Check your Understanding (Integration)

Section 1: Integration Techniques

1.	**ACJC JC1 Promo 8865/2019/Q3**
	(i) Find $\int e^{2-3x}\ dx$. [2]
	(ii) Find $\int \frac{1}{\sqrt{3x+1}}\ dx$. [3]
	(iii) Without using a calculator, find $\int_1^2 \left(\frac{2}{\sqrt{x}}+2\sqrt{x}\right)^2\ dx$. [4]
	ACJC JC1 Promo 8865/2019/Q3 (Solutions)
	(i) $\int e^{2-3x}\ dx = -\frac{1}{3}e^{2-3x}+c$
	(ii) $\int \frac{1}{\sqrt{3x+1}}\ dx = \frac{2}{3}\sqrt{3x+1}+c$
	(iii) $\int_1^2 \left(\frac{2}{\sqrt{x}}+2\sqrt{x}\right)^2\ dx = \int_1^2 \left(\frac{4}{x}+8+4x\right)\ dx$ $=\left[4\ln x+8x+\frac{4x^2}{2}\right]_1^2$ $=\left[4\ln 2+8(2)+2(2)^2-4\ln 1-8-2\right]=4\ln 2+14$

2.	**CJC JC1 Promo 8865/2019/Q6**
	Find (a) $\int \frac{2}{(x-1)^4}\,dx$, [3] (b) $\int \left(\frac{5}{1+5x}+e^{1-2x}\right)dx$. [4]
	CJC JC1 Promo 8865/2019/Q6 (Solutions)
(a)	$\int \frac{2}{(x-1)^4}\,dx$ $=2\int (x-1)^{-4}\,dx$ $=2\left(\frac{(x-1)^{-3}}{-3}\right)+\text{c}$ $=-\frac{2}{3(x-1)^3}+\text{c}$

(b)	$\int\left(\frac{5}{1+5x}+\mathrm{e}^{1-2x}\right)\mathrm{d}x$ $=\frac{5\ln(1+5x)}{5}+\frac{\mathrm{e}^{1-2x}}{-2}+\mathrm{c}$ $=\ln(1+5x)-\frac{1}{2}\mathrm{e}^{1-2x}+\mathrm{c}$

3.	**SAJC JC2 Prelim 8865/2019/Q1**
	Find the exact value of a such that $$\int_{\frac{2}{3}}^{a}\frac{1}{3x-1}\,\mathrm{d}x=\int_0^4\frac{1}{\sqrt{x}}\,\mathrm{d}x.$$ [3]
	SAJC JC2 Prelim 8865/2019/Q1 (Solutions)
	$\int_{\frac{2}{3}}^{a}\frac{1}{3x-1}\,\mathrm{d}x=\int_0^4\frac{1}{\sqrt{x}}\,\mathrm{d}x$ $\left[\frac{\ln(3x-1)}{3}\right]_{\frac{2}{3}}^{a}=\left[2\sqrt{x}\right]_0^4$ $\frac{\ln(3a-1)}{3}-0=4$ $3a-1=e^{12}$ $a=\frac{e^{12}+1}{3}$

4.	**JPJC JC2 Prelim 8865/2019/Q2**
	(a) Find $\int\left(\frac{2}{\sqrt{x}}-1\right)^2\mathrm{d}x$. [3] (b) Find the exact value of n given that $\int_0^n\frac{1}{(7-2x)^2}\,\mathrm{d}x=\frac{1}{14}$, where $x\neq\frac{7}{2}$. [3]
	JPJC JC2 Prelim 8865/2019/Q2 (Solutions)
(a)	$\int\left(\frac{2}{\sqrt{x}}-1\right)^2\mathrm{d}x=\int\left(\frac{4}{x}-\frac{4}{\sqrt{x}}+1\right)\,\mathrm{d}x=\int\frac{4}{x}-4x^{-\frac{1}{2}}+1\,\,\mathrm{d}x=4\ln x-8x^{\frac{1}{2}}+x+c$
(b)	$\int_0^n\frac{1}{(7-2x)^2}\,\mathrm{d}x=\int_0^n(7-2x)^{-2}\,\mathrm{d}x$ $=\left[\frac{(7-2x)^{-1}}{(-1)(-2)}\right]_0^n=\frac{1}{2}\left[\frac{1}{7-2x}\right]_0^n=\frac{1}{2}\left[\frac{1}{7-2n}-\frac{1}{7}\right]$

	$\frac{1}{2}\left[\frac{1}{7-2n}-\frac{1}{7}\right]=\frac{1}{14}$ $\frac{1}{7-2n}=\frac{2}{7}$ $14-4n=7$ $4n=7$ $n=\frac{7}{4}$

5.	**RI JC1 Promo 8865/2019/Q5**
	(a) Find $\int\left(\frac{1}{\sqrt{x}}-\sqrt{2x+e^{-3x}}\right)\left(\frac{1}{\sqrt{x}}+\sqrt{2x+e^{-3x}}\right)\mathrm{d}x$. [4] (b) Show that $\frac{2x}{\sqrt{2x+3}}=\sqrt{2x+3}-\frac{k}{\sqrt{2x+3}}$, where k is a positive integer to be determined. [1] Hence evaluate $\int_{-1}^{1}\frac{x}{\sqrt{2x+3}}\,\mathrm{d}x$, giving your answer in the form $A+B\sqrt{5}$, where A and B are constants to be determined. [4]
	RI JC1 Promo 8865/2019/Q5 (Solutions)
(a)	$\int\left(\frac{1}{\sqrt{x}}-\sqrt{2x+e^{-3x}}\right)\left(\frac{1}{\sqrt{x}}+\sqrt{2x+e^{-3x}}\right)\mathrm{d}x$ $=\int\frac{1}{x}-(2x+e^{-3x})\,\mathrm{d}x=\ln\lvert x\rvert-x^2+\frac{1}{3}e^{-3x}+C$
(b)	$\frac{2x}{\sqrt{2x+3}}=\frac{(2x+3)-3}{\sqrt{2x+3}}=\sqrt{2x+3}-\frac{3}{\sqrt{2x+3}}$ (shown) $\therefore\int_{-1}^{1}\frac{x}{\sqrt{2x+3}}\,\mathrm{d}x$ $=\frac{1}{2}\int_{-1}^{1}\sqrt{2x+3}-\frac{3}{\sqrt{2x+3}}\,\mathrm{d}x=\frac{1}{2}\left[\frac{(2x+3)^{\frac{3}{2}}}{\frac{3}{2}\times 2}-\frac{3(2x+3)^{\frac{1}{2}}}{\frac{1}{2}\times 2}\right]_{-1}^{1}$ $=\frac{1}{2}\left[\frac{(2x+3)^{\frac{3}{2}}}{3}-3(2x+3)^{\frac{1}{2}}\right]_{-1}^{1}=\frac{1}{2}\left[\frac{1}{3}(5)^{\frac{3}{2}}-3(5)^{\frac{1}{2}}-\frac{1}{3}+3\right]$

	$=\frac{1}{2}\left[\frac{5}{3}\sqrt{5}-3\sqrt{5}+\frac{8}{3}\right]=\frac{4}{3}-\frac{2}{3}\sqrt{5}$ where $A=\frac{4}{3}$, $B=-\frac{2}{3}$

6.	**NJC JC1 Promo 8865/2019/Q6(a)(b)**	
	(a) Show that $\int_{-1}^{0}\left(\mathrm{e}^{-x}+\sqrt{x}\right)\left(\mathrm{e}^{-x}-\sqrt{x}\right)\mathrm{d}x=\frac{\mathrm{e}^{2}}{2}$.	[2]
	(b) If $\int_{0}^{n}\frac{1}{(1-4x)^{2}}\,\mathrm{d}x=-\frac{1}{2}$, determine the value of n.	[3]

	NJC JC1 Promo 8865/2019/Q6 (Solutions)
(a)	$\int_{-1}^{0}\left(\mathrm{e}^{-x}+\sqrt{x}\right)\left(\mathrm{e}^{-x}-\sqrt{x}\right)\mathrm{d}x$ $=\int_{-1}^{0}\left(\mathrm{e}^{-2x}-x\right)\mathrm{d}x$ $=\left[\frac{\mathrm{e}^{-2x}}{-2}-\frac{x^{2}}{2}\right]_{-1}^{0}$ $=-\frac{1}{2}+\frac{\mathrm{e}^{2}}{2}+\frac{1}{2}$ $=\frac{\mathrm{e}^{2}}{2}$
(b)	$\int_{0}^{n}\frac{1}{(1-4x)^{2}}\,\mathrm{d}x$ $=\int_{0}^{n}(1-4x)^{-2}\,\mathrm{d}x$ $=\left[\frac{(1-4x)^{-1}}{(-1)(-4)}\right]_{0}^{n}$ $=\frac{1}{4(1-4n)}-\frac{1}{4}$ $\frac{1}{4(1-4n)}-\frac{1}{4}=-\frac{1}{2}$ $\frac{1}{4(1-4n)}=\frac{1}{4}-\frac{1}{2}$ $\frac{1}{4(1-4n)}=-\frac{1}{4}$ $(4n-1)=1$ $n=\frac{1}{2}$

Section 2: Reverse process of Differentiation is Integration

7.	**NJC JC1 Promo 8865/2019/Q6(c)**
	(c) Differentiate $\mathrm{e}^{\mathrm{e}x^2+x}$ with respect to x. Hence, find $\int 6x\mathrm{e}^{\mathrm{e}x^2+x+1}+3\mathrm{e}^{\mathrm{e}x^2+x}\ \mathrm{d}x$. [4]

	NJC JC1 Promo 8865/2019/Q6 (Solutions)
(c)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\mathrm{e}x^2+x}\right)=\left(2\mathrm{e}x+1\right)\mathrm{e}^{\mathrm{e}x^2+x}=2x\mathrm{e}^{\mathrm{e}x^2+x+1}+\mathrm{e}^{\mathrm{e}x^2+x}$ $\Rightarrow\int 2x\mathrm{e}^{\mathrm{e}x^2+x+1}+\mathrm{e}^{\mathrm{e}x^2+x}\ \mathrm{d}x=\mathrm{e}^{\mathrm{e}x^2+x}+C_1$ $\int 6x\mathrm{e}^{\mathrm{e}x^2+x+1}+3\mathrm{e}^{\mathrm{e}x^2+x}\ \mathrm{d}x$ $=3\int 2x\mathrm{e}^{\mathrm{e}x^2+x}+\mathrm{e}^{\mathrm{e}x^2+x}\ \mathrm{d}x$ $=3\mathrm{e}^{\mathrm{e}x^2+x}+C_2$

8.	**CJC JC2 Prelim 8865/2019/Q3(b)**
	(b) (i) By first expressing $\frac{2x+3}{x+4}$ in the form $A+\frac{B}{x+4}$ where A and B are constants, find $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x+3}{x+4}\right)$. [2] (ii) Hence find the exact value of $\int_{-1}^{5}\left(\frac{1}{(x+4)^2}+\frac{1}{2(x+4)}\right)\mathrm{d}x$. [4]

	CJC JC2 Prelim 8865/2019/Q3(b) (Solutions)
(b)(i)	$\frac{2x+3}{x+4}=A+\frac{B}{x+4}$ $2x+3=A(x+4)+B$ Method 1: Comparing coefficients, $A=2$ $4A+B=3\Rightarrow B=3-4(2)=-5$ $\therefore\frac{2x+3}{x+4}=2-\frac{5}{x+4}$ Method 2: $x+4\overline{\smash{)}2x+3}$ quotient 2 $\underline{2x+8}$ -5 $\therefore\frac{2x+3}{x+4}=2-\frac{5}{x+4}$

	$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x+3}{x+4}\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(2-\frac{5}{x+4}\right)$$ $$=\frac{5}{(x+4)^2}$$
(b)(ii)	$$\int_{-1}^{5}\left(\frac{1}{(x+4)^2}+\frac{1}{2(x+4)}\right)\mathrm{d}x$$ $$=\left[\frac{1}{5}\left(\frac{2x+3}{x+4}\right)+\frac{1}{2}\ln(x+4)\right]_{-1}^{5}$$ $$=\frac{1}{5}\left(\frac{13}{9}\right)+\frac{1}{2}\ln 9-\frac{1}{5}\left(\frac{1}{3}\right)-\frac{1}{2}\ln 3$$ $$=\frac{2}{9}+\frac{1}{2}\ln\frac{9}{3}$$ $$=\frac{2}{9}+\frac{1}{2}\ln 3$$

9.	**NYJC JC1 Promo 8865/2019/Q4**	
	(a) Differentiate $\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}-\frac{1}{3x}+\ln 2$ with respect to x.	[3]
	(b) Differentiate $(\ln 2x)^4$ with respect to x. Hence, find $\int \frac{x\mathrm{e}^{2x}+6(\ln 2x)^3}{x}\mathrm{d}x$.	[4]
	(c) Given that $\int_1^k \frac{5x^2-4}{\sqrt{x}}\,\mathrm{d}x=54$, find the value of k, where k is a real number.	[5]

	NYJC JC1 Promo 8865/2019/Q4 (Solutions)
(a)	$$\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}-\frac{1}{3x}+\ln 2=\mathrm{e}^{(2-x)^{-\frac{1}{2}}}-\frac{1}{3}x^{-1}+\ln 2$$ $$\frac{\mathrm{d}\left(\mathrm{e}^{(2-x)^{-\frac{1}{2}}}-\frac{1}{3}x^{-1}+\ln 2\right)}{\mathrm{d}x}$$ $$=-\frac{1}{2}(2-x)^{-\frac{3}{2}}(-1)\mathrm{e}^{(2-x)^{-\frac{1}{2}}}-\frac{1}{3}(-1)x^{-2}$$ $$=\frac{\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}}{2(2-x)^{\frac{3}{2}}}+\frac{1}{3x^2}$$
(b)	$$\frac{\mathrm{d}(\ln 2x)^4}{\mathrm{d}x}=4(\ln 2x)^3\frac{1}{2x}(2)$$ $$=\frac{4(\ln 2x)^3}{x}$$

	$\int \frac{xe^{2x}+6(\ln 2x)^3}{x}\mathrm{d}x = \int \mathrm{e}^{2x}+\frac{6(\ln 2x)^3}{x}\mathrm{d}x$ $=\int \mathrm{e}^{2x}\mathrm{d}x+\frac{6}{4}\int \frac{4(\ln 2x)^3}{x}\mathrm{d}x$ $=\frac{1}{2}\mathrm{e}^{2x}+\frac{3}{2}(\ln 2x)^4+C$
(c)	$\int_1^k \frac{5x^2-4}{\sqrt{x}}\ \mathrm{d}x=54$ $\int_1^k 5x^{\frac{3}{2}}-4x^{-\frac{1}{2}}\ \mathrm{d}x=54$ $\left[2x^{\frac{5}{2}}-8x^{\frac{1}{2}}\right]_1^k=54$ $\left(2k^{\frac{5}{2}}-8k^{\frac{1}{2}}\right)-(-6)=54$ $2k^{\frac{5}{2}}-8k^{\frac{1}{2}}=48$ $k^{\frac{5}{2}}-4k^{\frac{1}{2}}=24$ Let $y=k^{\frac{1}{2}}$ $y^5-4y=24$ Using G.C. $y=k^{\frac{1}{2}}=2$ $k=4$

10.	**DHS JC2 Prelim 8865/2019/Q2**	
	(i) Differentiate $\ln\left(\frac{\mathrm{e}}{\sqrt{1+\mathrm{e}^{2x}}}\right)$.	[2]
	(ii) Hence find the exact value of the constant a such that $\int_0^a \frac{1}{1+\mathrm{e}^{-2x}}\,\mathrm{d}x=\frac{1}{2}$.	[4]
	DHS JC2 Prelim 8865/2019/Q2 (Solutions)	
(i)	$y=\ln\left(\frac{\mathrm{e}}{\sqrt{1+\mathrm{e}^{2x}}}\right)$ $=\ln \mathrm{e}-\ln\sqrt{1+\mathrm{e}^{2x}}$ $=1-\frac{1}{2}\ln\left(1+\mathrm{e}^{2x}\right)$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2}\left(\frac{2\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\right)=\frac{-\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}$	

(ii)	$\int_0^a \frac{1}{1+e^{-2x}}\,dx$ $= -\int_0^a \frac{-e^{2x}}{1+e^{2x}}\,dx$ $= -\left[\ln \frac{e}{\sqrt{1+e^{2x}}}\right]_0^a$ $= -\left[\ln \frac{e}{\sqrt{1+e^{2a}}} - \ln \frac{e}{\sqrt{2}}\right]$ $= \ln \frac{e}{\sqrt{2}} - \ln \frac{e}{\sqrt{1+e^{2a}}}$ $= \ln \sqrt{\frac{1+e^{2a}}{2}}$ $\ln \sqrt{\frac{1+e^{2a}}{2}} = \frac{1}{2}$ $\ln\left(\frac{1+e^{2a}}{2}\right) = 1$ $\frac{1+e^{2a}}{2} = e$ $e^{2a} = 2e-1$ $a = \frac{1}{2}\ln(2e-1)$

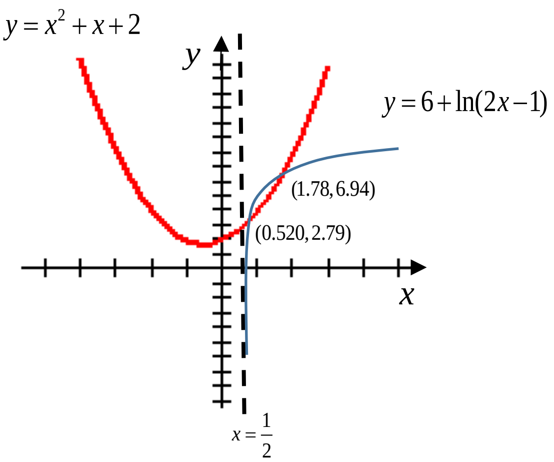
11.	**EJC JC1 Promo 8865/2019/Q6part**	
	(c) If $y = \ln(e^{4x}+3)^6$, find $\frac{dy}{dx}$. Hence evaluate $\int_0^1 \frac{3e^{4x}}{e^{4x}+3}\,dx$ in terms of e.	[5]
	EJC JC1 Promo 8865/2019/Q6 (Solutions)	
(c)	$y = \ln(e^{4x}+3)^6$ $\Rightarrow y = 6\ln(e^{4x}+3)$ $\frac{dy}{dx} = \left(\frac{6}{e^{4x}+3}\right)(4e^{4x}) = \frac{24e^{4x}}{e^{4x}+3}$ $\int_0^1 \frac{3e^{4x}}{e^{4x}+3}\,dx$ $= \frac{1}{8}\int_0^1 \frac{24e^{4x}}{e^{4x}+3}\,dx$ $= \frac{1}{8}\left[\ln(e^{4x}+3)^6\right]_0^1$ $\quad\quad$ $= \frac{1}{8}\left[6\ln(e^{4x}+3)^6\right]_0^1$ $= \frac{1}{8}\left[\ln(e^4+3)^6 - \ln(4)^6\right]$ $\quad$ or $\quad$ $= \frac{3}{4}\left[\ln(e^4+3) - \ln(4)\right]$ $= \frac{1}{8}\left(\ln(\frac{e^4+3}{4})^6\right)$ $\quad\quad$ $= \frac{3}{4}\left(\ln(\frac{e^4+3}{4})\right)$	

12.	TMJC JC2 Prelim 8865/2019/Q2
	(i) Differentiate $\ln\left(x^2+4x+5\right)$ with respect to x. [2]
	(ii) Hence find $\int\left(\frac{x+2}{x^2+4x+5}-3\sqrt{x}\right)\mathrm{d}x$. [3]

	TMJC JC2 Prelim 8865/2019/Q2 (Solutions)
(i)	$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x^2+4x+5\right)=\frac{2x+4}{x^2+4x+5}$
(ii)	$\int\left(\frac{x+2}{x^2+4x+5}-3\sqrt{x}\right)\mathrm{d}x$ $=\frac{1}{2}\int\frac{2(x+2)}{x^2+4x+5}\,\mathrm{d}x-\int 3\sqrt{x}\,\mathrm{d}x$ $=\frac{1}{2}\ln\left(x^2+4x+5\right)-\frac{3x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}+C$ $=\frac{1}{2}\ln\left(x^2+4x+5\right)-2x^{\frac{3}{2}}+C$

Section 3: Application of Integration – Area

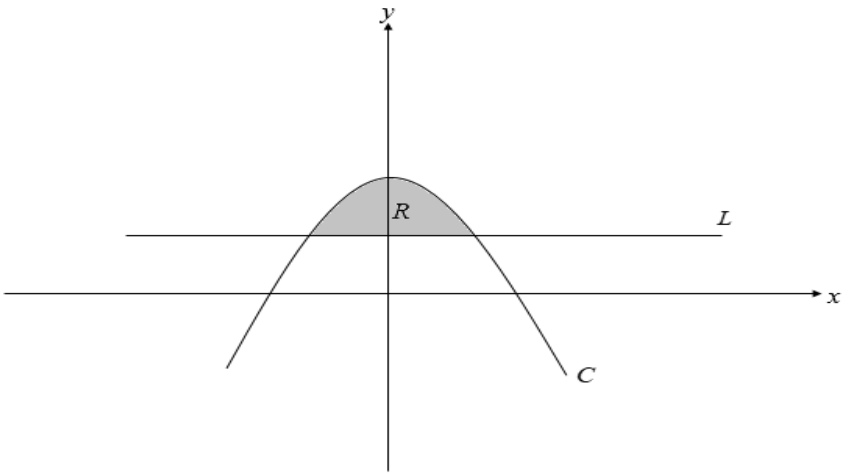
13.	HCI JC2 Prelim 8865/2019/Q1
	On the same axes, sketch the curves with the equations $y=x^2+x+2$ and $y=6+\ln(2x-1)$. State clearly the equations of any asymptotes and the coordinates of the points where the two curves intersect. Find the area of the region bounded by the two curves, giving your answer correct to 3 decimal places. [5]

	HCI JC2 Prelim 8865/2019/Q1 (Solutions)
	 When $x=0.52019$, $y=2.79$ When $x=1.77778$, $y=6.93827$

	Area $=\int_{0.52019}^{1.78788} 6+\ln(2x-1)-(x^2+x+2)\ \mathrm{d}x$ $=1.7655$ $\approx 1.766\ \text{units}^2$ (to 3 d.p.)

14. RVHS JC1 Promo 8865/2019/Q8

The diagram below shows the curve C with equation of $y=k^4-x^2$ and the line L with equation $y=\dfrac{15}{16}k^4$, where k is a positive constant. P and Q are the points of intersection where C and L intersect.



(i) Find the x-coordinates of P and Q in terms of k. [2]

(ii) Show that the area of shaded region R can be expressed as ck^6, where c is a constant to be determined. [5]

(iii) Suppose that the area of R is $\dfrac{4}{3}$ units^2, find the value of k and hence state the equation of the line L. [3]

RVHS JC1 Promo 8865/2019/Q8 (Solutions)

(i)	To find intersection of the curves: $\dfrac{15}{16}k^4=k^4-x^2$ $x^2=\dfrac{1}{16}k^4$ $x=\pm\dfrac{1}{4}k^2$
(ii)	Area of shaded region R $=2\int_0^{\frac{k^2}{4}} k^4-x^2\ \mathrm{d}x-2\left(\dfrac{k^2}{4}\right)\left(\dfrac{15}{16}k^4\right)$

	$=2\left[k^4x-\dfrac{x^3}{3}\right]_0^{\frac{k^2}{4}}-\dfrac{15}{32}k^6$ $=2\left[k^4\left(\dfrac{k^2}{4}\right)-\dfrac{1}{3}\left(\dfrac{k^2}{4}\right)^3\right]-\dfrac{15}{32}k^6$ $=\dfrac{1}{48}k^6$ Thus $c=\dfrac{1}{48}$
(iii)	$\dfrac{1}{48}k^6=\dfrac{4}{3}$ $k^6=64$ $k=2$ Therefore equation of line L is $y=\dfrac{15}{16}\left(2^4\right)=15$

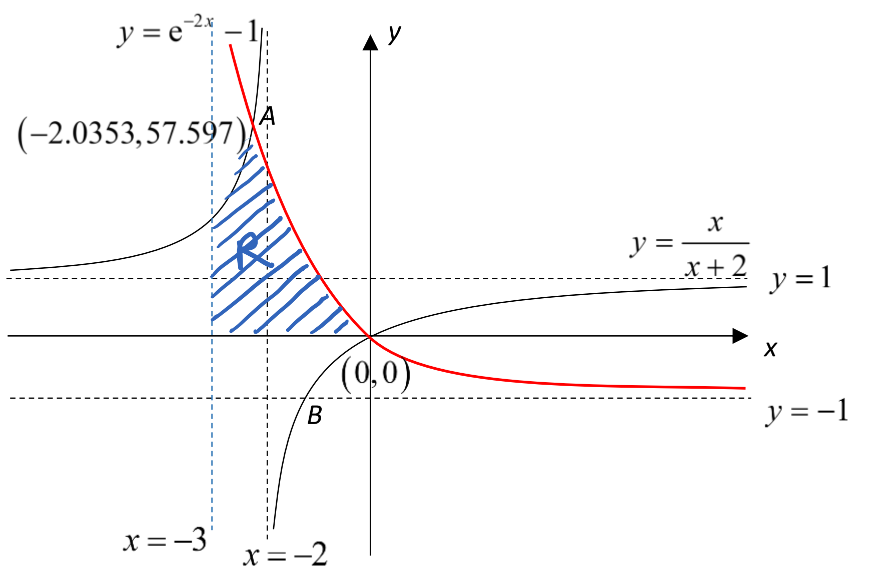
15. NYJC JC2 Prelim 8865/2019/Q4

On the same diagram, sketch the graphs of $y=\mathrm{e}^{-2x}-1$, and $y=\dfrac{x}{2+x}$, stating the coordinates of any points of intersection with the axes and the equations of any asymptotes. [4]

R is the region bounded by the two curves, the x-axis and line $x=-3$. Find the numerical value of the area of the region R. [4]

S is another region bounded by the two curves and the line $x=k$, where k is a positive integer, such that the area of region S is less than 75% of the area of the region R. Find the largest value of k, showing your workings clearly. [3]

NYJC JC2 Prelim 8865/2019/Q4 (Solutions)



Let A denote the point of intersection between the curves and $y=\mathrm{e}^{-2x}-1$.

At $A(-2.0353, 57.597)$

Area of region R = $\int_{-3}^{-2.0353} \frac{x}{x+2}\,dx + \int_{-2.0353}^{0} \left(e^{-2x} - 1\right) dx$

= 34.413

= 34.4 sq units

Area of region S = $\int_0^k \frac{x}{x+2} - \left(e^{-2x} - 1\right) dx < 25.80975$

Using GC,

k	$\int_0^k \frac{x}{x+2} - \left(e^{-2x} - 1\right) dx$
15	25.220
16	27.106

Hence the largest value of k is 15.

<u>Alternative method</u>

$\int_0^k \frac{x}{x+2} - \left(e^{-2x} - 1\right) dx < 25.80975$

$\int_0^k 1 - \frac{2}{x+2} - e^{-2x} + 1\,dx < 25.80975$

$\int_0^k 2 - \frac{2}{x+2} - e^{-2x}\,dx < 25.80975$

$\left[2x - 2\ln(x+2) + \frac{e^{-2x}}{2}\right]_0^k < 25.80975$

$2k - 2\ln(k+2) + \frac{e^{-2k}}{2} + 2\ln 2 - \frac{1}{2} < 25.80975$

Using GC,

k	$2k - 2\ln(k+2) + \frac{e^{-2k}}{2} + 2\ln 2 - \frac{1}{2}$
15	25.220
16	27.106

Hence the largest value of k is 15.

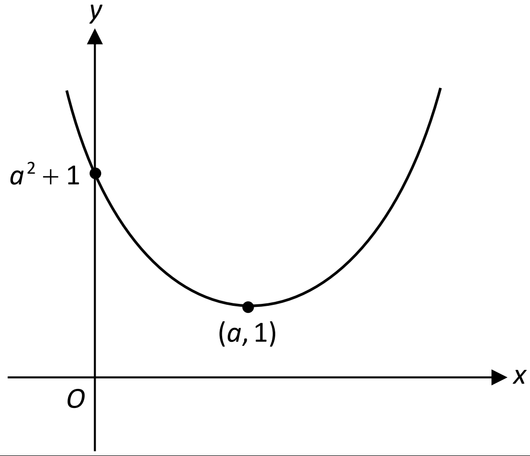
16. RI JC1 Promo 8865/2019/Q6

The curve C has equation $y = (x - a)^2 + 1$, where a is a positive constant.

(i) Sketch the graph of C, stating clearly, in terms of a, the axial intercepts and the coordinates of the turning point. [3]

(ii) Find, in terms of a, the area under the curve C between $x = 0$ and $x = a$. [3]

(iii) Hence, or otherwise, find, in terms of a, the area of the finite region bounded by the curve C and the line $y = a^2 + 1$. [3]

RI JC1 Promo 8865/2019/Q6 (Solutions)	
(i)	
(ii)	Area $= \int_0^a (x-a)^2 + 1 \, \mathrm{d}x = \left[\frac{(x-a)^3}{3} + x\right]_0^a = a + \frac{1}{3}a^3$
(iii)	Area $= 2 \times \left[a(a^2+1) - \left(a + \frac{1}{3}a^3\right)\right] = \frac{4}{3}a^3$

17. EJC JC2 Prelim 8865/2019/Q4

A curve C has equation $y = \frac{1}{x-9} + \ln(x-9)$.

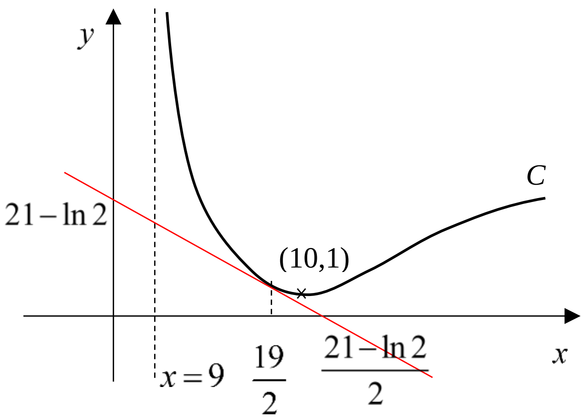
(i) Sketch the graph of C, stating the coordinates of any stationary point(s) and the equation of any asymptote(s). [3]

(ii) Find the exact equation of the tangent to C at the point where $x = \frac{19}{2}$, and sketch this tangent on the same diagram as (i). [5]

(iii) Hence, find the area of the region bound by C, the tangent at $x = \frac{19}{2}$, and the line $x = 10$. [2]

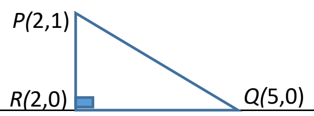
EJC JC2 Prelim 8865/2019/Q4 (Solutions)

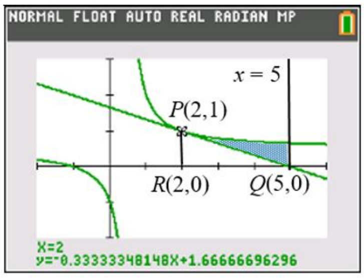
(i)



(ii)	$y=\dfrac{1}{x-9}+\ln(x-9)$ $\dfrac{dy}{dx}=-\dfrac{1}{(x-9)^2}+\dfrac{1}{(x-9)}$ At $x=\dfrac{19}{2}, y=2-\ln 2$ and $\dfrac{dy}{dx}=-2$ Equation of tangent: $y-(2-\ln 2)=-2\left(x-\dfrac{19}{2}\right)$ $y=-2x+21-\ln 2$
(iii)	Area $=\int_{\frac{19}{2}}^{10}\dfrac{1}{x-9}+\ln(x-9)-(-2x+21-\ln 2)\ dx = 0.136$ unit^2

18.	**ACJC JC2 Prelim 8865/2019/Q4**
(i)	Differentiate $A+\dfrac{B}{2x-1}$ with respect to x, where A and B are constants. [1]
(ii)	The curve C has equation $y=\dfrac{x+1}{2x-1}$. By substituting appropriate values of A and B into your answer in (i), or otherwise, show that the equation of the tangent to C at the point P where $x=2$ is given by $3y+x=5$. [3]
(iii)	The tangent to C at P cuts the x-axis at Q. Find the coordinates of Q. [1]
(iv)	The point R has coordinates $(2,0)$. Find the area of ΔPQR. [1]
(v)	Hence, or otherwise, find the exact area of the region bounded by C, the tangent in (ii) and the line $x=5$, simplifying your answer. [4]

	ACJC JC2 Prelim 8865/2019/Q4 (Solutions)
(i)	$\dfrac{d}{dx}\left(A+\dfrac{B}{2x-1}\right)=\dfrac{-2B}{(2x-1)^2}$
(ii)	$y=\dfrac{x+1}{2x-1}=\dfrac{1}{2}+\dfrac{3/2}{2x-1}\Rightarrow A=\dfrac{1}{2}\ \&\ B=\dfrac{3}{2}\Rightarrow\dfrac{dy}{dx}=\dfrac{-3}{(2x-1)^2}$ $\left(\text{or Quotient Rule: }\dfrac{dy}{dx}=\dfrac{(2x-1)(1)-(x+1)(2)}{(2x-1)^2}=\dfrac{-3}{(2x-1)^2}\right)$ when $x=2, \dfrac{dy}{dx}=\dfrac{-3}{(2(2)-1)^2}=-\dfrac{1}{3}\ \&\ y=\dfrac{2+1}{2(2)-1}=1$ Equation of tangent: $y-1=-\dfrac{1}{3}(x-2)\Rightarrow -3y+3=x-2\Rightarrow 3y+x=5$ (shown)
(iii)	when $y=0, x=5\Rightarrow Q(5,0)$
(iv)	Area of $\Delta PQR=\dfrac{1}{2}(3)(1)=\dfrac{3}{2}$ units^2 

(v)	$\text{Area} = \int_2^5 y_C \, dx - \frac{3}{2}$ $= \int_2^5 \frac{1}{2} + \frac{3/2}{2x-1} \, dx - \frac{3}{2}$ $= \left[\frac{1}{2}x + \frac{3}{4}\ln(2x-1)\right]_2^5 - \frac{3}{2}$ $= \left[\frac{5}{2} + \frac{3}{4}\ln 9 - 1 - \frac{3}{4}\ln 3\right] - \frac{3}{2} = \frac{3}{4}\ln 3 \text{ units}^2$ 

Section 4: Application Questions

19.	**DHS JC1 Promo 8865/2019/Q3**
	Find $\frac{d}{dx}\left[\ln\left(\frac{e}{1-2x^3}\right)\right]$. The gradient function of a curve is $x - \frac{6x^2}{1-2x^3}$. Given that the y-intercept for the curve is 5, find the equation of the curve. [5]
	DHS JC1 Promo 8865/2019/Q3 (Solutions)
	Let $y = \ln\left(\frac{e}{1-2x^3}\right) = \ln e - \ln\left(1-2x^3\right) = 1 - \ln\left(1-2x^3\right)$ $\frac{dy}{dx} = \frac{6x^2}{1-2x^3}$ $y = \int x - \frac{6x^2}{1-2x^3} dx$ $= \frac{x^2}{2} - \ln\left(\frac{e}{1-2x^3}\right) + c ----(1)$ Sub. $x=0, y=5$ into (1) $-\ln e + c = 5$ $c = 6$ $\therefore\ y = \frac{x^2}{2} - \ln\left(\frac{e}{1-2x^3}\right) + 6$

20.	**RVHS JC1 Promo 8865/2019/Q2**
	The gradient function of a curve is $\sqrt{8-ax}$, where a is a constant. Given that the curve has a stationary point at $(12, 3)$, find the equation of the curve. [6]
	RVHS JC1 Promo 8865/2019/Q2 (Solutions)
	Given $\frac{dy}{dx} = \sqrt{8-ax}$ $\frac{dy}{dx} = 0$ when $x = 12$: $\sqrt{8-a(12)} = 0$ $8 - a(12) = 0$

	$12a = 8$ $a = \frac{2}{3}$ Finding equation of the curve: $y = \int \sqrt{8 - \frac{2}{3}x}\ \mathrm{d}x$ $y = \frac{\left(8 - \frac{2}{3}x\right)^{\frac{3}{2}}}{\frac{3}{2}\left(-\frac{2}{3}\right)} + c$ $y = -\left(8 - \frac{2}{3}x\right)^{\frac{3}{2}} + c$ When $x = 12$, $y = 3$ $\therefore 3 = -\left(8 - \frac{2}{3}(12)\right)^{\frac{3}{2}} + c$ $c = 3$ Equation of the curve: $y = -\left(8 - \frac{3}{2}x\right)^{\frac{3}{2}} + 3$

21. MI PU2 Prelim 8865/2019/Q6

A new brand of Bubble Tea, MaxTea entered the retail market in Singapore. The rate, M hundred dollars per month, at which the revenue changes is tracked regularly over a period of t months.

The initial rate of change of revenue was recorded as 288 hundred dollars per month. With advertising efforts, the maximum value of M was 324 when $t = 6$. The manager of MaxTea believes that the connection between M and t can be modelled by the quadratic equation

$$M = at^2 + bt + c,$$

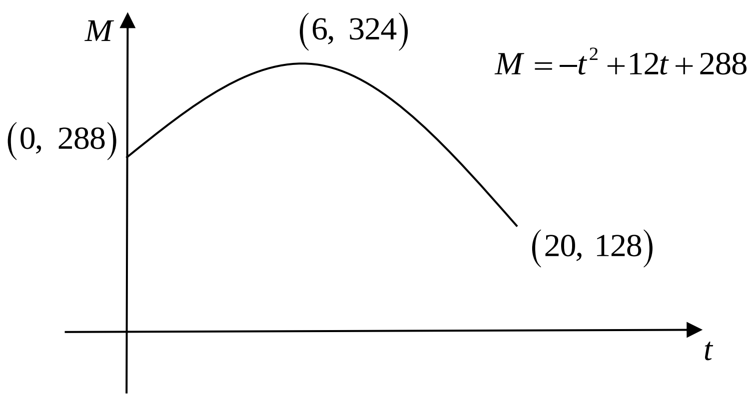
where a, b and c are constants.

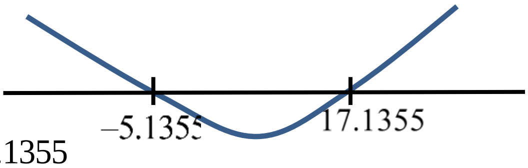
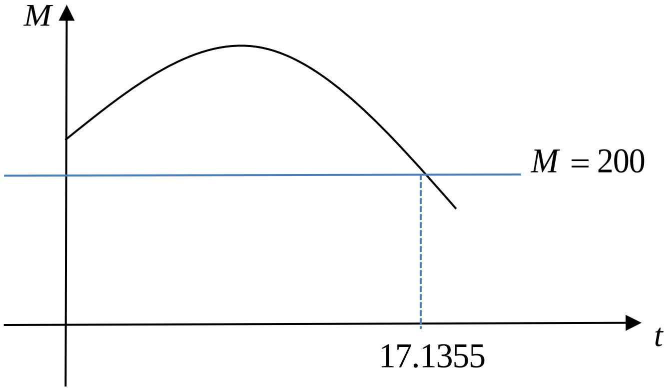
(i) Find the values of a, b and c. [4]

(ii) Sketch the graph of M against t for $0 \le t \le 20$, stating clearly the coordinates of any turning point. [2]

(iii) Write down as an integral an expression for the area of the region bounded by the curve, the t-axis and the lines $t = 0$ and $t = 12$. Evaluate this integral and give an interpretation of the area that you have found, in the context of the question. [3]

(iv) The manager of MaxTea would provide discount coupons for the products over a period of time if the rate of change of revenue is less than \$20000 per month. By writing an inequality involving t, find the least number of months it takes for the manager to do so. Give your answer correct to the nearest integer. [2]

	(v) Comment on the suitability of the quadratic model. [1]
	MI PU2 Prelim 8865/2019/Q6 (Solutions)
(i)	Let $M = at^2 + bt + c$ When $t = 0, M = 288$ $288 = c$ ---- (1) When $t = 6, M = 324$ $324 = 36a + 6b + c$ ---- (2) Sub (1) into (2): $324 = 36a + 6b + 288$ $36a + 6b = 36$ $6a + b = 6$ ---- (3) $\frac{\mathrm{d}M}{\mathrm{d}t} = 2at + b$ When $t = 6$, $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$ (Since it is a maximum value (stationary value)) $\frac{\mathrm{d}M}{\mathrm{d}t} = 2a(6) + b = 0$ $b = -12a$ ---- (4) Sub (4) into (3): $6a + (-12a) = 6$ $a = -1$ $b = 12$ Hence, $M = -t^2 + 12t + 288$.
(ii)	
(iii)	Area $= \int_0^{12} -t^2 + 12t + 288 \, \mathrm{d}t$

	$= \left[-\frac{t^3}{3} + 12\left(\frac{t^2}{2}\right) + 288t \right]_0^{12}$ $= \left(-\frac{12^3}{3} + 12\left(\frac{12^2}{2}\right) + 288(12) \right) - 0$ $= 3744$ (or use GC directly) The **total revenue** is **\$374400** over the first **12 months** (or **1 year**) of tracking.
(iv)	**<u>Method 1</u>** $-t^2 + 12t + 288 < 200$ $-t^2 + 12t + 88 < 0$ $t^2 - 12t - 88 > 0$ $t < -5.1355$ or $t > 17.1355$ (NA)  Least $t = 18$ **<u>Method 2</u>**  $-t^2 + 12t + 288 < 200$ $17.1355 < t \le 20$ Least $t = 18$
(v)	The quadratic model is **suitable** as it is expected that the rate of change of revenue of the MaxTea to decrease over a period of time due to a decrease in the interest of the tea/drinks. Alternatively: The model is **not suitable** in the long run as it means that the rate of change of revenue will be negative after some time which is not possible for the revenue to be negative.

22.	**NJC Prelim 8865/2018/Q5**
	A company produces three flavours of chocolate bars: Milk, Dark and White. Each chocolate bar weighs 100 grams. The manufacturing cost of 500 grams of Milk chocolate bars is the same as the manufacturing cost of a Dark Chocolate bar. The total manufacturing cost of 10 Dark chocolate

bars is \$16.70 more than the manufacturing cost of 8 White chocolate bars. The total manufacturing cost of 6 Milk, 5 Dark Chocolate and 3 White chocolate bars is \$25.08.

(i) By writing down three linear equations, find the manufacturing cost of each flavour of chocolate bars. [3]

The company is trialling a new flavour of chocolate bar, Caramel. A financial consultant for the company predicts that the profit \$$P$, generated by selling a Caramel chocolate bar, will be related to the manufacturing cost (\$$x$) by the equation

$$P = 6(3x)^{0.4} - 0.01x^4 - 8, \quad x \geq 0.$$

(ii) Sketch the graph of P against x, stating the coordinates of the points where the graph crosses the x- and P-axes. Find the maximum value of P. [3]

(iii) Find the value of $\frac{\mathrm{d}P}{\mathrm{d}x}$ at $x = 2.3$ and give an interpretation of the value found in the context of the question. [2]

(iv) Given that the manufacturing cost of a Caramel chocolate bar is \$2.30, state the selling price. $\left[\text{Profit} = \text{selling price} - \text{manufacturing cost}\right]$ [1]

(v) If the manufacturing cost of a Caramel chocolate bar is increased to \$6, would you advise the company to produce chocolate bar of this flavour? Justify your answer. [1]

The financial consultant predicts that the marginal cost of manufacturing a Caramel chocolate bar, C (in thousands) is related to the quantity manufactured, q (in thousands) by the equation

$$C = \frac{1}{8}q^2(q-4) + 3, \quad q > 0.$$

[Marginal cost = additional cost incurred in the production of one more unit of a good or service]

(vi) Use differentiation to find the minimum value of C, justifying that the value is minimum. Give an interpretation of the value found in the context of the question. [4]

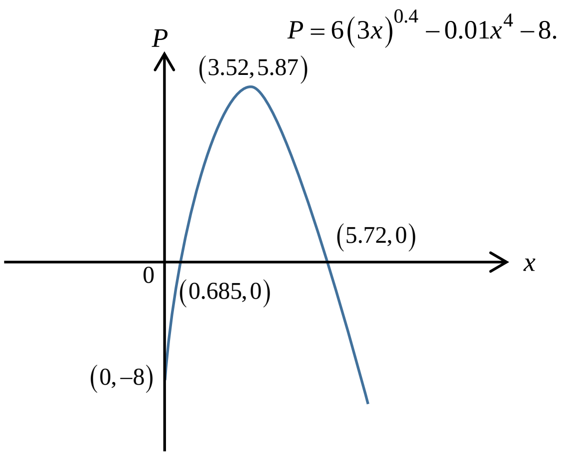
(vii) State the area bounded by the curve C, the line $q = 1$ and the axes. Give an interpretation of the value found in the context of the question. [2]

Answers

(i) $M = 0.63$, $D = 3.15$, $W = 1.85$; (ii) 5.87; (iii) 1.77; (iv) \$7.01; (vi) 1.81; (vii) 2.86

NJC Prelim 8865/2018/Q5 (Solutions)

(i)	Let M, D and W be the manufacturing cost (in \$) of a 100 gram Milk, Dark and White chocolate bar respectively.

	500 grams of Milk chocolate bars is equivalent to 5 Milk chocolate bars. $5M = D$ $10D - 8W = 16.7$ $6M + 5D + 3W = 25.08$ or $5M - D + 0W = 0$ $0M + 10D - 8W = 16.7$ $6M + 5D + 3W = 25.08$ By GC: $M = 0.63$ $D = 3.15$ $W = 1.85$
(ii)	 Maximum value of P is 5.87 (2.d.p.).
(iii)	By GC, $\frac{\mathrm{d}P}{\mathrm{d}x} = 1.77(2.\mathrm{d.p})$ at $x = 2.3$. When manufacturing cost to produce a Caramel chocolate bar is \$2.30, every dollar spent on manufacturing cost will generate a profit of \$1.77.
(iv)	$\text{Profit} = 6[3(2.3)]^{0.4} - 0.01(2.3)^4 - 8 = \$4.7126$ $\text{Selling price} = 4.7126 + 2.3 = \$7.01(2.\text{d.p.})$
(v)	When $x = 6$, $P = -1.89397$. This suggests that when there is no manufacturing cost, the company will make a loss at \$1.89 dollars per Caramel chocolate bar, which is not advisable. Or $\frac{\mathrm{d}P}{\mathrm{d}x}$ at $x = 6$ is $-7.37(2.\text{d.p.})$ which suggest that the company will make a loss of \$7.37 for every dollar spent when the manufacturing cost to produce a Caramel chocolate bar is at \$6.

(vi)

$C=\frac{1}{8}q^2(q-4)+3=\frac{1}{8}q^3-\frac{1}{2}q^2+3$

$\frac{\mathrm{d}C}{\mathrm{d}q}=\frac{3}{8}q^2-q=q\left(\frac{3}{8}q-1\right)$

To find the minimum value of C as q varies,

$\frac{\mathrm{d}C}{\mathrm{d}q}=q\left(\frac{3}{8}q-1\right)=0$

$q=0$ or $C=\frac{3}{8}q-1=0\Rightarrow q=\frac{8}{3}$

Reject $q=0$ since $q>0$

To calculate minimum value of C,

$C=\frac{1}{8}\left(\frac{8}{3}\right)^2\left(\frac{8}{3}-4\right)+3=\frac{49}{27}=1.81(2.\text{d.p.})$

To show that it is minimum value,

$q=\frac{8}{3}-0.01$ $=2.6567$	$q=\frac{8}{3}$	$q=\frac{8}{3}+0.01$ $=2.6767$
$\frac{\mathrm{d}C}{\mathrm{d}q}=-0.00996$	$\frac{\mathrm{d}C}{\mathrm{d}q}=0$	$\frac{\mathrm{d}C}{\mathrm{d}q}=0.0101$

OR

$\frac{\mathrm{d}^2C}{\mathrm{d}q^2}=\frac{3}{4}q-1$

when $q=\frac{8}{3}$, $\frac{\mathrm{d}^2C}{\mathrm{d}q^2}=\frac{3}{4}\left(\frac{8}{3}\right)-1=1>0$

Hence $C=1.8148=1.81(3.\text{s.f.})$

is a minimum value at $q=\frac{8}{3}$

The lowest marginal cost of $1814.81 is achieved when the quantity of chocolate bar manufactured is about 2666.

(vii)

By GC

$\int_0^1\left[\frac{1}{8}q^2(q-4)+3\right]\mathrm{d}t=2.864583=2.86(3.\text{s.f.})$

The total cost of manufacture for the first thousand of Caramel chocolate bars is $2864.58.