x^2+\frac{8\beta^3}{3}x^3+\cdots$
(c)	Let $\beta=3$ Equation of tangent to curve of $y=\tan\left(\frac{\pi}{4}+3x\right)$ is $y=1+6x$
(d)	When $x=0$, cot $3x$ is undefined. Hence Maclaurin expansion for $y=\cot 3x$ cannot be found.

Q9	Suggested Answers
(a)	$\overrightarrow{AC}=\begin{pmatrix}2\\ \alpha\\ \sqrt{2}\end{pmatrix}-\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}=\begin{pmatrix}-1\\ \alpha+1\\ \sqrt{2}\end{pmatrix}$ $\left\lvert\overrightarrow{AC}\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right\rvert=\left\lvert\overrightarrow{AC}\right\rvert\cdot 1\cdot\cos 60^\circ$ $\left\lvert\begin{pmatrix}-1\\ \alpha+1\\ \sqrt{2}\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix}\right\rvert=\sqrt{1+(\alpha+1)^2+2}\cdot\frac{1}{2}$ $\lvert -1\rvert=\sqrt{\alpha^2+2\alpha+4}\cdot\frac{1}{2}$ $\alpha^2+2\alpha=0$ $\alpha(\alpha+2)=0$ Since α is a non-zero constant, $\alpha=-2$
(b)	$\overrightarrow{AB}=\begin{pmatrix}5\\ 3\\ 4\end{pmatrix}-\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}=\begin{pmatrix}2\\ 4\\ 4\end{pmatrix}//\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$ Line l: $\mathbf{r}=\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}+\lambda\begin{pmatrix}1\\ 2\\ 2\end{pmatrix},\ \lambda\in\mathbb{R}$ Since Q lies on l, $\overrightarrow{OQ}=\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}+\lambda\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{AQ}=\lambda\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}$ $\left\lvert\overrightarrow{AQ}\right\rvert=3 \quad\Rightarrow\quad \left\lvert\lambda\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}\right\rvert=3$ $\lvert\lambda\rvert=1$ $\lambda=\pm 1$ $\lambda=1$: $\quad\overrightarrow{OQ}=\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}+\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=\begin{pmatrix}4\\ 1\\ 2\end{pmatrix}$ $\lambda=-1$: $\quad\overrightarrow{OQ}=\begin{pmatrix}3\\ -1\\ 0\end{pmatrix}-\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=\begin{pmatrix}2\\ -3\\ -2\end{pmatrix}$

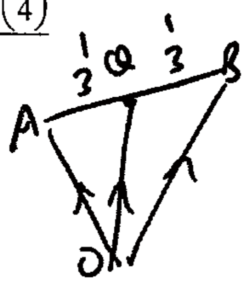
Alternative Method:

$$\overrightarrow{AB} = \begin{pmatrix}5\\3\\4\end{pmatrix} - \begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}2\\4\\4\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \left\|\begin{pmatrix}2\\4\\4\end{pmatrix}\right\| = 6$$

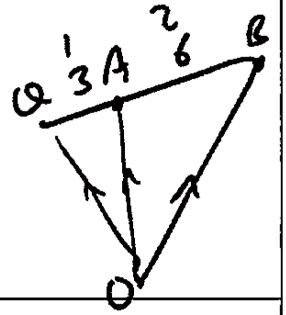
By ratio theorem,

$$\overrightarrow{OQ} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2} = \frac{\begin{pmatrix}3\\-1\\0\end{pmatrix}+\begin{pmatrix}5\\3\\4\end{pmatrix}}{2} = \begin{pmatrix}4\\1\\2\end{pmatrix}$$



$$\overrightarrow{OA} = \frac{2\overrightarrow{OQ}+\overrightarrow{OB}}{3}$$

$$\overrightarrow{OQ} = \frac{3\overrightarrow{OA}-\overrightarrow{OB}}{2} = \frac{3\begin{pmatrix}3\\-1\\0\end{pmatrix}-\begin{pmatrix}5\\3\\4\end{pmatrix}}{2} = \begin{pmatrix}2\\-3\\-2\end{pmatrix}$$



(c) **Standard Method:**

Line BF: $\mathbf{r} = \begin{pmatrix}5\\3\\4\end{pmatrix} + \mu\begin{pmatrix}1\\0\\2\end{pmatrix},\ \mu \in \mathbb{R}$ Plane π : $\mathbf{r} \cdot \begin{pmatrix}1\\0\\2\end{pmatrix} = 3$

$$\overrightarrow{OF} = \begin{pmatrix}5\\3\\4\end{pmatrix} + \mu\begin{pmatrix}1\\0\\2\end{pmatrix} \text{ for some } \mu \in \mathbb{R}$$

$$\left(\begin{pmatrix}5\\3\\4\end{pmatrix} + \mu\begin{pmatrix}1\\0\\2\end{pmatrix}\right) \cdot \begin{pmatrix}1\\0\\2\end{pmatrix} = 3$$

$$(5+8)+\mu(1+4) = 3$$

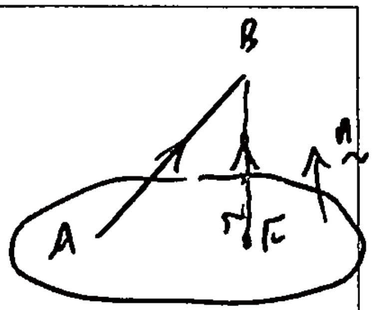
$$\mu = -2$$

$$\overrightarrow{OF} = \begin{pmatrix}5\\3\\4\end{pmatrix} + (-2)\begin{pmatrix}1\\0\\2\end{pmatrix} = \begin{pmatrix}3\\3\\0\end{pmatrix}$$

Alternative method: using projection

$$\overrightarrow{FB}=\left(\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right)\hat{\mathbf{n}}$$

$$=\left(\begin{pmatrix}2\\4\\4\end{pmatrix}\cdot\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}\right)\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}$$



$$=\frac{10}{5}\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$=\begin{pmatrix}2\\0\\4\end{pmatrix}$$

$$\overrightarrow{OB}=\overrightarrow{OF}+\overrightarrow{FB}$$

Hence $\overrightarrow{OF}=\overrightarrow{OB}-\overrightarrow{FB}$

$$=\begin{pmatrix}5\\3\\4\end{pmatrix}-\begin{pmatrix}2\\0\\4\end{pmatrix}$$

$$=\begin{pmatrix}3\\3\\0\end{pmatrix}$$

(d) Let B' be the reflection of B in π.

$$\overrightarrow{OB'}=\overrightarrow{OB}+2\overrightarrow{BF}$$

$$=\overrightarrow{OB}-2\overrightarrow{FB}$$

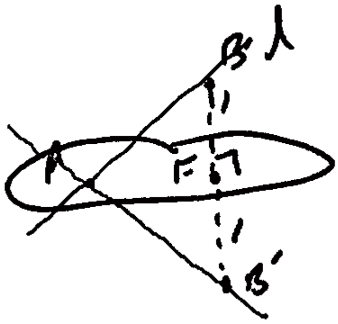
$$=\begin{pmatrix}5\\3\\4\end{pmatrix}-2\begin{pmatrix}2\\0\\4\end{pmatrix}$$

$$=\begin{pmatrix}1\\3\\-4\end{pmatrix}$$

(d) Let B' be the reflection of B in π.

$$\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{OB'}\right)$$

$$\overrightarrow{OB'}=2\overrightarrow{OF}-\overrightarrow{OB}$$



$$=2\begin{pmatrix}3\\3\\0\end{pmatrix}-\begin{pmatrix}5\\3\\4\end{pmatrix}$$

$$=\begin{pmatrix}1\\3\\-4\end{pmatrix}$$

	$\overrightarrow{AB'} = \overrightarrow{OB'} - \overrightarrow{OA}$ $= \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ $= \begin{pmatrix} -2 \\ 4 \\ -4 \end{pmatrix} // \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ Reflection of l in $\pi : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, t \in \mathbb{R}$

10	Suggested Answers

(a)

$6+(n-1)(8)=140$

$n=17.75$

Since n is not an integer, it is not possible to have a disc with that base area.

(b)

$h\left(\frac{22}{2}(2(6)+(22-1)(8))\right)=4950$

$h=2.5$

(c)

$3(0.95)^{n-1}\le 2$

$n\ge 8.90$

Hence he needs a minimum of 9 attempts

(d)

Day (n)	Prize Money at start of day
1	200
2	$200(0.8)+150$
3	$200(0.8)^2+150(0.8)+150$

Prize money on start of Day n

$=200(0.8)^{n-1}+150(0.8)^{n-2}+150(0.8)^{n-3}+...+150$

$=200(0.8)^{n-1}+\underbrace{150(0.8)^0+...+150(0.8)^{n-3}+150(0.8)^{n-2}}_{\text{GP with } n-1 \text{ terms}}$

$=200(0.8)^{n-1}+\left(\frac{150(1-0.8^{n-1})}{1-0.8}\right)$

$=200(0.8)^{n-1}+750-750(0.8)^{n-1}$

$=750-550(0.8)^{n-1}$

(e)

Prize money won by first winner on Day $k=750-550(0.8)^{k-1}$

Prize money won by second winner on Day 24

$=750-550(0.8)^{(24-k)-1}$

$=750-550(0.8)^{(23-k)}$

$\left[750-550(0.8)^{k-1}\right]-\left[750-550(0.8)^{23-k}\right]\ge 100$

$550(0.8)^{23-k}-550(0.8)^{k-1}-100\ge 0$

Using GC,

k	$550(0.8)^{23-k}-550(0.8)^{k-1}-100$
16	−4.0080
17	28.698

Hence the set of possible values of k is $\{k\in\mathbb{Z}^+ : 17\le k\le 23\}$

Q11	Suggested Answers
(a)(i)	By Pythagoras Theorem, $\left(\frac{1}{2}a\right)^2+\left(\frac{1}{2}a+h\right)^2=20^2$ $\left(\frac{1}{2}a+h\right)^2=20^2-\left(\frac{1}{2}a\right)^2$ $h=\sqrt{400-0.25a^2}-0.5a$ -------------- (*) Volume $V=a^2\left(\sqrt{400-0.25a^2}-0.5a\right)$ $=a^2\sqrt{400-0.25a^2}-0.5a^3$ (shown) <u>Alternatively</u> $a^2+(a+2h)^2=40^2$ $4h^2+4ah+2a^2-1600=0$ $h=\frac{-4a\pm\sqrt{16a^2-4(4)(2a^2-1600)}}{2(4)}$ $=-\frac{a}{2}\pm\frac{1}{8}\sqrt{16\times1600-16a^2}$ $=-\frac{a}{2}\pm\frac{1}{8}\sqrt{64\left(400-\frac{1}{4}a^2\right)}$ $=-\frac{a}{2}+\sqrt{400-0.25a^2}$ (reject $-\frac{a}{2}-\sqrt{400-0.25a^2}$ as $h>0$) Thus Volume $V=a^2\left(\sqrt{400-0.25a^2}-0.5a\right)$
(a)(ii)	$\frac{dV}{da}=2a\sqrt{400-0.25a^2}+a^2\times\frac{-0.5a}{2\sqrt{400-0.25a^2}}-1.5a^2$ At stationary points, $\frac{dV}{da}=0$ $2a\sqrt{400-0.25a^2}-\frac{0.5a^3}{2\sqrt{400-0.25a^2}}-1.5a^2=0$ Using GC, $a=19.585\approx19.6$ Using (*), $h=7.646\approx7.6$
(b)	$V=\frac{1}{3}\pi r^2x$ Using similar triangles, $\frac{r}{x}=\frac{10}{16}$

	$r=\frac{5}{8}x$ $V=\frac{1}{3}\pi\left(\frac{5}{8}x\right)^2x=\frac{25}{192}\pi x^3$ Differentiate w.r.t t $\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{25}{64}\pi x^2\frac{\mathrm{d}x}{\mathrm{d}t}$ When $x=12$ cm, $5=\frac{25}{64}\pi(12)^2\frac{\mathrm{d}x}{\mathrm{d}t}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{4}{45\pi}$ cm/s Or $\frac{\mathrm{d}x}{\mathrm{d}t}=0.0283$ cm/s