VICTORIA JUNIOR COLLEGE
Preliminary Examination

MATHEMATICS
(Higher 2)

9740/01

Paper 1

September 2015

3 hours

Additional Materials: Answer Paper
Graph Paper
List of Formulae (MF15)

READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of 5 printed pages

 VICTORIA JUNIOR COLLEGE **[Turn over**

1 A sequence $w_1,\ w_2,\ w_3, \ldots$ is such that

$$w_{n+1} = \frac{1}{n}[(n+1)w_n + 1], \quad \text{where } n \in \mathbb{Z}^+, \text{ and}$$

$$w_1 = a, \quad \text{where } a \text{ is a constant.}$$

Use the method of mathematical induction to prove that $w_n = an + (n-1)$. [5]

2 The function h is given by

$$\mathrm{h} : x \mapsto ax^3 + bx^2 + cx + d, \quad x \in \mathbb{R},$$

where $a,\ b,\ c$ and d are real constants.

The graph of $y = \mathrm{h}(x)$ passes through the point $(1,1)$. Given that $(2,2)$ is a stationary point, find three linear equations involving $a,\ b,\ c$ and d. [3]

By writing each of a, b and c in terms of d, find the exact set of values of d such that $\frac{ab}{c} \leq 0$. [3]

3 The curve C has equation $y = \dfrac{ax^2 + bx + d}{x-2}$, where a, b and d are constants.
Given that the line $y = 2x + 3$ is an asymptote to C, find the values of a and b. [3]

Given further that $d < -6$, find the coordinates of any points of intersection with the x- and y-axes, leaving your answer in terms of d. Hence sketch C, stating the equations of any asymptotes. [4]

4 The function g is defined by

$$\mathrm{g} : x \mapsto \mathrm{e}^x - 7x, \quad x > \lambda,$$

where λ is a real constant.

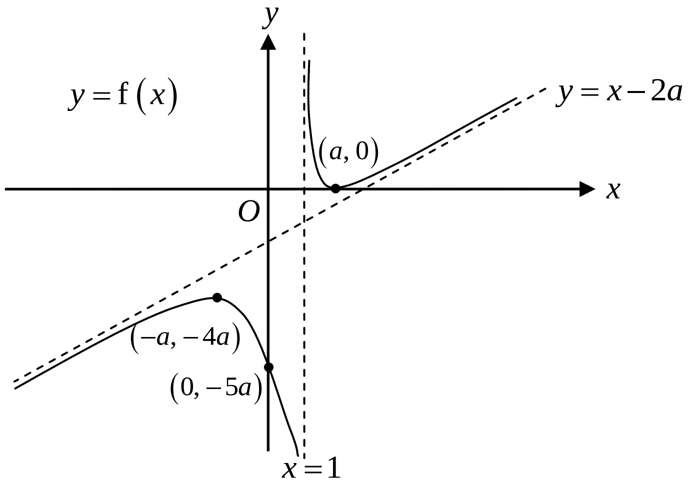
(i) Find the exact minimum value of λ such that the inverse of g exists. [2]

Using the value of λ found in **(i)**,

(ii) find $\mathrm{g}^{-1}(1)$, [2]

(iii) sketch the graphs of $y = \mathrm{g}(x)$, $y = \mathrm{g}^{-1}(x)$ and $y = \mathrm{gg}^{-1}(x)$ on a single diagram. [3]

5 The diagram shows the curve C with equation $y = \mathrm{f}(x)$. The lines $y = x - 2a$ and $x = 1$ are asymptotes to C. C has a minimum $(a, 0)$ and a maximum point $(-a, -4a)$, where $a > 2$. C cuts the y-axis at the point $(0, -5a)$.



On separate diagrams, sketch the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$. [3]

Find the value of $\int_{-a}^{0} \left[2 - \mathrm{f}'(x)\right] \mathrm{d}x$, leaving your answer in terms of a. [2]

6 On 1 January 2015, Mrs Koh put \$1000 into an investment fund which pays compound interest at a rate of 8% per annum on the last day of each year. She puts a further \$1000 into the fund on the first day of each subsequent year until she retires.

(i) If she retires on 31 December 2040, show that the total value of her investment on her retirement day is \$86351, correct to the nearest dollar. [4]

On 1 January 2015, Mr Woo put \$1000 into a savings plan that pays no interest. On the first day of each subsequent year, he saves \$80 more than the previous year. Thus, he saves \$1080 on 1 January 2016, \$1160 on 1 January 2017, and so on.

(ii) By forming a suitable inequality, find the year in which Mr Woo will first have saved over \$86351 in total. [4]

[Turn over

7 A tank initially contains 400 litres of solution with 100 kg of salt dissolved in it. A solution containing 0.125 kg of salt per litre flows into the tank at a rate of 12 litres per minute and the solution flows out at the same rate. You should assume that the inflow is instantaneously and thoroughly mixed with the contents of the tank. If the amount of salt in the tank is q kg at the end of t minutes, show that

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 1.5 - 0.03q\,. \qquad [2]$$

Find the time taken for the concentration of salt in the tank to reach 0.16 kg per litre. [5]
(Concentration of salt = the amount of salt per unit volume of solution in the tank.)

State what happens to q for large values of t. Sketch a graph of q against t. [3]

8 **(i)** If $z = x + \mathrm{i}\,y$, where $x, y \in \mathbb{R}$, prove that $\left(z^2\right)^* = \left(z^*\right)^2$. [2]

(ii) Solve the equation $z^2 = 1 - 4\sqrt{3}\,\mathrm{i}$, giving your answers exactly in the form $x + \mathrm{i}y$. [4]

(iii) Use your answers in part **(ii)** to solve the equation $w^2 = 4 + 16\sqrt{3}\,\mathrm{i}$. [2]

(iv) The roots in part **(ii)** are represented by z_1 and z_2. Given that $\arg\left(z^2\right) = \theta$, find $\arg\left(z_1 z_2\right)$, giving your answer in terms of θ. [2]

9 A curve C has parametric equations

$$x = \mathrm{e}^{\theta}\sin\theta, \quad y = \mathrm{e}^{\theta}\cos\theta,$$

where $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$.

(i) Sketch C, indicating clearly the axial intercepts. [2]

(ii) C cuts the y- and x-axes at points A and B respectively. A particle moves along C from A to B, with its x-coordinate increasing at a constant rate of 0.1 units per second. Find the exact rate of change of its y-coordinate when $x = \frac{1}{2}\mathrm{e}^{\frac{\pi}{6}}$. [3]

(iii) The tangent at the point P on C is parallel to the y-axis. Find the equation of this tangent. [4]

(iv) The point Q on C is such that angle $POQ = \frac{\pi}{2}$. Find the area of triangle OPQ. [5]

10 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_i = \frac{A^i}{i!} - \frac{2A^{i+1}}{(i+1)!} + \frac{A^{i+2}}{(i+2)!}, \quad \text{where } A \text{ is a constant and } i \in \mathbb{Z}^+.$$

Another sequence $v_1, v_2, v_3, \ldots$ is defined by

$$v_n = \sum_{i=1}^{n} u_i, \text{ where } n \in \mathbb{Z}^+.$$

(i) Show that $v_n = A - \frac{A^2}{2} - \frac{A^{n+1}}{(n+1)!} + \frac{A^{n+2}}{(n+2)!}$. [4]

(ii) Hence, find $\sum_{n=2}^{N} \left\{ \frac{1}{N}\left(v_n + \frac{A^{n+1}}{(n+1)!} - \frac{A^{n+2}}{(n+2)!} \right) + 7^{n-N} \right\}$. [4]

Hence explain why $\sum_{n=2}^{N} \left\{ \frac{1}{N}\left(v_n + \frac{A^{n+1}}{(n+1)!} - \frac{A^{n+2}}{(n+2)!} \right) + 7^{n-N} \right\}$ converges as $N \to \infty$, and write down the value of the limit in terms of *A*. [3]

11 The equations of the plane π, and the lines l_1 and l_2 are given by

$$\pi : ax - 2y + z = 13,$$

$$l_1 : \mathbf{r} = (2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + \lambda\left[2a\,\mathbf{i} + (a^2 - 1)\,\mathbf{j} - 2\mathbf{k}\right],$$

$$l_2 : x - 4 = y,\ z = 5,$$

where *a* is constant, and λ is a real parameter.

Given that the shortest distance from the point *P* with coordinates $(1,2,12)$ to π is 1, show that $a = 2$. [2]

(i) Given that *A* is a point on l_1 and *B* is a point on l_2, find the position vectors of *A* and *B* such that *AB* is perpendicular to both l_1 and l_2. [4]

(ii) Show that l_2 is in the plane π. [2]

(iii) Given that l_1 is parallel to plane π, find the vector equation of the line of reflection of l_1 in π. [3]

(iv) Find the cartesian equation of plane *p* which is perpendicular to plane π and also contains l_2. [3]