RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Tutorial 3B
Vectors II – Equations of Straight Lines

Section A (Basic Questions)

1 For each of the following, write down a vector equation of the line l and convert it to Cartesian form.

(a) l passes through the point with position vector $-\mathbf{i}+\mathbf{k}$ and is parallel to the vector $\mathbf{i}+\mathbf{j}$.

(b) l passes through the points $P(1,-1,3)$ and $Q(2,1,-2)$.

(c) l passes through the origin O and is parallel to the line $m:\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\3\end{pmatrix},\ \lambda\in\mathbb{R}$.

(d) l passes through the point $C(1,1,0)$ and is parallel to the z – axis.

[**(a)** $x+1=y,\ z=1$ **(b)** $x-1=\frac{y+1}{2}=\frac{3-z}{5}$ **(c)** $x=\frac{y}{2}=\frac{z}{3}$ **(d)** $x=1,\ y=1$]

2 For each of the following, find the acute angle between the lines l_1 and l_2.
Determine if l_1 and l_2 are parallel, intersecting or skew.
In the case of intersecting lines, find the position vector of the point of intersection.

(a) $l_1: x-1=-y=z-2$ and $l_2:\frac{x-2}{2}=-\frac{y+1}{2}=\frac{z-3}{2}$

(b) $l_1:\mathbf{r}=\begin{pmatrix}1\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix},\ \alpha\in\mathbb{R}$ and $l_2:\mathbf{r}=\begin{pmatrix}0\\10\\1\end{pmatrix}+\beta\begin{pmatrix}3\\8\\1\end{pmatrix},\ \beta\in\mathbb{R}$

(c) $l_1:\mathbf{r}=(\mathbf{i}-5\mathbf{k})+\lambda(\mathbf{i}-\mathbf{j}+\mathbf{k}),\ \lambda\in\mathbb{R}$ and $l_2:\mathbf{r}=(\mathbf{i}-\mathbf{j}+\mathbf{k})+\mu(5\mathbf{i}-4\mathbf{j}-\mathbf{k}),\ \mu\in\mathbb{R}$

[**(a)** $0°$, parallel **(b)** $81.3°$, skew **(c)** $44.5°$, intersecting, $\begin{pmatrix}6\\-5\\0\end{pmatrix}$]

Section B (Standard Questions)

3 9740/2015/01/Q7

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC : CA = 3 : 2$. Point D lies on OB, between O and B, such that $OD : DB = 5 : 6$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of a parameter μ. [3]

3 mark (iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE : ED$. [5]

[(i) $\frac{3}{5}\mathbf{a}, \frac{5}{11}\mathbf{b}$ (ii) $\mathbf{r} = \frac{5}{11}\mu\mathbf{b} + (1-\mu)\mathbf{a}$ (iii) $\frac{9}{20}\mathbf{a} + \frac{1}{4}\mathbf{b}$, 11 : 9]

4 The points P and Q have coordinates $(0,-1,-1)$ and $(3,0,1)$ respectively, and the equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R} \quad \text{and} \quad l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \ \mu \in \mathbb{R}.$$

(a) (i) Show that P lies on l_1 but does not lie on l_2.

(ii) Determine whether the lines l_1 and l_2 passes through Q.

(b) (i) Find the coordinates of the foot of perpendicular from P to l_2. Hence, or otherwise, find the perpendicular distance from P to l_2.

(ii) Using (b)(i) or otherwise, find the length of projection of $\overrightarrow{PQ}$ onto l_2.

[(a)(ii) Q is on l_2 but not l_1 (b)(i) $(1,1,1)$, 3 units (b)(ii) $\sqrt{5}$ units]

5 MI JC2CT1 9740/2011/02/Q4

The line l_1 passes through the points A and B which have position vectors $2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $3\mathbf{i}-\mathbf{j}$ respectively. The line l_2 has Cartesian equation $2x = y+7 = \frac{z-3}{2}$. Find

(i) the vector equations of l_1 and l_2, [2]

(ii) the acute angle between the two lines, [2]

(iii) the position vector of the point of intersection between l_1 and l_2. [3]

The line l_3 contains the point B. It also is perpendicular to and intersects l_2.

(iv) Find the coordinates of the point of intersection between l_2 and l_3. [4]

(v) Hence, or otherwise, find the area of the triangle bounded by the three lines. [2]

[(i) $l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $l_2 : \mathbf{r} = \begin{pmatrix} 0 \\ -7 \\ 3 \end{pmatrix} + \beta \begin{pmatrix} 0.5 \\ 1 \\ 2 \end{pmatrix}$ (ii) 84.9^o (iii) $\begin{pmatrix} 0 \\ -7 \\ 3 \end{pmatrix}$

(iv) $\left(\frac{1}{7}, -\frac{47}{7}, \frac{25}{7}\right)$ (v) $\frac{15}{14}\sqrt{5}$ units^2]

6 The points A and B have position vectors $\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}+\mathbf{j}-2\mathbf{k}$ respectively.

(i) Find a vector equation of the line l passing through the midpoint, M, of AB and the origin O.

(ii) Find the position vector of a point N where N is the foot of the perpendicular from A to l.

(iii) Find the position vector of a point C such that $AOCM$ is a parallelogram.

(iv) Show that the points A, N and C are collinear.

State, with a geometrical reason, the value of $\dfrac{CO}{CM}$.

[(i) $l: \mathbf{r} = t\begin{pmatrix}1\\1\\0\end{pmatrix}$, $t \in \mathbb{R}$ (ii) $\overrightarrow{ON} = \dfrac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}$ (iii) $\overrightarrow{OC} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$ (iv) 1]

7 **ACJC Prelim 9233/ 2000/02/Q15 (part) (modified)**

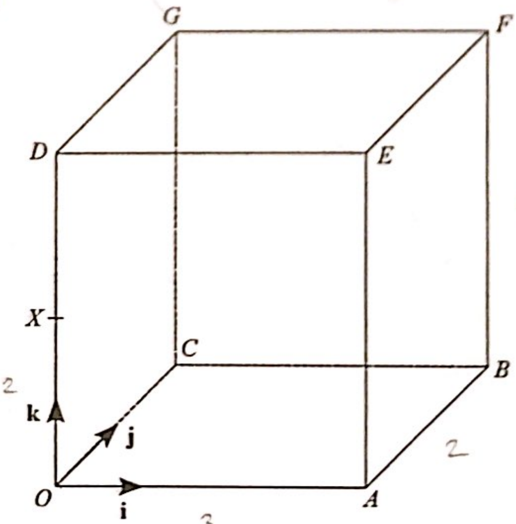
In the diagram on the right, O is the origin and **i**, **j** and **k** are the unit vectors along the x, y and z-axis respectively.

The length of OA, AB and BF of the cuboid $OABCDEFG$ is 3 units, 2 units and 4 units respectively.

The point X is the midpoint of OD.

l_1 is the line passing through the points X and F.

l_2 is the line passing through the points G and E.



(i) Find, to the nearest 0.1°, the acute angle between the lines l_1 and l_2.

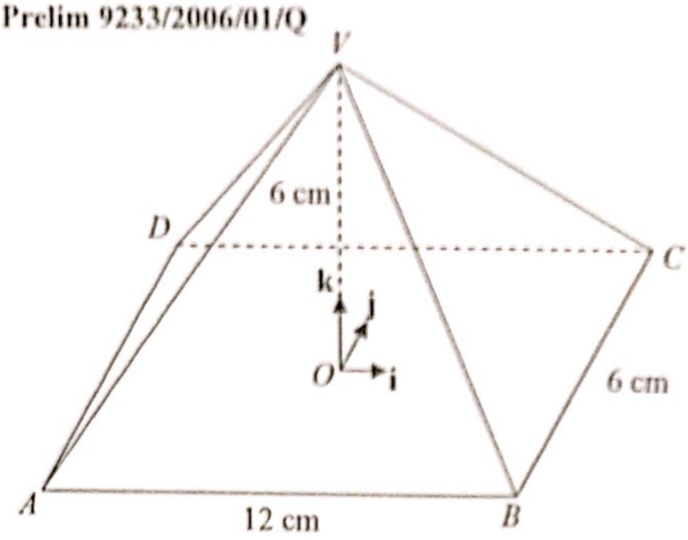
(ii) Verify that the point $P\left(\dfrac{3}{2}, 1, 4\right)$ is on the line l_2.

(iii) Find the length of the projection of $\overrightarrow{DP}$ onto l_1, leaving your answer in surd form.

(iv) Find the coordinates of the foot of the perpendicular from P to l_1, leaving your answer in fractional form.

[(i) 70.3° (iii) $\dfrac{13\sqrt{17}}{34}$ units (iv) $\left(\dfrac{63}{34}, \dfrac{21}{17}, \dfrac{55}{17}\right)$]

8 **HCI Prelim 9233/2006/01/Q**



In the diagram, O is centre of the rectangular base $ABCD$ of a right pyramid with vertex V.

Perpendicular unit vectors **i**, **j**, **k** are parallel to AB, BC and OV respectively. The length of AB, BC and OV are 12 cm, 6 cm and 6 cm respectively.

A line l has cartesian equation $\dfrac{-x-4}{10} = y+2 = \dfrac{t-z}{-2}$.

(i) Find the vector equation of line AV. [2]

(ii) If the line l intersects line AV at M, find the position vector of M and the value of t. [5]

(iii) Find the acute angle between line AV and the line l. Hence find the perpendicular distance from A to the line l. [5]

[(i) $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$ (ii) $\overrightarrow{OM} = \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix}$, $t = 2$ (iii) $60.8°$, 2.62]

Section C (Extension/Challenging Questions)

9 The four points A, B, C, D have position vectors $p\mathbf{i}$, $\mathbf{j}+q\mathbf{k}$, $\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ respectively, where p and q are positive real numbers. Find vector equations for the lines AB and CD.

(i) Given that the lines AB and CD intersect at a point R, express q in terms of p, and show that the two lines cannot be perpendicular. Show also that $\dfrac{AR}{AB} = \dfrac{CR}{CD}$.

(ii) Given that AB and CD are perpendicular, express q in terms of p, and show that the angle between AC and BD is equal to the angle between AD and BC.

[(i) $q = \dfrac{1}{p}$ (ii) $q = 1 - p$]

10 If **a**, **x** and **y** are three non-zero and mutually non-parallel vectors, and the vectors **b** and **c** are given by $\mathbf{b} = \mathbf{x} - (\mathbf{x} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}}$ and $\mathbf{c} = \mathbf{y} - (\mathbf{y} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} - (\mathbf{y} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$, where $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors along **a** and **b** respectively, show that **a**, **b** and **c** are mutually perpendicular.

Section D (Self-Practice Questions)

1 TPJC Prelim 9740/2013/01/Q10 (modified)

Referred to the origin O, the position vector of the point A is $2\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$ and the cartesian equation of the line l is $x - 1 = 2 - y = z + 6$. Find

(i) the position vector of the foot of the perpendicular from A to l, [3]

(ii) the perpendicular distance from A to l. [2]

[(i) $\frac{1}{3}\begin{pmatrix} 4 \\ 5 \\ -17 \end{pmatrix}$ (ii) $\frac{1}{3}\sqrt{6}$ or 0.816]

2 CJC Prelim 9233/2006/01/Q14

The points A and B have position vectors $\begin{pmatrix} 8 \\ 3 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$ respectively.

(i) Show that $AB = 2\sqrt{26}$. [1]

(ii) Find the cartesian equation for the line AB. [2]

(iii) The line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 6 \\ 5 \end{pmatrix}$.

Find the length of the projection of AB onto l. [2]

(iv) Calculate the acute angle between AB and l, giving your answer correct to the nearest degree. [3]

(v) Find the position vector of the foot N of the perpendicular from A to l. Hence find the position vector of the image of A in the line l. [4]

[(ii) $\frac{8-x}{10} = \frac{z-2}{2}$, $y = 3$ (iii) $\frac{10}{\sqrt{65}}$ (iv) 83° (v) $\frac{1}{13}\begin{pmatrix} -22 \\ 51 \\ 62 \end{pmatrix}$, $\frac{1}{13}\begin{pmatrix} -148 \\ 63 \\ 98 \end{pmatrix}$]

3 MI Prelim 9740/2014/01/Q1

The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \text{ respectively.}$$

(i) Show that l_1 and l_2 are skew lines. [2]

(ii) Find the acute angle between the lines l_1 and l_2. [2]

[(ii) 33.6°]

4 MJC Prelim 9233/2006/02/Q5

Relative to an origin O, points A and B have position vectors $\begin{pmatrix}3\\4\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\2\\0\end{pmatrix}$ respectively.

The line l has vector equation $\mathbf{r}=\begin{pmatrix}6\\a\\0\end{pmatrix}+t\begin{pmatrix}1\\3\\a\end{pmatrix}$, where t is a real parameter and a is a constant. The line m passes through the point A and is parallel to the line OB.

(i) Find the position vector of the point P on m such that OP is perpendicular to m. [4]

(ii) Show that the two lines l and m have no common point. [3]

(iii) If the acute angle between the line l and the $z-$axis is $60°$, find the exact values of the constant a. [3]

[(i) $\overrightarrow{OP}=\begin{pmatrix}4\\2\\1\end{pmatrix}$ (iii) $a=\pm\sqrt{\dfrac{10}{3}}$]

5 SRJC Prelim 9233/2006/02/Q2

The lines l_1 and l_2 have vector equations

$$\mathbf{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}0\\2\\1\end{pmatrix} \text{ and } \mathbf{r}=\begin{pmatrix}1\\0\\4\end{pmatrix}+\mu\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

respectively, where λ and μ are real parameters.

(i) Find the acute angle between the two lines l_1 and l_2, giving your answer to the nearest $0.1°$. [2]

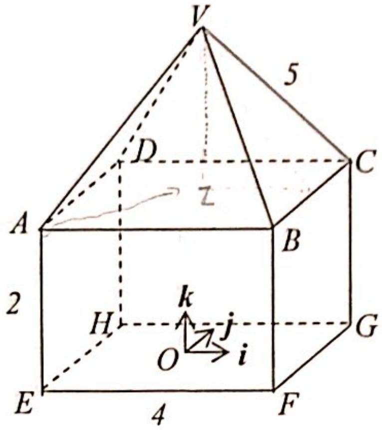
(ii) Show that l_1 passes through point P with position vector $\begin{pmatrix}1\\-4\\2\end{pmatrix}$.

Hence, show that the distance between point P and any point on the line l_2 is given by $\sqrt{6\mu^2-12\mu+20}$. Deduce the shortest distance between point P and the line l_2. [5]

[(i) 56.8° (ii) $\sqrt{14}$]

6 SAJC Prelim 9740/2013/01/Q5 (modified)

The figure shows a right pyramid *VABCD* with a square base *ABCD*, standing horizontally on a cuboid *ABCDEFGH*. It is given that $VA = VB = VC = VD = 5$ cm, $EF = FG = 4$ cm and $AE = 2$ cm, as shown in the diagram. *O* is the centre of the square base *EFGH*. Perpendicular unit vectors **i**, **j**, **k** are parallel to *EF*, *FG*, *EA* respectively.



(i) Show that the height of the figure, *OV* is $2+\sqrt{17}$. [2]

(ii) State the geometrical meaning of $\dfrac{\left|\overrightarrow{VA}\times\overrightarrow{OV}\right|}{2+\sqrt{17}}$. [1]

(iii) Find the equation of the line passing through *B* and *V*. [2]

$$\text{[(iii) } \mathbf{r}=\begin{pmatrix}0\\0\\2+\sqrt{17}\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\-\sqrt{17}\end{pmatrix}, \lambda\in\mathbb{R}\text{]}$$

THE END