SERANGOON JUNIOR COLLEGE

2018 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9758/1**

11 Sept 2018

3 hours

Additional materials: Writing paper

List of Formulae (MF 26)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets **[ ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

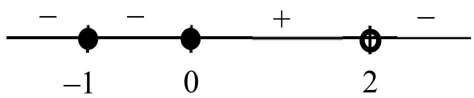
Total marks for this paper is 100 marks.

This question paper consists of 6 printed pages (inclusive of this page) and 2 blank pages.

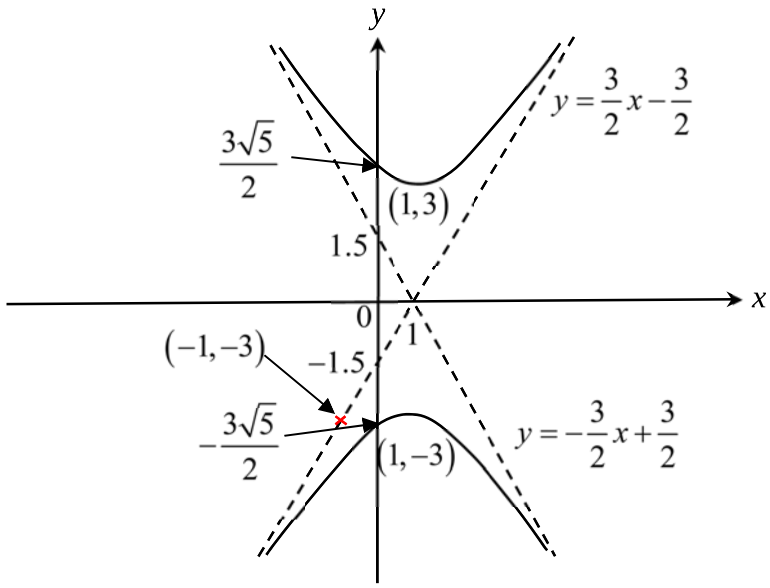
[TURN OVER]

Answer all questions [100 marks].

1	Find $\int \frac{x^2}{x^2+2x+5}\,\mathrm{d}x$.	[3]
	Solution	
	$\int \frac{x^2}{x^2+2x+5}\,\mathrm{d}x = \int 1 - \frac{2x+5}{x^2+2x+5}\,\mathrm{d}x$	
	$= \int 1\,\mathrm{d}x - \int \frac{2x+2}{x^2+2x+5}\,\mathrm{d}x - 3\int \frac{1}{(x+1)^2+4}\,\mathrm{d}x$	
	$= x - \ln\left(x^2+2x+5\right) - \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + c$	
2	The complex numbers z and w satisfy the equations	
	$zw^* + 2z = 15\mathrm{i}$ and $2w + 3z = 11$.	
	Find the complex numbers z and w.	[6]
	Solution	
	$zw^* + 2z = 15\mathrm{i} \ldots\ldots (1)$	
	$w = \frac{1}{2}(11-3z) \ldots\ldots (2)$	
	Subst (2) into (1) gives	
	$\frac{1}{2}z\left(11-3z^*\right) + 2z = 15\mathrm{i}$	
	$\frac{11}{2}z - \frac{3}{2}\lvert z\rvert^2 + 2z = 15\mathrm{i}$	
	$\frac{15}{2}z - \frac{3}{2}\lvert z\rvert^2 = 15\mathrm{i}$	
	Let $z = x + \mathrm{i}y$	
	$\frac{15}{2}(x+\mathrm{i}y) = \frac{3}{2}\left(x^2+y^2\right) + 15\mathrm{i}$	
	Comparing the real and Imaginary parts, we have	
	$\frac{15}{2}y = 15$ and $\frac{15}{2}x = \frac{3}{2}x^2 + \frac{3}{2}y^2 \ldots\ldots (3)$	
	$y = 2$	
	Subst into (3) gives	
	$x^2 - 5x + 4 = 0$	
	$(x-4)(x-1) = 0$	
	$\therefore x = 1$ or 4	
	When $z = 1 + 2\mathrm{i}$, $w = \frac{1}{2}(11-3-6\mathrm{i}) = 4 - 3\mathrm{i}$	
	When $z = 4 + 2\mathrm{i}$, $w = \frac{1}{2}(11-12-6\mathrm{i}) = -\frac{1}{2} - 3\mathrm{i}$	

3	Without the use of a graphing calculator, find the range of values of x for which $\dfrac{18}{2-x} \geq x^2+4x+9$.	[3]
	Hence find the exact range of values of x for which $\dfrac{18}{2-\mathrm{e}^x} \geq \mathrm{e}^{2x}+4\mathrm{e}^x+9$.	[3]
	Solution	
	$\dfrac{18}{2-x} \geq x^2+4x+9$	
	$\dfrac{18+(x-2)(x^2+4x+9)}{2-x} \geq 0$	
	$\dfrac{18+x^3+2x^2+x-18}{2-x} \geq 0$	
	$\dfrac{x(x+1)^2}{2-x} \geq 0$	
		
	$x=-1$ or $0 \leq x < 2$	
	Replace x by e^x	
	So $\mathrm{e}^x=-1$ or $0 \leq \mathrm{e}^x < 2$	
	No solution or $(0 \leq \mathrm{e}^x$ and $\mathrm{e}^x < 2)$	
	No solution or $(x \in \mathbb{R}$ and $x < \ln 2)$	
	$\therefore x < \ln 2$	
4	**(i)** The curve with equation $\dfrac{y^2}{9}-x^2=1$ undergoes a two-step transformations to become curve C with equation $\dfrac{y^2}{9}-\dfrac{(x-1)^2}{4}=1$. State the two transformations involved for the curve $\dfrac{y^2}{9}-x^2=1$.	[2]
	(ii) Draw a sketch of the curve C, labelling clearly the equation(s) of its asymptote(s), intersection with the axes and the coordinates of any turning points.	[3]
	(iii) Show that the point $(-1,-3)$ lies on the line $y=mx+m-3$ for all real values of m.	[1]
	(iv) Hence using the diagram drawn in **(ii)**, find the range of values of k such that the equation $\dfrac{(kx+k-3)^2}{9}-\dfrac{(x-1)^2}{4}=1$ has 2 negative real roots.	[2]
	Solution	
	(i) Method 1	

[TURN OVER]

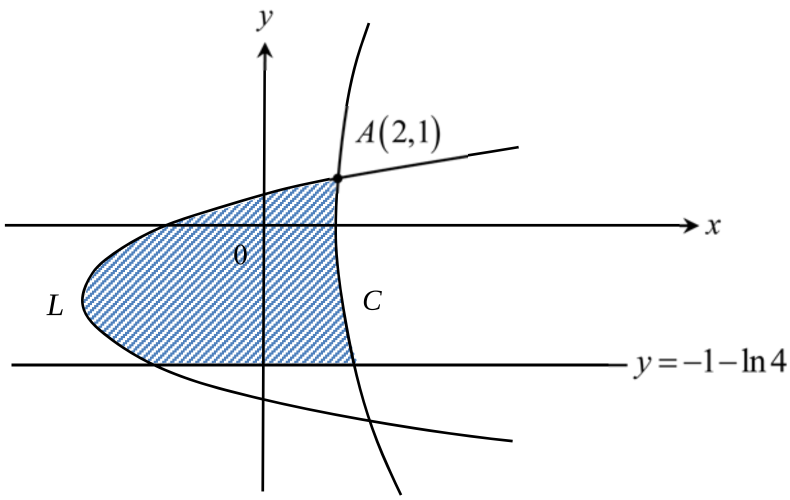
	$\frac{y^2}{9} - x^2 = 1 \rightarrow \frac{y^2}{9} - \frac{x^2}{4} = 1 \rightarrow \frac{y^2}{9} - \frac{(x-1)^2}{4} = 1$	
	(1) Scaling parallel to the *x* axis by a scale factor of 2.	
	(2) Translation of 1 unit in the positive *x* direction.	
	Method 2 $\frac{y^2}{9} - x^2 = 1 \rightarrow \frac{y^2}{9} - \left(x - \frac{1}{2}\right)^2 = 1 \rightarrow \frac{y^2}{9} - \frac{(x-1)^2}{4} = 1$	
	(1) Translation of 0.5 units in the positive *x* direction	
	(2) Scaling parallel to the *x* axis by a scale factor of 2.	
		
	(iii) When $x = -1$, $\text{RHS} = m(-1) + m - 3 = -3 = \text{LHS}$	
	Hence $(-1, -3)$ lies on the line $y = mx + m - 3$ for all *m*.	
	(iv) From the sketch, the range of values of *k* are $k > \frac{3\sqrt{5}}{2} + 3$ or $k < -\frac{3}{2}$	

5 **(i)** Find $\int \ln(x+1)\ \mathrm{d}x$ for $x>-1$. Show your working clearly. [2]

(ii) The curve C is defined by the parametric equations

$$x = 2t - 2\ln(t+1) + 2, \quad y = -2t - 2\ln(t+1) + 1 \qquad \text{where } t > -1.$$

Another curve L is defined by the equation $x = 2(y+1)^2 - 6$. The graphs of C and L intersect at the point $A(2,1)$ as shown in the diagram below.



Find the area of the shaded region bounded by C, L and the line $y=-1-\ln 4$, express your answer in the form $\frac{62}{3} + 4A - 2A^2 - \frac{16A^3}{3}$, where A is an exact real constant. [6]

Solution

(i) $\int \ln(1+x)\ \mathrm{d}x = x\ln(1+x) - \int \frac{x}{1+x}\ \mathrm{d}x$

$= x\ln(1+x) - \int 1 - \frac{1}{1+x}\ \mathrm{d}x$

$= x\ln(1+x) - x + \ln(1+x) + c$

$= (x+1)\ln(1+x) - x + c$

(ii) Area $= \int_{-1-\ln 4}^{1} x_C\ \mathrm{d}y - \int_{-1-\ln 4}^{1} \left[2(y+1)^2 - 6\right] \mathrm{d}y$

$= \int_1^0 \left[2t + 2 - 2\ln(1+t)\right]\left(-2 - \frac{2}{1+t}\right) \mathrm{d}t - \left[\frac{2(y+1)^3}{3} - 6y\right]_{-1-\ln 4}^{1}$

$= 4\int_0^1 \left[t + 1 - \ln(1+t)\right]\left(1 + \frac{1}{1+t}\right) \mathrm{d}t - \left\{\frac{16}{3} - 6 - \left[\frac{-2(\ln 4)^3}{3} + 6 + 6\ln 4\right]\right\}$

$= 4\int_0^1 t + 2\ \mathrm{d}t - 4\int_0^1 \ln(1+t)\ \mathrm{d}t - 4\int_0^1 \frac{\ln(1+t)}{1+t}\ \mathrm{d}t - \left[-\frac{20}{3} + \frac{2(\ln 4)^3}{3} - 6\ln 4\right]$

[TURN OVER]

	$=\left[2(t+2)^2\right]_0^1-4\left[(1+t)\ln(1+t)-t\right]_0^1-2\left[\left(\ln(1+t)\right)^2\right]_0^1-\left[-\dfrac{20}{3}+\dfrac{2(\ln 4)^3}{3}-6\ln 4\right]$	
	$=(18-8)-4[2\ln 2-1]-2(\ln 2)^2+\dfrac{20}{3}-\dfrac{2(\ln 4)^3}{3}+6\ln 4$	
	$=14-8\ln 2-2(\ln 2)^2+\dfrac{20}{3}-\dfrac{2(\ln 4)^3}{3}+6\ln 4$	
	$=\dfrac{62}{3}+4\ln 2-2(\ln 2)^2-\dfrac{16(\ln 2)^3}{3}$	
6	**(a)** **(i)** The twelfth, eighth and fifth terms of an arithmetic progression are three consecutive terms of a converging geometric progression of positive terms with common ratio *r*. Find the value of *r*.	[3]
	(ii) Take the value of *r* to be $\dfrac{3}{4}$. If the difference between the sum of the first *n* terms of the geometric progression and its sum to infinity is less than 0.3% of the sum to infinity, find the least value of *n*.	[3]
	(b) A convergent geometric sequence of positive terms, *G* has first non-zero term *a* and common ratio *r*.	
	(i) The sum of the first *n* odd-numbered terms of *G* is equal to the sum of all terms after the $(2n-1)^{\text{th}}$ term of *G*. Show that $2r^{2n}+r^{2n-1}-1=0$.	[2]
	(ii) In another sequence *H*, each term is the reciprocal of the corresponding term of *G*. If the *n*th term of *G* and *H* is denoted by u_n and v_n respectively, show that a new sequence whose *n*th term is $\ln\left(\dfrac{u_n}{v_n}\right)$, is an arithmetic progression.	[2]
	Solution	
	(ai) Let *b* and *d* be the first term and common difference of the arithmetic progression.	
	$ar^{n-1}=b+11d \ldots\ldots (1)$ $ar^{n}=b+7d \ldots\ldots (2)$ $ar^{n+1}=b+4d \ldots\ldots (3)$	
	Equation $(1)-(2)$ gives	
	$ar^{n-1}-ar^{n}=4d$	
	Equation $(2)-(3)$ gives	
	$ar^{n}-ar^{n+1}=3d$	
	Hence $4\left(ar^{n}-ar^{n+1}\right)=3\left(ar^{n-1}-ar^{n}\right)$	
	$4ar^{n+1}-7ar^{n}+3ar^{n-1}=0$	

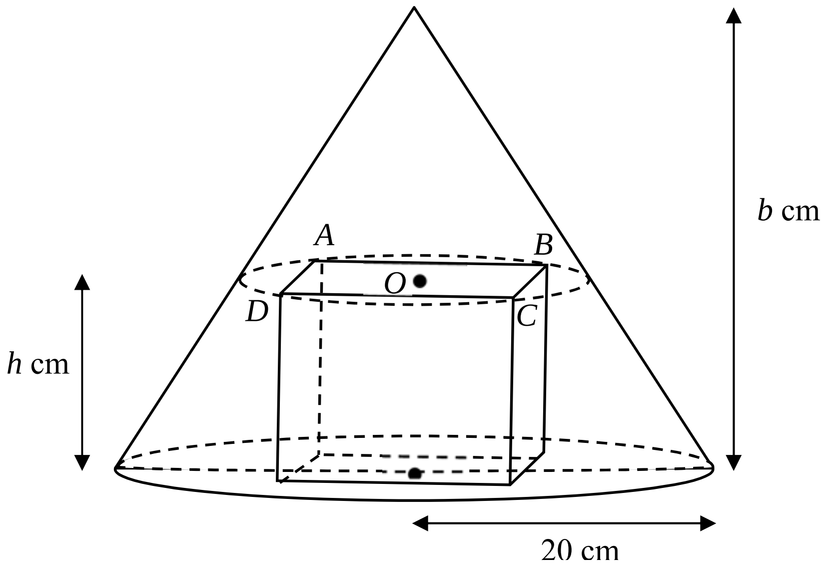
	$ar^{n-1}\left(4r^2-7r+3\right)=0$	
	$(4r-3)(r-1)=0$ Since *a* and *r* are non-zero	
	$r=\frac{3}{4}$ Since it is a converging GP.	
	(ii) $\left\|\frac{a\left(1-r^n\right)}{1-r}-\frac{a}{1-r}\right\|<0.003\left(\frac{a}{1-r}\right)$ or $\frac{a}{1-r}-\frac{a\left(1-r^n\right)}{1-r}<0.003\left(\frac{a}{1-r}\right)$ Since $S_\infty>S_n$ as *a* and *r* are positive.	
	$\left(\frac{3}{4}\right)^n<0.003$	
	$n>\frac{\ln(0.003)}{\ln\left(\frac{3}{4}\right)}$	
	$n>20.19$	
	Hence least *n* is 21.	
	(bi) $a+ar^2+ar^4+...+a\left(r^2\right)^{n-1}=ar^{2n-1}+ar^{2n}+...$	
	$\frac{a\left(1-r^{2n}\right)}{1-r^2}=\frac{ar^{2n-1}}{1-r}$ for sum of GP	
	Since $a\neq 0$, $r\neq 1$ $\frac{1-r^{2n}}{(1-r)(1+r)}=\frac{r^{2n-1}}{1-r}$	
	$1-r^{2n}=(1+r)r^{2n-1}$ $1-r^{2n}=r^{2n-1}+r^{2n}$ $2r^{2n}+r^{2n-1}-1=0(\text{shown})$	
	(ii) $\ln\left(\frac{u_n}{v_n}\right)-\ln\left(\frac{u_{n-1}}{v_{n-1}}\right)$	
	$=\ln u_n-\ln v_n-\ln u_{n-1}+\ln v_{n-1}$ $=\ln\left(\frac{u_n}{u_{n-1}}\right)+\ln\left(\frac{v_{n-1}}{v_n}\right)$	
	$=\ln r+\ln\left(\frac{1}{r}\right)^{-1}$ $=2\ln r$ (constant) Hence the new sequence is an arithmetic progression.	

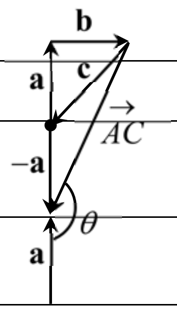
7	**(a)** The function f is defined by	
	$\mathrm{f}: x \mapsto x^2 - 2x - 8, \quad x \in \mathbb{R},\ x > k.$	
	(i) State the least value of k such that f^{-1} exists and find f^{-1} in a similar form.	[3]
	(ii) Using the value of k found in **(i)**, state the set of values of x such that $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x)$.	[1]
	(b) The functions g and h are defined by	
	$\mathrm{g}: x \mapsto \sqrt{x+41} + a, \quad x \geq -41,\ a \in \mathbb{R},$	
	$\mathrm{h}: x \mapsto x^2 + 10x - 16, \quad x \in \mathbb{R},\ x < -7.$	
	(i) Find the exact value of x for which $\mathrm{h}^{-1}(x) = \mathrm{h}(x)$.	[3]
	(ii) Explain clearly why the composite function gh exists.	[1]
	(iii) Find gh in the form $bx + c$, where b is a real constant and c is in terms of a. Explain your answers clearly.	[2]
	(iv) State the exact range of gh in terms of a.	[1]
	Solution	
	(a)(i) Least value of k is 1.	
	$y = (x-1)^2 - 9$	
	$x = 1 \pm \sqrt{y+9}$	
	$\therefore x = 1 + \sqrt{y+9} \quad (\because x > 1)$	
	So $\mathrm{f}^{-1}: x \mapsto 1 + \sqrt{x+9}, \quad x \in \mathbb{R},\ x > -9.$	
	(ii) $(1, \infty)$	
	(b)(i) $\mathrm{h}^{-1}(x) = \mathrm{h}(x)$	
	$\Rightarrow \mathrm{h}(x) = x$	
	$x^2 + 10x - 16 = x$	
	$x^2 + 9x - 16 = 0$	
	$x = \dfrac{-9 \pm \sqrt{81 - 4(-16)}}{2}$	
	$x = -\dfrac{9}{2} + \dfrac{\sqrt{145}}{2}$ (Rejected $\because x < -7$) or $-\dfrac{9}{2} - \dfrac{\sqrt{145}}{2}$	
	(ii) $\mathrm{R_h} = (-37, \infty)$ and $\mathrm{D_g} = [-41, \infty)$	
	Since $\mathrm{R_h} \subseteq \mathrm{D_g}$, so gh exists.	
	(iii) $\mathrm{gh}(x) = \mathrm{g}\left(x^2 + 10x - 16\right)$	
	$= \sqrt{x^2 + 10x + 25} + a$	
	$= \lvert x+5 \rvert + a$	
	$= -x + a - 5 \quad (\because x < -7)$	
	(iv) $(a+2, \infty)$	

8	A sequence $u_0, u_1, u_2 ...$ is such that $u_r = \dfrac{1}{r!}$ and $u_r = u_{r-2} + \dfrac{r+1-r^2}{r!}$, when $n \geq 2$.	
	(i) Show that $\displaystyle\sum_{r=2}^{n} \frac{r^2-r-1}{r!} = 2 - \frac{n+1}{n!}$.	[3]
	(ii) Hence find $\displaystyle\sum_{r=8}^{n+5} \frac{r^2-3r+1}{(r-1)!}$ in terms of n.	[3]
	Limit Comparison test states that for two series of the form $\displaystyle\sum_{r=k}^{n} a_r$ and $\displaystyle\sum_{r=k}^{n} b_r$ with $a_n, b_n \geq 0$ for all n, if $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} > 0$, then both $\displaystyle\sum_{r=k}^{\infty} a_r$ and $\displaystyle\sum_{r=k}^{\infty} b_r$ converges or both diverges.	
	(iii) Given that $\displaystyle\sum_{r=2}^{\infty} \frac{r^2-r-1}{r!}$ is convergent, using the test, explain why $\displaystyle\sum_{r=2}^{\infty} \frac{r-2}{(r-1)!}$ is convergent.	[2]
	(iv) Show that $e - 2 < \dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \dfrac{4}{5!} + ... < 2$.	[3]
	Solution	
	(i) $\displaystyle\sum_{r=2}^{n} \frac{r^2-r-1}{r!} = \sum_{r=2}^{n} (u_{r-2} - u_r)$	
	$= \left[\begin{array}{l} u_0 - u_2 \\ +u_1 - u_3 \\ +u_2 - u_4 \\ +u_3 - u_5 \\ +......... \\ +u_{n-4} - u_{n-2} \\ +u_{n-3} - u_{n-1} \\ +u_{n-2} - u_n \end{array}\right]$	
	$= u_0 + u_1 - u_{n-1} - u_n$ $= 2 - \dfrac{1}{(n-1)!} - \dfrac{1}{n!}$ $= 2 - \dfrac{n+1}{n!}$	
	(ii) $\displaystyle\sum_{r=8}^{n+5} \frac{r^2-3r+1}{(r-1)!} = \sum_{r=7}^{n+4} \frac{r^2-r-1}{r!}$ (Replace r by $r+1$)	

	$=\sum_{r=2}^{n+4}\frac{r^2-r-1}{r!}-\sum_{r=2}^{6}\frac{r^2-r-1}{r!}$ $=2-\frac{n+5}{(n+4)!}-\left(2-\frac{7}{6!}\right)$	
	$=\frac{7}{720}-\frac{n+5}{(n+4)!}$	
	(iii) Consider $\lim_{n\to\infty}\left[\frac{\left(\frac{n-2}{(n-1)!}\right)}{\left(\frac{n^2-n-1}{n!}\right)}\right]=\lim_{n\to\infty}\left[\frac{n-2}{(n-1)!}\times\frac{n!}{n^2-n-1}\right]$	
	$=\lim_{n\to\infty}\frac{n^2-2n}{n^2-n-1}=\lim_{n\to\infty}\left(1-\frac{n-1}{n^2-n-1}\right)=1$	
	Since $\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$ is convergent, So $\sum_{r=2}^{\infty}\frac{r-2}{(r-1)!}$ is convergent by the Limit Comparison Test.	
	(iv) Observe that $\sum_{r=2}^{\infty}\frac{r-1}{r!}=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+...$ and $\frac{r-1}{r!}<\frac{r^2-r-1}{r!}$ for all $r\geq 3$,	
	$\therefore\sum_{r=3}^{n}\frac{r-1}{r!}<\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	but $\frac{1}{2}+\sum_{r=3}^{n}\frac{r-1}{r!}<\frac{1}{2}+\sum_{r=3}^{n}\frac{r^2-r-1}{r!}$	
	$\Rightarrow\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	Method 1	
	Also, $\frac{1}{r!}<\frac{r-1}{r!}$ for all $r>2$ So $\sum_{r=3}^{n}\frac{1}{r!}<\sum_{r=3}^{n}\frac{r-1}{r!}$ $\Rightarrow\frac{1}{2!}+\sum_{r=3}^{n}\frac{1}{r!}<\frac{1}{2!}+\sum_{r=3}^{n}\frac{r-1}{r!}$ $\therefore\sum_{r=2}^{n}\frac{1}{r!}<\sum_{r=2}^{n}\frac{r-1}{r!}$	
	$\therefore\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}<\sum_{r=2}^{n}\frac{r-1}{r!}<\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	
	$\therefore\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}-\frac{1}{0!}-\frac{1}{1!}\right)<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r-1}{r!}<\lim_{n\to\infty}\sum_{r=2}^{n}\frac{r^2-r-1}{r!}$	

	$\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2$	
	Method 2	
	Since $\mathrm{e}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots$ So $\mathrm{e}-2=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\ldots<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots$	
	And $\sum_{r=2}^{\infty}\frac{r-1}{r!}<\sum_{r=2}^{\infty}\frac{r^2-r-1}{r!}$	
	So $\mathrm{e}-2<\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\ldots<2$	
9	The position vectors of points *A*, *B*, *C* with respect to the origin *O* are given by $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ respectively. The non-zero vectors **a**, **b** and **c** satisfy the equation $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$.	
	(i) By considering the plane *OAB* or otherwise, explain clearly why *O*, *A*, *B* and *C* lies on the same plane. Show that $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$.	[4]
	(ii) Show that the area of triangle *ABC* is given by $k\lvert\mathbf{a}\times\mathbf{b}\rvert$ where *k* is a constant to be determined.	[3]
	(iii) If $\mathbf{b}$ is a unit vector and $\mathbf{a}$ is perpendicular to $\mathbf{b}$, find the length of projection of $\overrightarrow{AC}$ onto $\overrightarrow{OB}$. Given that the magnitude of $\overrightarrow{AC}$ is 2 units, deduce the angle between $\overrightarrow{AC}$ and $\overrightarrow{OA}$.	[5]
	Solution	
	(i) Eqn of plane *OAB*: $\mathbf{r}=\lambda\mathbf{a}+\mu\mathbf{b}\quad \lambda,\mu\in\mathbb{R}$	
	Since $\mathbf{c}=-\mathbf{a}-\mathbf{b}$ where $\lambda=\mu=-1$, hence *C* lies on plane *OAB*. Thus *O*, *A*, *B* and *C* lies on the same plane.	
	OR $(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=(\mathbf{a}\times\mathbf{b})\bullet[-(\mathbf{a}+\mathbf{b})]$ $=-(\mathbf{a}\times\mathbf{b})\bullet\mathbf{a}-(\mathbf{a}\times\mathbf{b})\bullet\mathbf{b}$ $=0$ Thus *O*, *A*, *B* and *C* lies on the same plane.	
	Since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $\mathbf{b}=-\mathbf{a}-\mathbf{c}$	
	$\mathbf{a}\times\mathbf{b}=\mathbf{a}\times(-\mathbf{a}-\mathbf{c})$	
	$=-\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{c}$	
	$=\mathbf{c}\times\mathbf{a}$ $(\because \mathbf{a}\times\mathbf{a}=\mathbf{0}\ \&\ \mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c})$	
	$\mathbf{b}\times\mathbf{c}=(-\mathbf{a}-\mathbf{c})\times\mathbf{c}$	
	$=-\mathbf{a}\times\mathbf{c}-\mathbf{c}\times\mathbf{c}$	
	$=\mathbf{c}\times\mathbf{a}$ $(\because \mathbf{c}\times\mathbf{c}=\mathbf{0}\ \&\ \mathbf{c}\times\mathbf{a}=-\mathbf{a}\times\mathbf{c})$	
	$\therefore\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$, shown	
	(ii) Area of triangle *ABC*	

	$=\frac{1}{2}\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|$	
	$=\frac{1}{2}\left\|(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\right\|$	
	$=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}-\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{a}\right\|$	
	$=\frac{1}{2}\left\|\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}\right\|$	
	$=\frac{3}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|$ where k = $\frac{3}{2}$ (Since $\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}$, $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$)	
	(iii) $\left\|\overrightarrow{AC}\cdot\hat{\mathbf{b}}\right\|=\left\|(\mathbf{c}-\mathbf{a})\cdot\mathbf{b}\right\|$	
	$=\left\|(-\mathbf{b}-2\mathbf{a})\cdot\mathbf{b}\right\|$	
	$=\left\|-\left\|\mathbf{b}\right\|^2-2\mathbf{a}\cdot\mathbf{b}\right\|$	
	$=1 \quad (\because \mathbf{a}\cdot\mathbf{b}=0, \left\|\mathbf{b}\right\|=1)$	
	Let θ be the angle between $\overrightarrow{AC}$ and $\mathbf{a}$.	
	$\theta=180^\circ-\sin^{-1}\left(\frac{1}{2}\right)$	
	$=150^\circ$ or $\frac{5\pi}{6}$ rad	
10	The diagram below shows a rectangular container of variable height *h* cm, inscribed in a right circular cone with fixed height *b* cm and a radius of 20 cm. The four corners of the rectangular container's upper surface *ABCD* is always in contact with the conical surface. The point *O* is at the centre of the rectangle *ABCD*. The rectangular container is made with material of negligible thickness.	
		



	(i) If the fixed angle *BOC* is θ, show that the storage volume of the rectangular container in the shape of a cuboid is given by $V = 800b^{-2}(b-h)^2 h\sin\theta$.	[3]
	(ii) Find the value of *h* such that the rectangular container has a maximum storage volume, leaving your answer in terms of *b*.	[5]
	The rectangular container is opened at its upper surface *ABCD* and completely filled with a type of liquid perfume. A fragrance chemist placed the container in a room and allowed the liquid perfume to evaporate. It is known that the heat energy of the liquid perfume in the container, *E* joules, is related to the height of the container by the equation $$E = h - 3h^{-\frac{1}{2}}.$$	
	(iii) Given that the perfume evaporates at a rate of 0.08 c every hour and that the value of θ is $\frac{\pi}{6}$, calculate the rate of change of heat energy of the perfume when the height of the perfume in the rectangular container is $\frac{b}{5}$ cm, leaving your answer in the form $p+q\left(\frac{5}{b}\right)^{\frac{3}{2}}$, where *p* and *q* are exact constants to be determined.	[4]
	Solution	
	(i) Let the length of *OB* be *r*.	
	Method 1	
	$V = 4\left(r\sin\frac{\theta}{2}\right)\left(r\cos\frac{\theta}{2}\right)h$	
	$= 2r^2 h\sin\theta$	
	Method 2	
	$V = 2\left[\frac{1}{2}r^2\sin\theta + \frac{1}{2}r^2\sin(\pi-\theta)\right]h$	
	$= \left[r^2\sin\theta + r^2\sin(\pi-\theta)\right]h$	
	$= 2r^2 h\sin\theta$ (since $\sin(\pi-\theta) = \sin\theta$)	
	Using similar triangle,	
	$\frac{r}{20} = \frac{b-h}{b}$	
	$r = \frac{20}{b}(b-h)$	
	$V = 2\left(\frac{20}{b}\right)^2 (b-h)^2 h\sin\theta$	
	$= 800b^{-2}(b-h)^2 h\sin\theta$ (Shown)	
	(ii) $\frac{dV}{dh} = 800b^{-2}(b-h)^2\sin\theta + 800b^{-2}h\sin\theta(2)(b-h)(-1)$	

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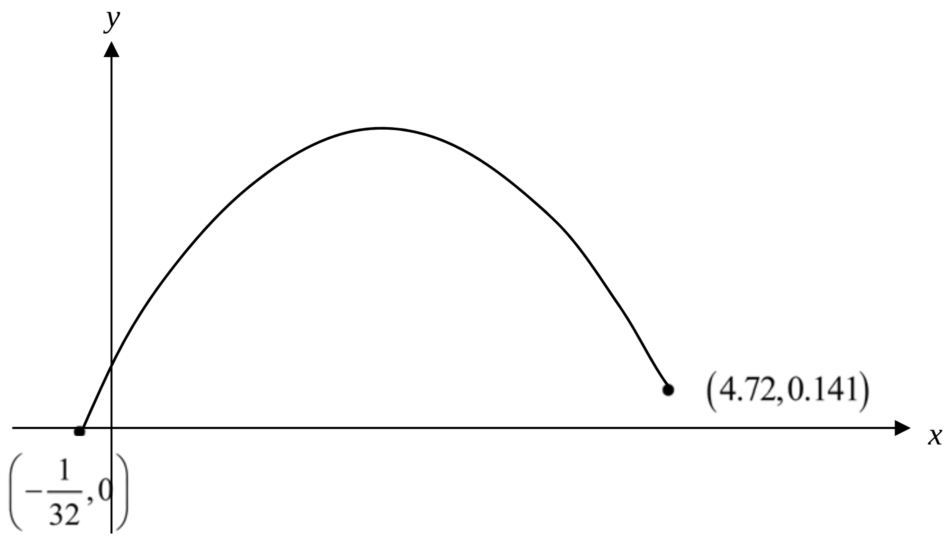
	$= 800b^{-2}(b-h)(\sin\theta)(b-h-2h)$	
	$= 800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	Setting $\frac{dV}{dh}=0$,	
	$800b^{-2}(\sin\theta)(b-h)(b-3h)=0$	
	$(b-h)(b-3h)=0$	
	$h=b$ (rejected since $h\neq b$) or $h=\frac{b}{3}$	
	Method 1 (Using second derivative test)	
	$\frac{d^2V}{dh^2}=800b^{-2}(\sin\theta)[(b-h)(-3)+(b-3h)(-1)]$	
	$=800b^{-2}(\sin\theta)(6h-4b)$	
	When $h=\frac{b}{3}$, $\frac{d^2V}{dh^2}=800b^{-2}(\sin\theta)(-2b)<0$	
	Hence, V is maximum when $h=\frac{b}{3}$.	
	Method 2 (Using first derivative test)	
	$\frac{dV}{dh}=800b^{-2}(\sin\theta)(b-h)(b-3h)$	
	When $h<\frac{b}{3}$, $800b^{-2}(\sin\theta)>0$, $(b-h)>0$ and $(b-3h)>0$, $\frac{dV}{dh}>0$	
	When $h>\frac{b}{3}$, $800b^{-2}(\sin\theta)>0$, $(b-h)>0$ and $(b-3h)<0$, $\frac{dV}{dh}<0$	
	Hence, V is maximum when $h=\frac{b}{3}$.	
	(iii) Since $\theta=\frac{\pi}{6}$, $\frac{dV}{dh}=\frac{400}{b^2}(b-h)(b-3h)$	
	$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$	
	When $h=\frac{b}{5}$,	
	$-0.08=\frac{400}{b^2}\left(\frac{4}{5}b\right)\left(\frac{2}{5}b\right)\times\frac{dh}{dt}$	
	$\frac{dh}{dt}=-\frac{1}{1600}$	
	$E=h-3h^{-\frac{1}{2}}$	
	$\frac{dE}{dh}=1+\frac{3}{2}h^{-\frac{3}{2}}$	

	$\frac{dE}{dt}=\frac{dE}{dh}\times\frac{dh}{dt}$	
	$=\left(1+\frac{3}{2}h^{-\frac{3}{2}}\right)\left(-\frac{1}{1600}\right)$	
	$=\left[1+\frac{3}{2}\left(\frac{b}{5}\right)^{-\frac{3}{2}}\right]\left(-\frac{1}{1600}\right)$	
	$=-\frac{1}{1600}-\frac{3}{3200}\left(\frac{5}{b}\right)^{\frac{3}{2}}$	
	Hence, $p=-\frac{1}{1600}$, $q=-\frac{3}{3200}$	
11	A charged particle is placed in a varying magnetic field. A researcher decides to fit a mathematical model for the path of the fast-moving charged particle under the influence of the magnetic field. The particle was observed for the first 1.5 seconds. The displacement of the particle measured with respect to the origin in the horizontal and vertical directions, at time t seconds, is denoted by the variables x and y respectively. It is given that when $t=0$, $x=-\frac{1}{32}$, $y=0$ and $\frac{dx}{dt}=3$. The variables are related by the differential equations $$(\cos t)\frac{dy}{dt}+y\sin t=4\cos^2 t-y^2 \text{ and } \frac{d^2x}{dt^2}=\cos 3t\cos t.$$	
	(i) Using the substitution $y=v\cos t$, show that $\frac{dv}{dt}=4-v^2$ and hence find y in terms of t.	[7]
	(ii) Show that $x=-\frac{1}{32}\cos 4t-\frac{1}{8}\cos 2t+3t+\frac{1}{8}$.	[4]
	(iii) Sketch the path travelled by the particle for the first 1.5 seconds, labelling the coordinates of the end points of the path. The evaluation of the y-intercept is not needed.	[2]
	Solution	
	(i) $y=v\cos t$	
	$\frac{dy}{dt}=-v\sin t+\frac{dv}{dt}\cos t$	
	$(\cos t)\frac{dy}{dt}+y\sin t=4\cos^2 t-y^2$	
	$(\cos t)\left(-v\sin t+\frac{dv}{dt}\cos t\right)+(v\cos t)\sin t=4\cos^2 t-v^2\cos^2 t$	

	$\frac{dv}{dt}\left(\cos^2 t\right)=\left(\cos^2 t\right)\left(4-v^2\right)$	
	$\frac{dv}{dt}=4-v^2$ (Shown)	
	$\int \frac{1}{4-v^2}\ dv=\int 1\ dt$	
	$\int \frac{1}{2^2-v^2}\ dv=\int 1\ dt$	
	$\frac{1}{2(2)}\ln\left\|\frac{2+v}{2-v}\right\|=t+d$	
	$\left\|\frac{2+v}{2-v}\right\|=e^{4t+4d}$	
	$\frac{2+v}{2-v}=\pm e^{4t+4d}$	
	$\frac{2+v}{2-v}=Ae^{4t}$, where $A=\pm e^{4d}$	
	$2+v=2Ae^{4t}-Ave^{4t}$	
	$v+Ave^{4t}=2\left(Ae^{4t}-1\right)$	
	$v=\frac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$	
	$\frac{y}{\cos t}=\frac{2\left(Ae^{4t}-1\right)}{Ae^{4t}+1}$	
	$y=\frac{2\left(Ae^{4t}-1\right)\cos t}{Ae^{4t}+1}$	
	When $t=0$, $y=0$	
	$0=\frac{2(A-1)}{A+1}$	
	$A=1$	
	$y=\frac{2\left(e^{4t}-1\right)\cos t}{e^{4t}+1}$	
	(ii) $\frac{d^2x}{dt^2}=\cos 3t\cos t$	
	$\frac{d^2x}{dt^2}=\frac{1}{2}\left(\cos 4t+\cos 2t\right)$	
	$\frac{dx}{dt}=\frac{1}{2}\int \cos 4t+\cos 2t\ dt$	
	$=\frac{1}{2}\left(\frac{1}{4}\sin 4t+\frac{1}{2}\sin 2t\right)+c$	
	$=\frac{1}{8}\sin 4t+\frac{1}{4}\sin 2t+c$	

	When $t = 0$, $\dfrac{dx}{dt} = 3$	
	$3 = \dfrac{1}{8}\sin 4(0) + \dfrac{1}{4}\sin 2(0) + c$	
	$c = 3$	
	$x = \int \dfrac{1}{8}\sin 4t + \dfrac{1}{4}\sin 2t + 3 \; dt$	
	$x = -\dfrac{1}{32}\cos 4t - \dfrac{1}{8}\cos 2t + 3t + k$	
	When $t = 0$, $x = -\dfrac{1}{32}$	
	$-\dfrac{1}{32} = -\dfrac{1}{32}\cos 4(0) - \dfrac{1}{8}\cos 2(0) + 3(0) + k$	
	$k = \dfrac{1}{8}$	
	Hence, $x = -\dfrac{1}{32}\cos 4t - \dfrac{1}{8}\cos 2t + 3t + \dfrac{1}{8}$	
	(iii) 	

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