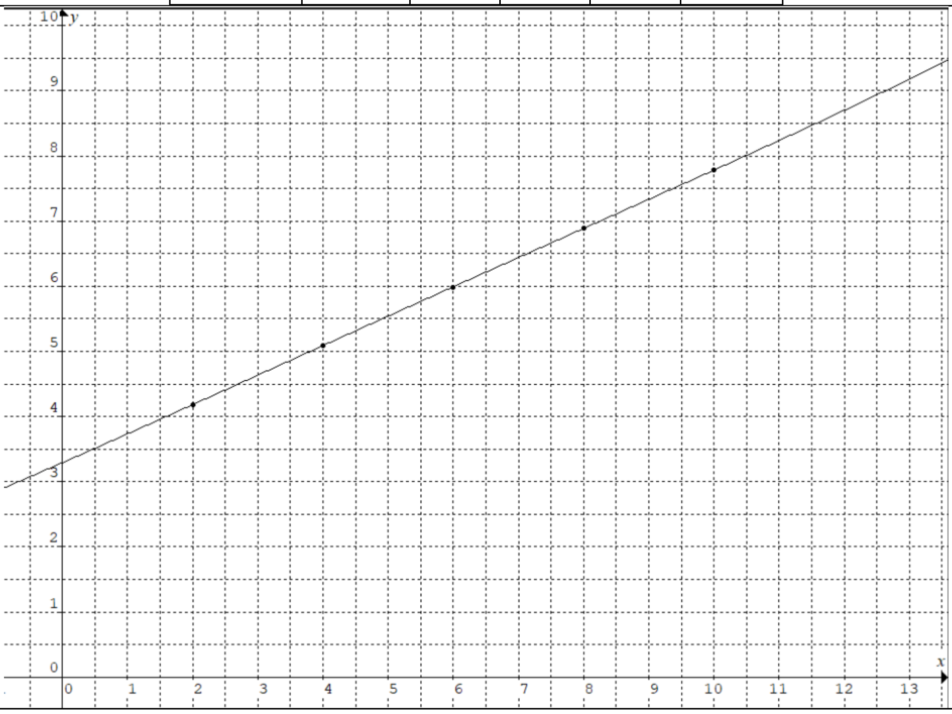
2021 PHSS Prelim AMATH Paper 2 Solutions

Q	SOLUTIONS	MARK	REMARKS
1	Express $\frac{1-2x^2+x^3}{x^3+3x}$ in partial fractions.		
	$\frac{1-2x^2+x^3}{x^3+3x}=1+\frac{1-3x-2x^2}{x^3+3x}$		
	$\frac{1-3x-2x^2}{x^3+3x}=\frac{1-3x-2x^2}{x(x^2+3)}$		
	$\frac{1-3x-2x^2}{x(x^2+3)}=\frac{A}{x}+\frac{Bx+C}{x^2+3}$		
	$1-3x-2x^2=A(x^2+3)+(Bx+C)x$		
	Sub x=0, $A=\frac{1}{3}$		
	Sub $A=\frac{1}{3}$,		
	$3-9x-6x^2=x^2+3+3Bx^2+3Cx$		
	Comparing coefficients of x^2,		
	$-6=1+3B$		
	$B=-\frac{7}{3}$		
	$C=-3$		
	$\frac{1-2x^2+x^3}{x^3+3x}=1+\frac{1}{3x}-\frac{7x+9}{3(x^2+3)}$		
2i	Explain clearly how N_0 and b can be calculated when a straight line graph of $\ln N$ against t is drawn.		
	$\frac{N}{N_0}=e^{bt}$		
	$N=N_0e^{bt}$		
	$\ln N=\ln(N_0e^{bt})$		
	$\ln N=\ln N_0+\ln e^{bt}$		
	$\ln N=\ln N_0+bt$		
	Gradient = b		
	Intercept = $\ln N_0$		
	Convert logarithmic to exponential form/ Solve logarithmic equation to find N_0		

2ii			

t	2	4	6	8	10
$\ln N$	4.19	5.09	5.99	6.89	7.79



2iii	Use your graph to estimate the values of N_0 and of b.		
	$\ln N_0 = 3.29$ (± 0.05) $N_0 = 26.8$		
	b = 0.5 (± 0.1)		
2iv	A state of emergency will be announced in Tiger City when the number of the virus reaches 5000. There are rumours online claiming that the announcement will happen after 13 hours. Using your graph, explain clearly if this is true.		
	$\ln(5000) = 8.517$		
	From graph, t = 11.5		
	The state of emergency will be announced between 11^{th} and 12^{th} hour. The rumours are false.		
2v	The World Health Organisation (WHO) updated the health advisory and suggests that due to a mutation in the virus, N and t are now related by the equation $3\ln N - 150 = 6N_o + 2bt$, where N_0 and b are constants. The new straight line graph of $\ln N$ against t has a $\ln N$ -intercept of 180.16. Find the new value of N_0.		

	$3\ln N - 150 = 6N_o + 2bt$		
	$\ln N - 50 = 2N_o + \frac{2bt}{3}$		
	Intercept $= 50 + 2N_0$		
	$N_0 = \frac{180.16 - 50}{2}$		
	$N_0 = 65.08$		
	OR		
	Sub $t = 0$, $N_o = \frac{3\ln N - 150}{6}$		
	$N_0 = 65.08$		
3	The diagram shows part of the curve $y = \sqrt{3+2x}$, meeting the tangent at P, where $x = 3$.		
3i	Find the equation of the tangent.		
	$y = \sqrt{3+2x}$		
	$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(3+2x)^{-\frac{1}{2}}(2)$		
	$\frac{\mathrm{d}y}{\mathrm{d}x} = (3+2x)^{-\frac{1}{2}}$		
	At P, gradient of tangent, $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{x=3} = \frac{1}{3}$		
	At P, $y = 3$		
	Equation of the tangent is $y = \frac{1}{3}x + 2$		
3ii	The area bounded by PQ, the line $x = 3$ and the line $x = a$ is given as 24 units^2. Show that $a = 9$.		
	Area under line PQ $\Rightarrow$ $\frac{1}{2}\left[3 + \left(\frac{a}{3} + 2\right)\right](a-3) = 24$		
	OR		
	$\left[\frac{x^2}{(3)(2)} + 2x\right]_3^a = 24$		
	$a^2 + 12a - 189 = 0$ $(a+21)(a-9) = 0$		

	$a = 9\ \ (a \neq -21)$		
3iii	Find the area of the shaded region bounded by *PQ*, the curve and the line $x = a$.		
	Area under the curve PQ = $\int_3^9 \sqrt{3+2x}\,\mathrm{d}x$		
	$= \left[\dfrac{(3+2x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(2)}\right]_3^9$		
	$= \dfrac{1}{3}\left[(21)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$		
	Shaded region = $24 - \dfrac{1}{3}\left[(21)^{\frac{3}{2}} - (9)^{\frac{3}{2}}\right]$		
	0.922 units^2		
4a	A curve has equation $y-(k+1)=\dfrac{2}{x}$ and a line has equation $y+1=kx$, where *k* is a constant. Find the set of values of *k* for which the curve meets the line.		
	$\dfrac{2}{x}+(k+1)=kx-1$		
	$2+(k+1)x=kx^2-x$		
	$kx^2-(k+2)x-2=0$		
	$\left[-(k+2)\right]^2-4(k)(-2)\geq 0$		
	$k^2+4k+4+8k\geq 0$		
	$k^2+12k+4\geq 0$		
	At $k^2+12k+4=0$		
	$k=\dfrac{-12\pm\sqrt{12^2-4(1)(4)}}{2}$		
	$k=-6\pm\sqrt{32}$		
	$k\leq -6-\sqrt{32}$ or $k\geq -6+\sqrt{32}$		

4b	The height above the ground, in metres, of the rooftop of a building is modelled as $P(x) = -x^2 + 6x + 11$, where x is the horizontal distance from the main office. The rooftop in the neighbouring school is modelled by $R(x) = 17 - x$. Find the values of x, in metres, for which the rooftop of the building is above that of the school.		
	$-x^2 + 6x + 11 > 17 - x$		
	$x^2 - 7x + 6 < 0$		
	$(x-1)(x-6) < 0$		
	$1 < x < 6$		
4c	The curve with equation $y = (a+1)x^2 + bx + 2 + b$, where a and b are constants, is always above the x-axis. Write down two conditions which apply to a and b.		
	$a > -1$		
	$b^2 - 4(a+1)(2+b) < 0$		
5a	Express $\frac{2x}{2x-3}$ in the form $a + \frac{b}{2x-3}$ where a and b are constants. Hence, $\int \frac{2x}{2x-3}\,dx$.		
	$\frac{2x-3+3}{2x-3} = 1 + \frac{3}{2x-3}$		
	$\int 1 + \frac{3}{2x-3}\,dx = x + \frac{3}{2}\ln(2x-3) + C_1$		
5b	Differentiate $\frac{x\ln(2x-3)}{2}$ with respect to x.		
	$\frac{d}{dx}\left[\frac{x\ln(2x-3)}{2}\right] = \frac{\ln(2x-3)}{2} + \frac{x}{2} \times \frac{1}{2x-3} \times 2$		
	$= \frac{\ln(2x-3)}{2} + \frac{x}{2x-3}$		
5c	Using the results from **(a)** and **(b)**, find $\int \frac{2\ln(2x-3)}{3}\,dx$.		
	From (b),		

	$\int \frac{\ln(2x-3)}{2} + \frac{x}{2x-3} \, dx = \frac{x\ln(2x-3)}{2} + C_2$		
	$\int \frac{\ln(2x-3)}{2} dx = \frac{x\ln(2x-3)}{2} - \int \frac{x}{2x-3} \, dx + C_2$		
	$\int \frac{2\ln(2x-3)}{3} dx = \frac{4}{3}\left[\frac{x\ln(2x-3)}{2}\right] - \frac{4}{3}\int \frac{x}{2x-3} \, dx + C_3$		
	From (a), $\int \frac{2x}{2x-3} \, dx = x + \frac{3}{2}\ln(2x-3) + C_1$		
	$\frac{2}{3}\int \frac{2x}{2x-3} \, dx = \frac{2x}{3} + \ln(2x-3) + C_4$		
	From (a) and (b), $\int \frac{2\ln(2x-3)}{3} dx = \frac{2x\ln(2x-3)}{3} - \frac{2x}{3} \; - \ln(2x-3) + C_5$		
6i	A guide brings her visitors to walk along a trail consisting of pathways *AB*, *BM*, *MC* and *CB*. Show that *T* m, the distance of the trail, can be expressed in the form $T = 720 + 360\sin\theta + 360\cos\theta$.		
	ABC is an isosceles triangle. Perpendicular (90^o) from *B* to *M*, the midpoint of *AC*.		
	$\cos\theta = \frac{CM}{360}$		
	$CM = 360\cos\theta$		
	$\sin\theta = \frac{BM}{360}$		
	$BM = 360\sin\theta$		
	Total distance = $720 + 360\sin\theta + 360\cos\theta$		
6ii	Express *T* in the form $p + R\sin(\theta + \alpha)$, where $R > 0$ and α is an acute angle.		
	$T = 720 + R\sin(\theta + \alpha)$		
	$R = \sqrt{360^2 + 360^2}$		
	= $\sqrt{259200}$		
	= $360\sqrt{2}$		
	$\tan\alpha = \frac{360}{360} = 1$		

	$\alpha = 45^{\circ}$		
	$T = 720 + 360\sqrt{2}\sin(\theta + 45^{\circ})$		
6iii	Given that the trail in (i) is 1.2 km, find the value of θ.		
	$720 + 360\sqrt{2}\sin(\theta + 45^{\circ}) = 1200$		
	$\sin(\theta + 45^{\circ}) = \dfrac{480}{360\sqrt{2}}$		
	$\theta + 45^{\circ} = 70.5287^{\circ}$		
	$\theta = 25.5^{\circ}$		
7	The equation of a curve is $y = e^{3x-2x^2}$.		
7a	Find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.		
	$\dfrac{dy}{dx} = e^{3x-2x^2} \times (3-4x)$		
	$\dfrac{d^2y}{dx^2} = e^{3x-2x^2} \times (-4) + e^{3x-2x^2} \times (3-4x) \times (3-4x)$		
	$\dfrac{d^2y}{dx^2} = -4e^{3x-2x^2} + e^{3x-2x^2} \times (3-4x)^2$		
7b	Find the exact value of the coordinates of the stationary point.		
	$e^{3x-2x^2} \times (3-4x) = 0$		
	$(3-4x) = 0 \Rightarrow x = \dfrac{3}{4}$		
	Reject $e^{3x-2x^2} = 0$		
	If $x = \dfrac{3}{4}$, $y = e^{\frac{9}{8}}$		
7c	Find the nature of the stationary point.		

	If $x=\frac{3}{4}$, $\frac{d^2y}{dx^2}=-4e^{\frac{9}{8}}$		
	This is always **negative**. Therefore, it is a maximum turning point.		
8	A circle, C_1, has a diameter *AB* where *A* is the point (–1, 1) and the tangent at *B* is the line $3y-4x+43=0$.		
8i	Find the equation of the diameter *AB* and hence the coordinates of *B*.		
	Gradient of diameter = $-\frac{3}{4}$		
	$y=-\frac{3}{4}x+c$		
	Equation of the diameter *AB* is $y=-\frac{3}{4}x+\frac{1}{4}$		
	Point of intersection , *B* : $3\left(-\frac{3}{4}x+\frac{1}{4}\right)-4x+43=0$		
	$-\frac{25}{4}x=-\frac{175}{4}$		
	Coordinates of *B* is (7, –5)		
8ii	Find the equation of the circle, C_1.		
	Centre of C_1.= $\left(\frac{-1+7}{2},\frac{1-5}{2}\right)$		
	$(3,-2)$		
	Radius = $\sqrt{(3-(-1))^2+(-2-1)^2}$ =5 units		
	Equation is $(x-3)^2+(y+2)^2=25$		
8iii	A second circle, C_2, has equation $x^2+y^2-x+4y=12$.		
	Find the coordinates of the centre and the radius of C_2,		
	Centre = $\left(\frac{1}{2},-2\right)$		
	Radius = $\sqrt{\left(-\frac{1}{2}\right)^2+(2)^2-(-12)}$		

	= $\sqrt{\frac{65}{4}}$ units		
8iv	Explain if the circle C_2 lies entirely within the circle C_1 .		
	Both centre lie on the same horizontal line.		
	Radius is 5 units, so (**3–5**, –2).		
	(–2, –2) lies on the circumference of C_1		
	Similarly, $\left(\frac{1}{2}-\sqrt{\frac{65}{4}}, -2\right)$ lies on the circumference of C_2		
	Since $\frac{1}{2}-\sqrt{\frac{65}{4}}$ on the circumference of C_2 is out of circle C_1 , C_2 cannot lie entirely within C_1.		
9a	Use the substitution of $U = 3^x$ to solve the equation $2(3^{2x}) - 3^{x+1} = 5$.		
	$2(U^2) - 3U - 5 = 0$		
	$(U+1)(2U-5) = 0$		
	$U = \frac{5}{2}$ or $U = -1$ (no solution)		
	$3^x = \frac{5}{2}$		
	$x = 0.834$		
9b	Solve the equation $\log_2 x - \log_4 2 = \log_x 8$.		
	$\log_2 x - \frac{1}{\log_2 4} = \frac{\log_2 8}{\log_2 x}$		
	Let $X = \log_2 x$		
	$X - \frac{1}{2} = \frac{3}{X}$		
	$2X^2 - X - 6 = 0$		
	$(2X+3)(X-2) = 0$		
	$\log_2 x = -\frac{3}{2}$ or $\log_2 x = 2$		
	$x = 0.354$ or 4		

9c	Explain why $\lg(x+1)+\lg(x-2)=\lg(x-5)$ has no real solution.		
	$(x+1)(x-2)=(x-5)$		
	$x^2-2x+3=0$		
	Discriminant = –8 (<0)		
	Discriminant is negative. Hence, the equation has no solution.		