Additional Practice Questions (H2 Chap 6A Integration Techniques)

1. **[2008/Promo/AJC/Q8(b)]**

(i) Given that $y=(\sin x)e^{\sin x}$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Hence, or otherwise, show that $\int(\sin 2x)e^{\sin x}\ \mathrm{d}x=2(\sin x)e^{\sin x}-2e^{\sin x}+C$, where C is an arbitrary constant. [3]

(ii) Find $\int(\sin x)(\sin 2x)e^{\sin x}\ \mathrm{d}x$. [3]

Solution:

(i) $\frac{dy}{dx}=\frac{d}{dx}(\sin x)e^{\sin x}=(\cos x)(e^{\sin x})+(\sin x)(\cos x)e^{\sin x}$

(ii) Method 1

Integrating both sides w.r.t x:

$$(\sin x)e^{\sin x}=\int(\cos x)(e^{\sin x})dx+\int(\sin x)(\cos x)e^{\sin x}dx$$

$$(\sin x)e^{\sin x}=e^{\sin x}+\frac{1}{2}\int(\sin 2x)e^{\sin x}dx$$

$$\int(\sin 2x)e^{\sin x}dx=2(\sin x)e^{\sin x}-2e^{\sin x}\ +\ \mathrm{C}\quad\text{(shown)}$$

Method 2 (by parts)

$$\int(\sin 2x)e^{\sin x}dx=2\int(\sin x)(\cos x)e^{\sin x}dx$$

Let u = sin$x$ $\Rightarrow\frac{du}{dx}=\cos x$ and $\frac{dv}{dx}=(\cos x)e^{\sin x}\Rightarrow v=e^{\sin x}$

$$\int(\sin 2x)e^{\sin x}dx=2(\sin x)e^{\sin x}-2\int(\cos x)e^{\sin x}dx$$

$$=2(\sin x)e^{\sin x}-2e^{\sin x}\ +\ \mathrm{C}\quad\text{(shown)}$$

(iii) (by parts and using (ii))

$$\int(\sin x)(\sin 2x)e^{\sin x}dx$$

$$=(\sin x)[2(\sin x)e^{\sin x}-2e^{\sin x}]$$

$$-\int 2(\sin x)(\cos x)e^{\sin x}-2(\cos x)e^{\sin x}dx$$

$$=(\sin x)[2(\sin x)e^{\sin x}-2e^{\sin x}]-[2(\sin x)e^{\sin x}-2e^{\sin x}]+2e^{\sin x}+c$$

$$=2e^{\sin x}[\sin^2 x-2\sin x+2]+c$$

2. **[2008/Promo/ACJC/Q4]**

Express $\dfrac{6x^3-x^2+4x-1}{(x^2+1)(3x^2+2)}$ in the form $\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{3x^2+2}$ where A, B, C and D are constants to be determined. [3]

Hence, or otherwise, find $\displaystyle\int \frac{6x^3-x^2+4x-1}{(x^2+1)(3x^2+2)}\,dx$. [3]

<u>Solution:</u>

$$\frac{6x^3-x^2+4x-1}{(x^2+1)(3x^2+2)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{3x^2+2}$$

$$6x^3-x^2+4x-1=(Ax+B)(3x^2+2)+(Cx+D)(x^2+1)$$

coefficient of x^3: $3A+C=6$

coefficient of x^2: $3B+D=-1$

coefficient of x: $2A+C=4$

constant: $2B+D=-1$

Solving using G.C. or manually: $A=2, B=0, C=0, D=-1$

$$\begin{aligned}\int\frac{6x^3-x^2+4x-1}{(x^2+1)(3x^2+2)}dx&=\int\frac{2x}{x^2+1}-\frac{1}{3x^2+2}\,dx\\&=\int\frac{2x}{x^2+1}dx-\frac{1}{3}\int\frac{1}{x^2+\frac{2}{3}}dx\\&=\ln\left|x^2+1\right|-\frac{1}{3}\left(\sqrt{\frac{3}{2}}\right)\tan^{-1}\sqrt{\frac{3}{2}}x+c\\&\text{or }\ln\left|x^2+1\right|-\frac{\sqrt{6}}{6}\tan^{-1}\frac{\sqrt{6}}{2}x+c\end{aligned}$$

3. **[2008/Promo/CJC/Q14]**

(a) (i) Express $\dfrac{x^2-3x+7}{(1-x)(4+x^2)}$ in partial fractions. [4]

(ii) Hence find $\int \mathrm{f}(x)\,\mathrm{d}x$, where $\mathrm{f}(x)=\dfrac{x^2-3x+7}{(1-x)(4+x^2)}$. [2]

(b) By substituting $u=\sqrt{3x^2+5}$, find the integral $\int 3x\sqrt{3x^2+5}\,\mathrm{d}x$. [3]

Solution:

a) i) $\dfrac{x^2-3x+7}{(1-x)(4+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{4+x^2}$

$x^2-3x+7 = A(4+x^2)+(1-x)(Bx+C)$

$x=1$: $5 = 5A \Rightarrow A=1$

$x=0$: $7 = 4A+C \Rightarrow C=3$

$x=-1$: $11 = 5A+2(-B+3) \Rightarrow B=0$

$\therefore \dfrac{x^2-3x+7}{(1-x)(4+x^2)} = \dfrac{1}{1-x} + \dfrac{3}{4+x^2}$ #

ii) $\displaystyle\int \frac{x^2-3x+7}{(1-x)(4+x^2)}\,dx = \int \frac{1}{1-x} + \frac{3}{4+x^2}\,dx$

$= -\ln|1-x| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$ #

b) $\int 3x\sqrt{3x^2+5}\,dx$

$= \int u^2\,du$

$= \dfrac{u^3}{3} + C$

$= \frac{1}{3}\left[\sqrt{3x^2+5}\right]^3 + C$ #

$u = \sqrt{3x^2+5}$

$\dfrac{du}{dx} = \frac{1}{2}(3x^2+5)^{-\frac{1}{2}}(6x)$

$= \dfrac{3x}{\sqrt{3x^2+5}} = \dfrac{3x}{u}$

$\therefore u\,du = 3x\,dx$

4. **[2008/Promo/HCI/Q2]**

Solve $\int x\cos\left(\ln x\right)\ \mathrm{d}x$. [5]

Solution:

$$u = \cos\left(\ln x\right) \qquad v' = x$$

$$u' = -\frac{1}{x}\sin\left(\ln x\right) \qquad v = \frac{1}{2}x^2$$

$$\int x\cos\left(\ln x\right)\ \mathrm{d}x$$

$$= \frac{1}{2}x^2\cos\left(\ln x\right) + \frac{1}{2}\int x\sin\left(\ln x\right)\ \mathrm{d}x$$

$$u = \sin\left(\ln x\right) \qquad v' = x$$

$$u' = \frac{1}{x}\cos\left(\ln x\right) \qquad v = \frac{1}{2}x^2$$

$$\int x\cos\left(\ln x\right)\ \mathrm{d}x$$

$$= \frac{1}{2}x^2\cos\left(\ln x\right) + \frac{1}{2}\left[\frac{1}{2}x^2\sin\left(\ln x\right) - \int \frac{1}{2}x\cos\left(\ln x\right)\ \mathrm{d}x\right]$$

$$\frac{5}{4}\int x\cos\left(\ln x\right)\ \mathrm{d}x = \frac{1}{2}x^2\cos\left(\ln x\right) + \frac{1}{4}x^2\sin\left(\ln x\right)$$

$$\int x\cos\left(\ln x\right)\ \mathrm{d}x = \frac{2}{5}x^2\cos\left(\ln x\right) + \frac{1}{5}x^2\sin\left(\ln x\right) + C$$

5. **[2008/Promo/NJC/Q11(a)]**

By considering $x+1=A(-1-2x)+B$, where A and B are constants, or otherwise, find $\int \frac{x+1}{\sqrt{3-x-x^2}}\,\mathrm{d}x$. [4]

Solution:

$$\int \frac{x+1}{\sqrt{3-x-x^2}}\,\mathrm{d}x=-\frac{1}{2}\int \frac{-2x-2}{\sqrt{3-x-x^2}}\,\mathrm{d}x$$

$$=-\frac{1}{2}\int \frac{-2x-1}{\sqrt{3-x-x^2}}\,\mathrm{d}x+\frac{1}{2}\int \frac{1}{\sqrt{3-x-x^2}}\,\mathrm{d}x$$

$$=-\frac{1}{2}\int \frac{-2x-1}{\sqrt{3-x-x^2}}\,\mathrm{d}x+\frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{13}}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}}\,\mathrm{d}x$$

$$=-\sqrt{3-x-x^2}+\frac{1}{2}\sin^{-1}\left(\frac{2x+1}{\sqrt{13}}\right)+C$$

6. **[2008/Promo/SAJC/Q10]**

Find the following integrals :

(a) $\displaystyle\int \frac{4}{\sqrt{9-4x^2}}\,dx$ [3]

(b) $\displaystyle\int \left(\cos 2x + e^{\sin 2x}\right)\cos 2x\ dx$ [3]

(c) $\displaystyle\int \frac{2x-1}{x^2+4x+5}\,dx$ [4]

<u>Solution:</u>

10 (a)	$\displaystyle\int \frac{4}{\sqrt{9-4x^2}}\,dx$ $\displaystyle= \int \frac{4}{2\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}\,dx = 2\sin^{-1}\frac{x}{3/2} + C = 2\sin^{-1}\frac{2}{3}x + C$
10 (b)	$\displaystyle\int \left(\cos 2x + e^{\sin 2x}\right)\cos 2x\ dx$ $\displaystyle= \int \left(\cos^2 2x + e^{\sin 2x}\cos 2x\right)\ dx$ $\displaystyle= \int \frac{1}{2}(1+\cos 4x)\ dx + \int e^{\sin 2x}\cos 2x\ dx$ $\displaystyle= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right) + \frac{1}{2}e^{\sin 2x} + C$ $\displaystyle= \frac{1}{2}x + \frac{1}{8}\sin 4x + \frac{1}{2}e^{\sin 2x} + C$
10 (c)	$\displaystyle\int \frac{2x-1}{x^2+4x+5}\,dx$ $\displaystyle= \int \frac{2x+4}{x^2+4x+5} + \frac{-5}{x^2+4x+5}\,dx$ $\displaystyle= \int \frac{2x+4}{x^2+4x+5}\,dx - 5\int \frac{1}{(x+2)^2+1}\,dx$ $\displaystyle= \ln\left(x^2+4x+5\right) - 5\tan^{-1}(x+2) + C$

7. **[2008/Promo/TJC/Q11]**

Find

(a) $\int \cot^2 3x \, \mathrm{d}x$. [3]

(b) Find $\int x\sin^{-1}\left(x^2\right) \mathrm{d}x$. [4]

(c) Given that $3-2x = A(2x-4)+B$ for all values of x, find the constants A and B. Hence or otherwise, find $\int \frac{3-2x}{x^2-4x+6}\mathrm{d}x$. [4]

<u>Solution:</u>

(a) $$\int \cot^2 3x \, dx = \int \left(\operatorname{cosec}^2 3x - 1\right) dx$$
$$= -\frac{1}{3}\cot 3x - x + c$$

(b) $$\int x\sin^{-1}\left(x^2\right) \mathrm{d}x = \frac{x^2}{2}\sin^{-1}\left(x^2\right) - \int\left(\frac{x^2}{2}\right)\frac{2x}{\sqrt{1-x^4}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}\left(x^2\right) - \int\frac{x^3}{\sqrt{1-x^4}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{4}\int\frac{-4x^3}{\sqrt{1-x^4}}\, dx$$
$$= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{4}\left(2\sqrt{1-x^4}\right) + c$$
$$= \frac{x^2}{2}\sin^{-1}\left(x^2\right) + \frac{1}{2}\left(\sqrt{1-x^4}\right) + c$$

(c) By comparing coefficients, $2A = -2 \Rightarrow A = -1$
$3 = B - 4A \Rightarrow B = -1$

$$\int \frac{3-2x}{x^2-4x+6}dx = \int\frac{-(2x-4)-1}{x^2-4x+6}dx$$
$$= -\int\frac{2x-4}{x^2-4x+6}dx - \int\frac{1}{(x-2)^2+2}dx$$
$$= -\ln\left(x^2-4x+6\right) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x-2}{\sqrt{2}} + c$$

8. **[2010/Promo/HCI/Q12]**

(a) Write down the constants A and B such that, for all values of x,

$$2x+5 = A(x-1)+B.$$

Hence find $\displaystyle\int \frac{2x+5}{x^2-2x+5}\,dx$. [4]

(b) Find the derivative of $\tan\left(x^2\right)$. Hence find $\displaystyle\int x^3 \sec^2\left(x^2\right)\,dx$. [4]

(c) By using the substitution $x=\dfrac{1}{u}$, find the exact value of $\displaystyle\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx$. [5]

Solution:

8(a)	$A=2, B=7$ $\displaystyle\int \frac{2x+5}{x^2-2x+5}\,dx = \int \frac{2x-2+7}{x^2-2x+5}\,dx$ $\displaystyle= \int \frac{2x-2}{x^2-2x+5}\,dx + \int \frac{7}{x^2-2x+5}\,dx$ $\displaystyle= \ln\left\|x^2-2x+5\right\| + 7\int \frac{1}{(x-1)^2+4}\,dx$ $\displaystyle= \ln\left\|x^2-2x+5\right\| + \frac{7}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+C$
(b)	$\displaystyle\frac{d}{dx}\tan\left(x^2\right) = 2x\sec^2\left(x^2\right)$ $\displaystyle\int x^3\sec^2\left(x^2\right)\,dx = \int x^2\cdot x\sec^2\left(x^2\right)\,dx$ $u=x^2 \qquad v' = x\sec^2\left(x^2\right)$ $u'=2x \qquad v=\dfrac{1}{2}\tan\left(x^2\right)$ $\displaystyle\int x^3\sec^2\left(x^2\right)\,dx = \frac{x^2}{2}\tan\left(x^2\right) - \int x\tan\left(x^2\right)\,dx$ $\displaystyle= \frac{x^2}{2}\tan\left(x^2\right) + \frac{1}{2}\int \frac{-2x\sin\left(x^2\right)}{\cos\left(x^2\right)}\,dx = \frac{x^2}{2}\tan\left(x^2\right) + \frac{1}{2}\ln\left\|\cos\left(x^2\right)\right\|+C$
(c)	$\displaystyle x=\frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}\,du$ $\displaystyle x=2\sqrt{2} \Rightarrow u=\frac{1}{2\sqrt{2}}$ and $\displaystyle x=4 \Rightarrow u=\frac{1}{4}$ $\displaystyle\int_{2\sqrt{2}}^{4} \frac{1}{x\sqrt{x^2-4}}\,dx = \int_{\frac{1}{4}}^{\frac{1}{2\sqrt{2}}} \frac{1}{\sqrt{1-4u^2}}\cdot du$ $\displaystyle= \frac{1}{2}\left[\sin^{-1}(2u)\right]_{\frac{1}{4}}^{\frac{1}{2\sqrt{2}}}$ $\displaystyle= \frac{1}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\pi}{24}$

9. **[2010/Promo/RVHS/Q11]**

(a) Using the substitution $u = \mathrm{e}^x$, find $\int \frac{1}{\mathrm{e}^x + 2\mathrm{e}^{-x}}\,\mathrm{d}x$. [4]

(b) By expressing $4x - 5$ in the form $A(2 - 2x) + B$, show that

$\int_0^1 \frac{4x-5}{\sqrt{3+2x-x^2}}\,\mathrm{d}x = \frac{a\sqrt{3}+b-\pi}{6}$, where a and b are constants to be found.

[6]

<u>Solution:</u>

a.	$u = e^x \Rightarrow \frac{du}{dx} = e^x = u$. Then: $$\int \frac{1}{e^x + 2e^{-x}}\,dx = \int \frac{1}{u + \frac{2}{u}}\left(\frac{du}{u}\right)$$ $$= \int \frac{1}{u^2+2}\,du$$ $$= \int \frac{1}{\left(\sqrt{2}\right)^2 + u^2}\,du$$ $$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$ $$= \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{e^x}{\sqrt{2}}\right) + c$$
b.	$$\int_0^1 \frac{4x-5}{\sqrt{3+2x-x^2}}\,dx = \int_0^1 \frac{-2(2-2x)-1}{\sqrt{3+2x-x^2}}\,dx$$ $$= -2\int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}}\,dx - \int_0^1 \frac{1}{\sqrt{3+2x-x^2}}\,dx$$ $$= -2\int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}}\,dx - \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx$$ $$= -2\left[2\sqrt{3+2x-x^2}\right]_0^1 - \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^1$$ $$= -2\left[2\sqrt{4} - 2\sqrt{3}\right]_0^1 - \left[\sin^{-1}\left(\frac{1-1}{2}\right) - \sin^{-1}\left(\frac{0-1}{2}\right)\right]$$ $$= 4\sqrt{3} - 8 - 0 - \frac{\pi}{6}$$ $$= \frac{24\sqrt{3} - 48 - \pi}{6}$$

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(a) Find $\int \tan^2(3x)\ \mathrm{d}x$. [2]

(b) Find $\int \dfrac{2x+3}{x^2-2x+5}\ \mathrm{d}x$. [4]

(c) Differentiate $\sin^{-1}(x^2)$ with respect to x. [1]

Hence find the exact value of $\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \dfrac{x}{\sin^{-1}(x^2)\sqrt{1-x^4}}\ \mathrm{d}x$, simplifying your answer.

[4]

Solution:

10(a)	$\int \tan^2(3x)\ \mathrm{d}x = \int \left[\sec^2(3x)-1\right]\ \mathrm{d}x$ $= \frac{1}{3}\tan(3x) - x + C$
(b)	$\int \frac{2x+3}{x^2-2x+5}\ \mathrm{d}x$ $= \int \frac{(2x-2)+5}{x^2-2x+5}\ \mathrm{d}x$ $= \int \frac{2x-2}{x^2-2x+5}\ \mathrm{d}x + 5\int \frac{1}{(x-1)^2+4}\ \mathrm{d}x$ $= \ln\left|x^2-2x+5\right| + 5\times\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$ $= \ln\left(x^2-2x+5\right) + \frac{5}{2}\tan^{-1}\left(\frac{x-1}{2}\right) + C$ $\left(\because (x-1)^2+4>0 \text{ for all real values of } x\right)$
(c)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(x^2\right)\right) = \frac{2x}{\sqrt{1-x^4}}$ $\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \frac{x}{\sin^{-1}\left(x^2\right)\sqrt{1-x^4}}\ \mathrm{d}x = \frac{1}{2}\int_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}} \frac{\frac{2x}{\sqrt{1-x^4}}}{\sin^{-1}\left(x^2\right)}\ \mathrm{d}x$ $= \frac{1}{2}\left[\ln\left\|\sin^{-1}\left(x^2\right)\right\|\right]_{\frac{1}{\sqrt{2}}}^{\left(\frac{3}{4}\right)^{\frac{1}{4}}}$ $= \frac{1}{2}\left[\ln\left\|\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right\| - \ln\left\|\sin^{-1}\left(\frac{1}{2}\right)\right\|\right] = \frac{1}{2}\left[\ln\frac{\pi}{3} - \ln\frac{\pi}{6}\right]$ $= \frac{1}{2}\ln 2$

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(a) Find $\int \frac{1}{x\ln(2014x)}\,\mathrm{d}x$. [2]

(b) Use the substitution $u = \mathrm{e}^x + 2$ to find $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x + 2}\,\mathrm{d}x$. [4]

(c) (i) Find $\int x\cos 2x\ \mathrm{d}x$. [2]

(ii) Hence find $\int_0^{\frac{\pi}{4}} x\sin^2 x\ \mathrm{d}x$, giving your answer in exact form. [3]

(a)	$\int \frac{1}{x\ln(2014x)}\,\mathrm{d}x = \int \frac{1/x}{\ln(2014x)}\,\mathrm{d}x$ $= \ln\left\|\ln(2014x)\right\| + C$
(b)	$u = \mathrm{e}^x + 2$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x = u-2 \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{u-2}$ $\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^x+2}\,\mathrm{d}x = \int \frac{(u-2)^2}{u}\cdot\frac{1}{u-2}\,\mathrm{d}u$ $= \int \frac{u-2}{u}\,\mathrm{d}u = \int 1 - \frac{2}{u}\,\mathrm{d}u$ $= u - 2\ln\left\|u\right\| + C$ $= \mathrm{e}^x + 2 - 2\ln\left\|\mathrm{e}^x+2\right\| + C$ $= \mathrm{e}^x - 2\ln\left(\mathrm{e}^x+2\right) + C$
(c)	(i) $\int x\cos 2x\ \mathrm{d}x = \frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\ \mathrm{d}x$ $= \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$ (ii) $\int_0^{\frac{\pi}{4}} x\sin^2 x\ \mathrm{d}x$ $= \frac{1}{2}\int_0^{\frac{\pi}{4}} x(1-\cos 2x)\ \mathrm{d}x = \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} x\ \mathrm{d}x - \int_0^{\frac{\pi}{4}} x\cos 2x\ \mathrm{d}x\right]$ $= \frac{1}{2}\left[\frac{1}{2}x^2 - \frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x\right]_0^{\frac{\pi}{4}}$ $= \frac{1}{2}\left[\frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \frac{1}{2}\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) - \frac{1}{4}\cos\left(\frac{\pi}{2}\right) + \frac{1}{4}\cos 0\right]$ $= \frac{\pi^2}{64} - \frac{\pi}{16} + \frac{1}{8}$