1	$\mathrm{p}(x)=(x+a)(x+b)(x+c)-(x-d)(x-e)(x-f)$ $=\left[x^3+(a+b+c)x^2+(ab+bc+ca)x+abc\right]$ $-\left[x^3-(d+e+f)x^2+(de+ef+fd)x-def\right]$ $=(a+b+c+d+e+f)x^2+(ab+bc+ca-de-ef-fd)x$ $+abc+def$ $=Sx^2+(ab+bc+ca-de-ef-fd)x+abc+def$ Since $S\mid abc+def$ and $S\mid ab+bc+ca-de-ef-fd$, $S\mid \mathrm{p}(x)$ for any integer x. In particular, $S\mid \mathrm{p}(d)$. Now, $S\mid \mathrm{p}(d)\Rightarrow S\mid (d+a)(d+b)(d+c)$. If S is prime, by Euclid's lemma, $S\mid d+a$ or $S\mid d+b$ or $S\mid d+c$. WLOG, suppose that $S\mid d+a$. Then, $S\le d+a$ as both S and $d+a$ are positive. However, we now have $S=a+b+c+d+e+f>a+d\ge S$, which is a contradiction.

2(a)	Out of (28 + 2) symbols in a line, choose 28 to be balls and 2 to be dividers. $\binom{28+2}{2}=435$
2(b)	Let $y_1=x_1-12$. The problem is equivalent to finding the number of solutions to the equation $y_1+x_2+x_3=28-12$ where y_1, x_2 and x_3 are non-negative integers. Hence, $\binom{28-12+2}{2}=153$
2(c)	Let $y_i=11-x_i$ for $i=1,2,3$. Then $x_1+x_2+x_3=28$ becomes $y_1+y_2+y_3=5$ where y_i is non-negative for $i=1,2,3$. $\binom{5+2}{2}=21$.

2(d)	$x_1 + x_2 + \ldots + x_k = n$ ------- (*) Let $\lvert S\rvert$ be the number of solutions to (*) where $x_1, x_2, \ldots, x_k$ are non-negative integers less than *r*. Generalising from (c), let $y_i = r - 1 - x_i$ for $i = 1, 2, \ldots, k$. Then (*) becomes $y_1 + y_2 + \ldots + y_k = k(r-1) - n$ where y_i is non-negative for $i = 1, 2, \ldots, k$. Thus, $\lvert S\rvert = \binom{k(r-1)-n+(k-1)}{k-1} = \binom{kr-n-1}{k-1}$. -----(1) Let X_i be the set of solutions to (*) where $x_1, x_2, \ldots, x_k$ are non-negative integers and $x_i \ge r$ for some $i = 1, 2, \ldots, k$. Using the Principle of Inclusion and Exclusion, $\lvert S\rvert$ $= \lvert S_o\rvert - \sum_{i=1}^{k} \lvert X_i\rvert + \sum_{1 \le j_1 < j_2 \le k} \lvert X_{j_1} \cap X_{j_2}\rvert - \sum_{1 \le j_1 < j_2 < j_3 \le k} \lvert X_{j_1} \cap X_{j_2} \cap X_{j_3}\rvert$ $+ \ldots. + (-1)^k \lvert X_1 \cap X_2 \cap X_3 \cap \ldots \cap X_k\rvert$, where $\lvert S_o\rvert$ is the number of solutions to (*) where $x_1, x_2, \ldots, x_k$ are non-negative integers. Generalising from (a), $\lvert S_o\rvert = \binom{n+k-1}{k-1}$. Generalising from (b), $\lvert X_i\rvert = \binom{n-r+k-1}{k-1}$ for $i = 1, 2, \ldots, k$

To find $\left|X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_m}\right|$, where $1 \le j_1 < j_2 < \ldots < j_m \le k$,

let $z_{j_i} = x_{j_i} - r$ if $x_{j_i} \ge r$. Then (*) becomes

$$z_{j_1} + z_{j_2} + \ldots + z_{j_m} + \underbrace{x_{j_{m+1}} + \ldots + x_{j_k}}_{\text{remaining values of } x_{j_i}} = n - mr$$

where all terms on the left are non-negative integers.

Note that $\left|X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_m}\right| = 0$ if $n - mr < 0$ i.e. $m > \dfrac{n}{r}$.

[As an illustration, let's consider finding $\left|X_2 \cap X_3\right|$ for the equation in 2(c).

Let $z_2 = x_2 - 12$ and $z_3 = x_3 - 12$.

Then $x_1 + x_2 + x_3 = 28$ becomes

$z_2 + z_3 + x_1 = 28 - 2(12) = 4$.]

Hence,

$$\left|X_{j_1} \cap X_{j_2} \cap \ldots \cap X_{j_m}\right|$$
$$= \begin{cases} \dbinom{n - mr + k - 1}{k - 1} & \text{if } m \le \dfrac{n}{r}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by the Principle of Inclusion and Exclusion,

$$|S|$$
$$= \binom{n+k-1}{k-1} - \binom{k}{1}\binom{n-r+k-1}{k-1} + \binom{k}{2}\binom{n-2r+k-1}{k-1} - \binom{k}{3}\binom{n-3r+k-1}{k-1}$$
$$+ \ldots. + (-1)^m \binom{k}{m}\binom{n-mr+k-1}{k-1}, \text{ where } m = \left\lfloor \frac{n}{r} \right\rfloor$$
$$= \sum_{m=0}^{\left\lfloor \frac{n}{r} \right\rfloor} (-1)^m \binom{k}{m}\binom{n-mr+k-1}{k-1} \quad ------(2)$$

Hence, by (1) and (2),

$$\sum_{m=0}^{\left\lfloor \frac{n}{r} \right\rfloor} (-1)^m \binom{k}{m}\binom{n-mr+k-1}{k-1} = \binom{kr-n-1}{k-1}.$$

3(i)	$u^4+1=\left(u^2+Cu+1\right)\left(u^2-Cu+1\right)$ $=\left(u^2+1\right)^2-C^2u^2$ $=u^4+2u^2+1-C^2u^2$ Hence $2-C^2=0\Rightarrow C=\sqrt{2}$ $\dfrac{Au+B}{u^2+\sqrt{2}u+1}-\dfrac{Au-B}{u^2-\sqrt{2}u+1}$ $=\dfrac{(Au+B)\left(u^2-\sqrt{2}u+1\right)-(Au-B)\left(u^2+\sqrt{2}u+1\right)}{u^4+1}$ In numerator: Constant term: $B+B=1\Rightarrow B=\dfrac{1}{2}$ Coefficient of u^2: $-\sqrt{2}A+B-\left(\sqrt{2}A-B\right)=0$ $A=\dfrac{2B}{2\sqrt{2}}=\dfrac{1}{2\sqrt{2}}$ Hence $\dfrac{1}{u^4+1}=\dfrac{\frac{1}{2\sqrt{2}}u+\frac{1}{2}}{u^2+\sqrt{2}u+1}-\dfrac{\frac{1}{2\sqrt{2}}u-\frac{1}{2}}{u^2-\sqrt{2}u+1}$.
3(ii)	$u^2=\tan x$ $2u\dfrac{\mathrm{d}u}{\mathrm{d}x}=\sec^2 x=u^4+1$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}=\dfrac{u^4+1}{2u}$

	$\int \frac{1}{\sqrt{\tan x}}\,\mathrm{d}x = \int \frac{1}{u}\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u$ $= \int \frac{2}{u^4+1}\,\mathrm{d}u$ $= \int \frac{\frac{1}{\sqrt{2}}u+1}{u^2+\sqrt{2}u+1} - \frac{\frac{1}{\sqrt{2}}u-1}{u^2-\sqrt{2}u+1}\,\mathrm{d}u$ $= \int \frac{\frac{1}{2\sqrt{2}}\left(2u+\sqrt{2}\right)+\frac{1}{2}}{u^2+\sqrt{2}u+1} - \frac{\frac{1}{2\sqrt{2}}\left(2u-\sqrt{2}\right)-\frac{1}{2}}{u^2-\sqrt{2}u+1}\,\mathrm{d}u$ $= \frac{1}{2\sqrt{2}}\int \frac{2u+\sqrt{2}}{u^2+\sqrt{2}u+1} - \frac{2u-\sqrt{2}}{u^2-\sqrt{2}u+1}\,\mathrm{d}u$ $+\frac{1}{2}\int \frac{1}{u^2+\sqrt{2}u+1} + \frac{1}{u^2-\sqrt{2}u+1}\,\mathrm{d}u$ $= \frac{1}{2\sqrt{2}}\left(\ln\left\|u^2+\sqrt{2}u+1\right\| - \ln\left\|u^2-\sqrt{2}u+1\right\|\right)$ $+\frac{1}{2}\int \frac{1}{\left(u+\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}} + \frac{1}{\left(u-\frac{1}{\sqrt{2}}\right)^2+\frac{1}{2}}\,\mathrm{d}u$ $= \frac{1}{2\sqrt{2}}\ln\left\|\frac{u^2+\sqrt{2}u+1}{u^2-\sqrt{2}u+1}\right\|$ $+\frac{\sqrt{2}}{2}\left(\tan^{-1}\left(\frac{u+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)+\tan^{-1}\left(\frac{u-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)\right)+C$ $= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left\|\frac{u^2+\sqrt{2}u+1}{u^2-\sqrt{2}u+1}\right\| + \tan^{-1}\left(\sqrt{2}u+1\right)+\tan^{-1}\left(\sqrt{2}u-1\right)\right)+C$ $= \frac{1}{\sqrt{2}}\left(\frac{1}{2}\ln\left\|\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right\| + \tan^{-1}\left(\sqrt{2\tan x}+1\right)+\tan^{-1}\left(\sqrt{2\tan x}-1\right)\right)+C$

4(a) (i)	$\dfrac{x_{n+1}}{x_n}=\dfrac{(n+1)^2}{(2n+1)!}\times\dfrac{(2n-1)!}{n^2}$ $=\dfrac{\left(\frac{n+1}{n}\right)^2}{2n(2n+1)}$ $=\dfrac{\left(1+\frac{1}{n}\right)^2}{2n(2n+1)}$ For all positive integers n, $n\ge 1$. Thus, $\left(1+\frac{1}{n}\right)^2\le 2^2=4$ and $2n(2n+1)\ge 2\times 3=6$ So $\dfrac{x_{n+1}}{x_n}\le\dfrac{4}{6}=\dfrac{2}{3}$ (shown)
4(a) (ii)	$x_n\le\dfrac{2}{3}x_{n-1}\le\left(\dfrac{2}{3}\right)^2x_{n-2}\le\cdots\le\left(\dfrac{2}{3}\right)^{n-1}x_1$. Thus, $\displaystyle\sum_{n=1}^{N}x_n\le\sum_{n=1}^{N}x_1\left(\frac{2}{3}\right)^{n-1}$ $=x_1\dfrac{1-\left(\frac{2}{3}\right)^N}{1-\frac{2}{3}}$ $=3x_1\left[1-\left(\dfrac{2}{3}\right)^N\right]\to 3x_1$ as $N\to\infty$, which is a finite number, hence the series $\displaystyle\sum_{n=1}^{\infty}x_n$ converges.
4(a) (iii)	Since $l<1$, $1-l>0\Rightarrow\dfrac{1-l}{2}>0$. Take $k=\dfrac{1-l}{2}$. Then we can find a sufficiently large N such that if $n>N$, then $\dfrac{v_{n+1}}{v_n}<l+k=l+\dfrac{1-l}{2}=\dfrac{l+1}{2}$, which is a positive number less than 1. Let $r=\dfrac{l+1}{2}$. Then for all positive integers m, $\dfrac{v_{N+m+1}}{v_{N+m}}<r\Rightarrow v_{N+m+1}<rv_{N+m}<r^2v_{N+m-1}<\cdots r^mv_{N+1}$.

	Hence, for large integers M (larger than N), $$\begin{aligned}\sum_{n=1}^{M} v_n &= \sum_{n=1}^{N} v_n + \sum_{n=N+1}^{M} v_n \\ &< \sum_{n=1}^{N} v_n + \sum_{n=N+1}^{M} r^m v_{N+1} \\ &= \sum_{n=1}^{N} v_n + v_{N+1} \sum_{n=N+1}^{M} r^m \\ &= \sum_{n=1}^{N} v_n + v_{N+1}\left(\frac{1-r^{M-N}}{1-r}\right) \\ &\to \sum_{n=1}^{N} v_n + \frac{v_{N+1}}{1-r} \text{ as } M \to \infty, \text{ since } 0<r<1.\end{aligned}$$ Since $\sum_{n=1}^{M} v_n$ is bounded above by a finite value for all positive integers N and the terms of the series are all positive, the series $\sum_{n=1}^{\infty} v_n$ must be convergent.
4(b)	$$\mathrm{e} = \mathrm{e}^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \cdots \qquad (1)$$ $$\mathrm{e}^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \qquad (2)$$ $$(1)-(2): \mathrm{e} - \mathrm{e}^{-1} = 2\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right)$$ $$1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots = \frac{\mathrm{e} - \mathrm{e}^{-1}}{2} = \frac{\mathrm{e}^2 - 1}{2\mathrm{e}}$$

5(i)(a)	Partition the set $\{1,2,3,\ldots,2m\}$ to m sets as follows: $$\{1,2\},\{3,4\},\{5,6\},\ldots,\{2m-1,2m\}.$$ It can be seen that both elements in each set differ by 1 and are therefore coprime. By the Pigeonhole principle, since $m+1$ integers are chosen, at least one of these sets will have both its elements chosen, which gives rise to the required pair of coprime integers.
5(i)(b)	The largest odd divisor of each of the $m+1$ integers can only possibly be from the set $\{1,3,5,\ldots,2m-1\}$ which has m elements. By the Pigeonhole Principle, at least two of these $m+1$ integers will share the same largest odd divisor, call it *d*. Thus, these two integers are of the form $2^i d$ and $2^j d$ where $i, j \in \mathbb{Z}^+$. Regardless of the relative values of *i* and *j*, one will divide the other.
5(ii)	Let $\frac{a}{b}$ be an irreducible fraction in *I* with $1 \le b \le n$. Let $\frac{c}{kb}$, with $k \in \mathbb{Z}^+$ and $1 \le kb \le n$, be an irreducible fraction distinct from $\frac{a}{b}$. Then, $$\left\|\frac{c}{kb}-\frac{a}{b}\right\|=\left\|\frac{c-ak}{kb}\right\| \ge \frac{1}{kb} \ge \frac{1}{n}.$$ Thus, $\frac{c}{kb} \notin I$ as *I* is an *open* interval of length of $\frac{1}{n}$.
5(iii)	From **(ii)**, it can be seen that every irreducible fraction in *I* with denominator at least 1 and at most *n* will have different denominators such that no two denominators have the property that one divides the other. If *n* is even, i.e., $n=2m$, by **(i)(b)**, there can be at most $m=\frac{n}{2}$ such irreducible fractions in *I*. If *n* is odd, i.e., $n=2m+1$, by the given result, there can be at most $m+1=\frac{2m+1+1}{2}=\frac{n+1}{2}>\frac{n}{2}$ such irreducible fractions in *I*. Therefore, *I* contains at most $\frac{n+1}{2}$ irreducible fractions with denominator between 1 and *n* inclusive.

5(iv)	No. Counterexample: Let $n=2$ and consider the closed interval $\left[\frac{1}{2},1\right]$ of length $\frac{1}{2}$ which contains the two irreducible fractions $\frac{1}{2}$ and $\frac{1}{1}$, each with denominator between 1 and 2 inclusive. Since the interval contains more than $\frac{2+1}{2}=\frac{3}{2}$ of such irreducible fractions, the part **(iv)** result does not hold.

6(i)	Let $\mathrm{f}(x)=ax^4+bx^3+cx^2+dx+e$. Scale parallel to y-axis by a factor of $\frac{1}{a}$. Get $y=\mathrm{f}_1(x)=x^4+\frac{b}{a}x^3+\frac{c}{a}x^2+\frac{d}{a}x+\frac{e}{a}$ Translate the graph of $y=\mathrm{f}_1(x)$ by $\frac{b}{4a}$ units in the x-direction. Get $y=\mathrm{f}_2(x)=\left(x-\frac{b}{4a}\right)^4+\frac{b}{a}\left(x-\frac{b}{4a}\right)^3+\frac{c}{a}\left(x-\frac{b}{4a}\right)^2+\frac{d}{a}\left(x-\frac{b}{4a}\right)+\frac{e}{a}$ Coefficient of x^3 is $\binom{4}{1}\left(-\frac{b}{4a}\right)+\frac{b}{a}=0$ Constant term is $\left(\frac{b}{4a}\right)^4-\frac{b}{a}\left(\frac{b}{4a}\right)^3+\frac{c}{a}\left(\frac{b}{4a}\right)^2-\frac{d}{a}\left(\frac{b}{4a}\right)+\frac{e}{a}$ Hence $\mathrm{f}_2(x)=x^4+px^2+qx+r$ where $r=\left(\frac{b}{4a}\right)^4-\frac{b}{a}\left(\frac{b}{4a}\right)^3+\frac{c}{a}\left(\frac{b}{4a}\right)^2-\frac{d}{a}\left(\frac{b}{4a}\right)+\frac{e}{a}$ Finally, translate the graph of $y=\mathrm{f}_2(x)$ by $-r$ units in the y-direction. Get $y=\mathrm{g}(x)=x^4+px^2+qx$
6(ii)	$\mathrm{g}(x)=x^4+px^2+qx$ $\mathrm{g}'(x)=4x^3+2px+q$ $\mathrm{g}''(x)=12x^2+2p$ At a point of inflection, there is a change in sign of $\mathrm{g}''(x)$, in other words, $\mathrm{g}''(x)$ is positive for some values of x and negative for others. If p were positive or zero, $\mathrm{g}''(x)\geq 0$ for all real values of x, which is a contradiction. Hence, p is negative.

6(iii)	If p is negative then the two solutions are $x=\pm\sqrt{-\frac{p}{6}}$. $\mathrm{g}\left(\pm\sqrt{-\frac{p}{6}}\right)=\frac{p^2}{36}-\frac{p^2}{6}\pm q\sqrt{-\frac{p}{6}}=-\frac{5p^2}{36}\pm q\sqrt{-\frac{p}{6}}$ Equation of ℓ: $\frac{y-\left(-\frac{5p^2}{36}+q\sqrt{-\frac{p}{6}}\right)}{x-\sqrt{-\frac{p}{6}}}=\frac{\left(-\frac{5p^2}{36}+q\sqrt{-\frac{p}{6}}\right)-\left(-\frac{5p^2}{36}-q\sqrt{-\frac{p}{6}}\right)}{\sqrt{-\frac{p}{6}}-\left(-\sqrt{-\frac{p}{6}}\right)}$ $=q$ $y-\left(-\frac{5p^2}{36}+q\sqrt{-\frac{p}{6}}\right)=q\left(x-\sqrt{-\frac{p}{6}}\right)$ $y=qx-\frac{5p^2}{36}$
6(iv)	Find where ℓ intersects $y=\mathrm{g}(x)$: $x^4+px^2+qx=qx-\frac{5p^2}{36}$ $x^4+px^2+\frac{5p^2}{36}=0$ $\left(x^2+\frac{p}{6}\right)\left(x^2+\frac{5p}{6}\right)=0$ The four solutions are $x_D=-\sqrt{-\frac{5p}{6}}, x_B=-\sqrt{-\frac{p}{6}}, x_A=\sqrt{-\frac{p}{6}}, x_C=\sqrt{-\frac{5p}{6}}$ Since p is negative, these four solutions are all real and are listed above in increasing order. The two x-values in (iii), which are the points of inflection, are in the middle (x_B and x_A) and the other two values are outside. Hence, the order of points along the line is *DABC*.

6(v)	Draw triangles with BA and DB as hypotenuses and the other two sides are parallel to the x- and y-axes. These two triangles are similar. By similar triangles, $\dfrac{BA}{DB}=\dfrac{x_A-x_B}{x_B-x_D}$ $=\dfrac{2\sqrt{-\frac{p}{6}}}{-\sqrt{-\frac{p}{6}}-\left(\sqrt{-\frac{5p}{6}}\right)}$ $=\dfrac{2}{-1+\sqrt{5}}$ $=\dfrac{2}{-1+\sqrt{5}}\dfrac{1+\sqrt{5}}{1+\sqrt{5}}$ $=\dfrac{1+\sqrt{5}}{2}$ Similarly, $\dfrac{BA}{AC}=\dfrac{x_A-x_B}{x_C-x_A}$ $=\dfrac{2\sqrt{-\frac{p}{6}}}{\sqrt{-\frac{5p}{6}}-\sqrt{-\frac{p}{6}}}$ $=\dfrac{2}{-1+\sqrt{5}}$ $=\dfrac{1+\sqrt{5}}{2}$

7(i)(a)	$(i,0)$ is along the bottom row of the network and can only be reached by going right directly from $(0,0)$. Hence $a_{i,0}=1$.
7(i)(b)	(i,j) can only be reached from $(i-1,j)$ if it exists in the network, and $(i,j-1)$ (which must be in the network since $i>j-1\geq 0$). Since $a_{i-1,j}$ is the number of ways to reach $(i-1,j)$ if it is in the network, and 0 otherwise, $a_{i,j}=a_{i-1,j}+a_{i,j-1}$ must be true by the additive principle.
7(ii)	Let $P_{i,j}$ be the statement that $a_{i,j}=\frac{i-j+1}{i+1}\binom{i+j}{i}$ for non-negative integers i, j where $i\geq j$. Base case: To show $P_{i,0}$ is true for all non-negative integers *i*: LHS: $a_{i,0}=1$ from (i)(a) RHS: $\frac{i-0+1}{i+1}\binom{i+0}{i}=\frac{i+1}{i+1}\binom{i}{i}=1$ Hence $P_{i,0}$ is true for all non-negative integers *i*. Suppose $P_{q,r}$ is true for some non-negative integer *r*, and all *q* where $q\geq r$. First we show that $P_{r+1,r+1}$ is true i.e. $a_{r+1,r+1}=\frac{1}{r+2}\binom{2r+2}{r+1}$. $a_{r+1,r+1}=a_{r,r+1}+a_{r+1,r}$ $=0+\frac{r+1-r+1}{r+1+1}\binom{r+1+r}{r+1}$ $=\frac{2}{r+2}\binom{2r+1}{r+1}$ $=\frac{2}{r+2}\frac{(2r+1)!}{(r+1)!r!}$ $=\frac{2(r+1)}{r+2}\frac{(2r+1)!}{(r+1)!(r+1)!}$ $=\frac{1}{r+2}\frac{(2r+2)!}{(r+1)!(r+1)!}$ $=\frac{1}{r+2}\binom{2r+2}{r+1}$

	Now suppose $P_{r+k,r+1}$ is true for some positive integer k. We show that $P_{r+k+1,r+1}$ is true i.e. $a_{r+k+1,r+1}=\frac{k+1}{r+k+2}\binom{2r+k+2}{r+k+1}$. $a_{r+k+1,r+1}=a_{r+k,r+1}+a_{r+k+1,r}$ $=\frac{k}{r+k+1}\binom{2r+k+1}{r+k}+\frac{k+2}{r+k+2}\binom{2r+k+1}{r+k+1}$ $=\frac{k}{r+k+1}\frac{(2r+k+1)!}{(r+k)!(r+1)!}+\frac{k+2}{r+k+2}\frac{(2r+k+1)!}{(r+k+1)!r!}$ $=\frac{k(2r+k+1)!}{(r+k+1)!(r+1)r!}+\frac{k+2}{r+k+2}\frac{(2r+k+1)!}{(r+k+1)!r!}$ $=\left(\frac{k}{r+1}+\frac{k+2}{r+k+2}\right)\frac{(2r+k+1)!}{(r+k+1)!r!}$ $=\left(\frac{k(r+k+2)+(k+2)(r+1)}{(r+1)(r+k+2)}\right)\frac{(2r+k+1)!}{(r+k+1)!r!}$ $=\frac{k^2+(2r+3)k+2r+2}{r+k+2}\frac{(2r+k+1)!}{(r+k+1)!(r+1)!}$ $=\frac{(k+1)(2r+k+2)}{r+k+2}\frac{(2r+k+1)!}{(r+k+1)!(r+1)!}$ $=\frac{k+1}{r+k+2}\frac{(2r+k+2)!}{(r+k+1)!(r+1)!}$ $=\frac{k+1}{r+k+2}\binom{2r+k+2}{r+k+1}$ Since $P_{r+1,r+1}$ is true and $P_{r+k,r+1}$ is true $\Rightarrow P_{r+k+1,r+1}$ is true, $P_{q,r+1}$ is true for all q where $q\ge r+1$. Since $P_{i,0}$ is true for all $i\ge 0$ and $P_{q,r}$ is true $\Rightarrow P_{q,r+1}$ is true for all $q\ge r+1\ge 0$, $P_{i,j}$ is true for all non-negative integers i, j where $i\ge j$.

7(iii)(a)	$\binom{i+j}{i}-\binom{i+j}{i+1}=\frac{(i+j)!}{i!\,j!}-\frac{(i+j)!}{(i+1)!(j-1)!}$ $=\frac{(i+j)!}{i!(j-1)!}\left(\frac{1}{j}-\frac{1}{i+1}\right)$ $=\frac{(i+j)!}{i!(j-1)!}\frac{i-j+1}{j(i+1)}$ $=\frac{(i+j)!}{i!\,j!}\frac{i-j+1}{i+1}$ $=\binom{i+j}{i}\frac{i-j+1}{i+1}$ $=a_{i,j}$
7(iii)(b)	$\sum_{r=0}^{n}a_{2n-r,r}=1+\sum_{r=1}^{n}\binom{2n}{2n-r}-\binom{2n}{2n-r+1}$ $=1$ $+\cancel{\binom{2n}{2n-1}}-\binom{2n}{2n}$ $+\cancel{\binom{2n}{2n-2}}-\cancel{\binom{2n}{2n-1}}$ $\vdots$ $+\cancel{\binom{2n}{2n-(n-1)}}-\cancel{\binom{2n}{2n-(n-1)+1}}$ $+\binom{2n}{2n-n}-\cancel{\binom{2n}{2n-n+1}}$ $=1-\binom{2n}{2n}+\binom{2n}{n}$ $=\binom{2n}{n}$

8(i)(a)	We see that $3^1 = 3 \equiv 3 \pmod 4$, $3^2 = 9 \equiv 1 \pmod 4$. So, order of 3 modulo 4 is 2.
8(i)(b)	We see that $2^1 = 2 \equiv 2 \pmod 5$, $2^2 = 4 \equiv 4 \pmod 5$, $2^3 = 8 \equiv 3 \pmod 5$, $2^4 = 16 \equiv 1 \pmod 5$. So, order of 2 modulo 5 is 4.
8(ii)	Since p is a prime, $\gcd(g, p) = 1$ for any $g \in S_{p-1}$. As such, $g^{p-1} \equiv 1 (\text{mod } p)$. Let d be the order of g. So, from above $1 \le d \le p-1$. As such, $g^d \equiv 1 (\text{mod } p)$. Suppose d does not divide $p-1$. Then $p-1 = dm + r$, for some $0 < r < d$. Thus, $g^{p-1} \equiv g^{dm+r} \pmod p$ $1 \equiv \left(g^d\right)^m g^r \pmod p$ $1 \equiv 1^m g^r \pmod p$ $1 \equiv g^r \pmod p$, which implies that there is an integer r smaller than d such that g^r is congruent to 1 mod p. This contradicts our assumption that d is the order of g. Thus, d must divide $p-1$.

8(iii)(a)(A)	We see that $2^1 \equiv 2(\text{mod}11)$, $2^2 \equiv 4(\text{mod}11)$, $2^3 \equiv 8(\text{mod}11)$, $2^4 \equiv 5(\text{mod}11)$, $2^5 \equiv 10(\text{mod}11)$, $2^6 \equiv 9(\text{mod}11)$, $2^7 \equiv 7(\text{mod}11)$, $2^8 \equiv 3(\text{mod}11)$, $2^9 \equiv 6(\text{mod}11)$, $2^{10} \equiv 1(\text{mod}11)$. So, for any $1 \le k \le 10$, there exists a natural number n such $2^n \equiv k(\text{mod}11)$
8(iii)(a)(B)	We see that $2^1 \equiv 2(\text{mod}13)$, $2^2 \equiv 4(\text{mod}13)$, $2^3 \equiv 8(\text{mod}13)$, $2^4 \equiv 3(\text{mod}13)$, $2^5 \equiv 6(\text{mod}13)$, $2^6 \equiv 12(\text{mod}13)$, $2^7 \equiv 11(\text{mod}13)$, $2^8 \equiv 9(\text{mod}13)$, $2^9 \equiv 5(\text{mod}13)$, $2^{10} \equiv 10(\text{mod}13)$, $2^{11} \equiv 7(\text{mod}13)$, $2^{12} \equiv 1(\text{mod}13)$ So, for any $1 \le k \le 12$, there exists a natural number n such $2^n \equiv k(\text{mod}13)$

8(iii)(b)	Let $1 \le a,b \le p-1$. Suppose that $$g^a \equiv g^b (\text{mod } p)$$ Since g is coprime to p, $$g^a \equiv g^b (\text{mod } p)$$ $$\frac{g^a}{g^b} \equiv \frac{g^b}{g^b} (\text{mod } p)$$ $$g^{a-b} \equiv 1 (\text{mod } p)$$ By part (ii), the order of g is $p-1$, as such $p-1 \| a-b$. Now, since $1 \le a,b \le p-1$, $-\left(p-2\right) \le a-b \le p-2$. Consequently, the only integer in the range such that $a-b$ is a multiple of $p-1$ is 0. Thus, $a-b=0$, i.e. $a=b$. So, there are no elements in $\left\{g, g^2, \cdots . g^{p-1}\right\}$ that are congruent to each other modulo p. In other words, the set $\left\{g, g^2, \cdots . g^{p-1}\right\}$ consists of $p-1$ distinct elements modulo p
8(iii)(c)	Let d be the order of the element g modulo p. By (b), $d \| p-1=2q..$ As q is prime, d is either 1 , 2 , q or $2q$. Case 1. $d=1$. This means that $g^1 \equiv 1\left(\text{mod } p\right)$. A contradiction. Case 2. $d=2$. This means that $g^2 \equiv 1\left(\text{mod } p\right)$. This implies $g \equiv \pm 1\left(\text{mod } p\right)$. But this is a contradiction also.

	Case 3. $d = q$. This means $g^q \equiv 1 \pmod{p}$. This is yet another contradiction. Case 4. $d = 2q$ is the only possible case then. Therefore, the order of g is $2q = p-1$. Consider the set $\{g, g^2, \cdots . g^{p-1}\}$. By (b), each of them are distinct elements modulo p. Let k be such that $1 \le k \le p-1$. Since $\{1, 2, \cdots, p-1\}$ is also a set of distinct elements modulo p, $k \equiv g^n \pmod{p}$ for some $1 \le n \le p-1$. Consequently, g is a primitive root modulo p.