2023 Term 4 Timed Practice (Structured Remedial Session 2)

Question 1 (**N16/I/9)**

A stone is held on the surface of a pond and released. The stone falls vertically through the water and the distance, x metres, that the stone has fallen in time t seconds is measured. It is given that $x=0$ and $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ when $t=0$.

(i) The motion of the stone is modelled by the differential equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}=10.$$

(a) By substituting $y=\frac{\mathrm{d}x}{\mathrm{d}t}$, show that the differential equation can be written as $\frac{\mathrm{d}y}{\mathrm{d}t}=10-2y$. [1]

Solution	***Learning Points and self mark***
$y=\frac{\mathrm{d}x}{\mathrm{d}t}$ Differentiate with respect to t, (*Use the substitution given and diff wrt t*) $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. Sub $\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ and $y=\frac{\mathrm{d}x}{\mathrm{d}t}$, *(use above 2 results to now substitute into the original equation, with objective that your* ***new equation is one that involves only y and t****, free of x)* $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+2\frac{\mathrm{d}x}{\mathrm{d}t}=10$ $\frac{\mathrm{d}y}{\mathrm{d}t}+2y=10$ $\frac{\mathrm{d}y}{\mathrm{d}t}=10-2y$ (Shown)	(Need know this substitution is not the typical one, but one that helps you step down from 2nd derivative equation to a 1st derivative equation. The concept and process is similar to the typical substitution. In a nutshell, you must follow the prompt "By substituting…., show that …." , otherwise its impossible to solve! 1 mark to show you have the steps leading to the shown result

(b) Find y in terms of t and hence find x in terms of t. [6]

Solution	
$\frac{dy}{dt}=10-2y$ $\frac{1}{10-2y}\frac{dy}{dt}=1$ $\int\frac{1}{10-2y}\,dy=\int 1\,dt$ $-\frac{1}{2}\ln\lvert 10-2y\rvert=t+c$ $\ln\lvert 10-2y\rvert=-2t+C,\quad C=-2c$ $10-2y=\pm e^{-2t+C}=Ae^{-2t},\qquad A=\pm e^{C}$ $y=5-\frac{A}{2}e^{-2t}$	1 mark- do separable variables. ***(these 7 steps on left are standard process - pls learn)*** 1 mark for ln | | expression 1 mark – arrive at expression
When $t=0$, we have $x=0$ and $y=\frac{dx}{dt}=0$, $0=5-\frac{A}{2}e^{0}$ $A=10$	1 mark – substitute initial conditions and attempt find the constant A.
Therefore, $y=5-5e^{-2t}$ $\frac{dx}{dt}=5-5e^{-2t}$ $x=\int 5-5e^{-2t}\,dt$ $x=5t+\frac{5}{2}e^{-2t}+D$ When $t=0$, we have $x=0$, $0=5(0)+\frac{5}{2}e^{0}+D$ $D=-\frac{5}{2}$	1 mark- method for integrating one more time wrt t after rewriting y as $\frac{dx}{dt}$. Cautious of the various negative signs!
Therefore, $x=5t+\frac{5}{2}e^{-2t}-\frac{5}{2}$	1 mark- final answer (all terms correct)

(ii) A second model for the motion of the stone is suggested, given by the differential equation

$$\frac{d^2x}{dt^2} = 10 - 5\sin\frac{1}{2}t.$$

Find x in terms of t for this model. [3]

Solution	
$\frac{d^2x}{dt^2} = 10 - 5\sin\frac{1}{2}t$ $\frac{dx}{dt} = \int 10 - 5\sin\frac{1}{2}t \ dt$ $\frac{dx}{dt} = 10t + 10\cos\frac{1}{2}t + E$ When $t = 0$, we have $\frac{dx}{dt} = 0$, $0 = 10(0) + 10\cos 0 + E$ $E = -10$ $x = \int 10t + 10\cos\frac{1}{2}t - 10 \ dt$ $x = 5t^2 + 20\sin\frac{1}{2}t - 10t + F$ When $t = 0$, we have $x = 0$, $0 = 5(0^2) + 20\sin 0 - 10(0) + F$ $F = 0$ Therefore, $x = 5t^2 + 20\sin\frac{1}{2}t - 10t.$	(note this is a second order DE, need do twice integrating to get to x) 1 mark – integrating sine function correctly. Need arbitrary constant E here. Note the negative sign. 1 mark – integrate one more time AND attempt to find the 2 arbitrary constants 1 mark – accurate answer

Question 2

(i) Given that $y=\sqrt{1+\sin x}$, prove that $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$. By repeated differentiation of this result expand y as a series of ascending powers of x up to and including the x^3 term. [5]

(ii) Using the standard series from the List of Formulae (MF26), verify that the same result can be obtained for y, up to the x^2 term, for small x. [3]

(i)	**Method 1** $y=\sqrt{1+\sin x}$ Differentiate with respect to x, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos x}{2\sqrt{1+\sin x}}$ $2\sqrt{1+\sin x}\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ (Shown) **Method 2 (preferred)** $y=\sqrt{1+\sin x}$ $y^2=1+\sin x$ Differentiate with respect to x, *(must use implicit diffn)* $2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x$ (Shown) Diff implicitly wrt x, $2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\sin x$ $4\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\cos x$ $6\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)+2y\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\cos x$ When $x=0$,	This method requires you to rewrite $\sqrt{1+\sin x}$ back as y. 1 mark 1 mark- get this 2nd derivative expression correct. *(implicit diffn must be fluent. Note that Product rule applies. View the two terms as* $2y$ *and* $\frac{\mathrm{d}y}{\mathrm{d}x}$*)* 1 mark - get 3rd derivative expression correct.

	$y=\sqrt{1+0}=1,$ $2(1)\frac{\mathrm{d}y}{\mathrm{d}x}=\cos 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2},$ $2\left(\frac{1}{2}\right)^2+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0 \Rightarrow \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{1}{4},$ $6\left(\frac{1}{2}\right)\left(-\frac{1}{4}\right)+2\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-1 \Rightarrow \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-\frac{1}{8}$ $\sqrt{1+\sin x}=1+\left(\frac{1}{2}\right)x+\frac{\left(-\frac{1}{4}\right)}{2!}x^2+\frac{\left(-\frac{1}{8}\right)}{3!}x^3+\ldots$ $\approx 1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{48}$	1 mark (for all values of derivatives evaluated at x=0) ***Refer/consult MF26 to quote Maclaurin formula.*** 1 (final answer) *(do the above steps systematically to find the derivatives when x=0, don't skip essential working steps although its worth 1 mark only)*
(ii)	***Using the MF26* sin *x series, observe that* sin *x* ≈ *x* .** ***(Alternate, By small angle approximation, since* sin *x* ≈ *x)*** $\sqrt{1+\sin x}\approx\sqrt{1+x}$ $=(1+x)^{\frac{1}{2}}$ $=1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\ldots$ $\approx 1+\frac{x}{2}-\frac{x^2}{8}$ Comparing with (i), the terms agree up to the square term for small *x*. (***verified***)	1 mark apply approx 1 mark apply binomial 1 final expression and statement to conclude that its verified.