SERANGOON JUNIOR COLLEGE

2012 JC2 PRELIMINARY EXAMINATION

MATHEMATICS

Higher 2 **9740/1**

Wednesday **15 Aug 2012**

Additional materials: Writing paper

List of Formulae (MF15)

**TIME : ** 3 hours

READ THESE INSTRUCTIONS FIRST

Write your name and class on the cover page and on all the work you hand in.
Write in dark or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.

Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

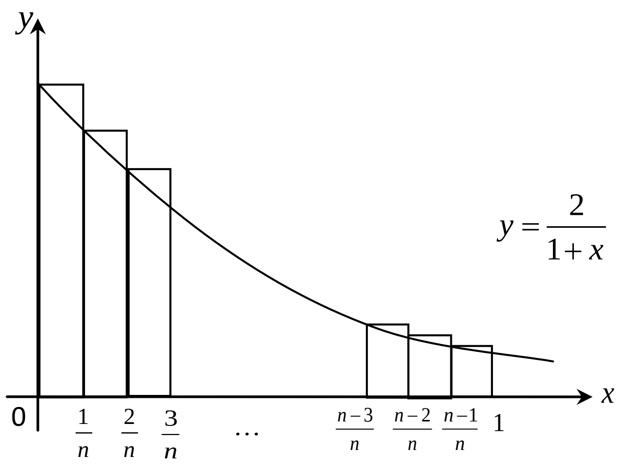
The number of marks is given in brackets **[ ]** at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

Total marks for this paper is 100 marks.

This question paper consists of 6 printed pages (inclusive of this page) and no blank page.

[TURN OVER]

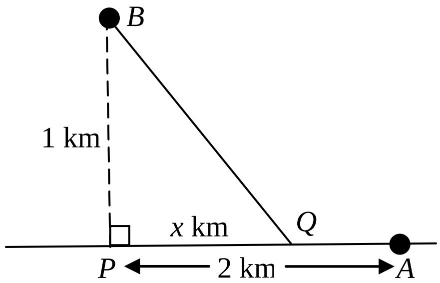
Answer all questions [100 marks].

1	The graph of $y=\dfrac{2}{1+x}$ for $x\geq 0$, is shown in the diagram below. Region R is bounded by the x-axis, the y-axis, the line $x=1$ and the curve $y=\dfrac{2}{1+x}$. The area of region R may be approximated by the total area, A, of n rectangles, each of width $\dfrac{1}{n}$, as shown in the diagram. 	
	(i) Show that $A=\sum_{r=0}^{n-1}\dfrac{2}{n+r}$.	**[2]**
	(ii) By considering the exact area of region R, show that $\sum_{r=0}^{n-1}\dfrac{1}{n+r}>\ln 2$.	**[2]**
	Solution	
	(i) Total area of all the n rectangles, $$A=\frac{1}{n}\left[\frac{2}{1+0}+\frac{2}{1+\frac{1}{n}}+\frac{2}{1+\frac{2}{n}}+\ldots+\frac{2}{1+\frac{n-1}{n}}\right]$$ $$=\frac{1}{n}\left[\frac{2}{0+1}+\frac{2n}{1+n}+\frac{2n}{2+n}+\ldots+\frac{2n}{(n-1)+n}\right]$$	
	$$=\frac{2}{0+n}+\frac{2}{1+n}+\frac{2}{2+n}+\ldots+\frac{2}{(n-1)+n}$$	
	$$=\sum_{r=0}^{n-1}\frac{2}{n+r}\quad (shown)$$	
	(ii) $$\int_0^1 \frac{2}{1+x}\,\mathrm{d}x=\left[2\ln\lvert 1+x\rvert\right]_0^1=2\ln 2$$	
	Area of the n rectangles > Area of region R	

	$1 \quad \sum_{r=0}^{n-1} \frac{2}{n+r} > 2\ln 2$	
	$1 \quad \sum_{r=0}^{n-1} \frac{1}{n+r} > \ln 2$ (*Shown*)	
2	A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation $x_{n+1} = \frac{n+2}{3} x_n$. Given that $x_1 = \frac{2}{3}$, write down x_2, x_3, x_4 in the form of $\frac{(n+a)!}{b^n}$, where a and b are positive integers.	**[2]**
	Hence make a conjecture for x_n and prove the conjecture by Mathematical Induction.	**[4]**
	Solution	
	$x_1 = \frac{2}{3}$, $x_2 = \frac{1+2}{3} x_1 = \frac{2}{3}$ $x_3 = \frac{2+2}{3} x_2 = \frac{8}{9}$ $x_4 = \frac{3+2}{3} x_3 = \frac{40}{27}$	
	$x_1 = \frac{2}{3} = \frac{1.2}{3} = \frac{(1+1)!}{3^1} = \frac{2!}{3^1}$ $x_2 = \frac{2}{3} = \frac{1.2.3}{3^2} = \frac{(2+1)!}{3^2} = \frac{3!}{3^2}$ $x_3 = \frac{8}{9} = \frac{1.2.3.4}{3^3} = \frac{(3+1)!}{3^3} = \frac{4!}{3^3}$ $x_4 = \frac{40}{27} = \frac{1.2.3.4.5}{3^4} = \frac{(4+1)!}{3^4} = \frac{5!}{3^4}$	
	$\therefore$ Conjecture: $x_n = \frac{(n+1)!}{3^n}$	
	Let P_n be the statement $x_n = \frac{(n+1)!}{3^n}$ for all $n \in \mathbb{Z}^+$. $n = 1, \text{LHS} = x_1 = \frac{2}{3}(\text{given})$ $\text{RHS} = \frac{(1+1)!}{3^1} = \frac{2}{3} = \text{LHS}$ $\therefore P_1$ is true.	
	Assume P_k is true for some $k \in \mathbb{Z}^+$ i.e. $x_k = \frac{(k+1)!}{3^k}$	

	To show P_{k+1} is true i.e. $x_{k+1} = \frac{(k+2)!}{3^{k+1}}$ $\text{LHS} = x_{k+1} = \frac{k+2}{3} x_k$ $= \frac{k+2}{3}\left(\frac{(k+1)!}{3^k}\right)$	
	$= \frac{(k+2)!}{3^{k+1}} = \text{RHS}$ $\therefore P_{k+1}$ is true if P_k is true. Since P_1 is true and P_{k+1} is true if P_k is true, P_n is true for all $n \in \mathbb{Z}^+$.	
3	A geometric series, *G*, has common ratio *r*, $r \neq 1$, and an arithmetic series, *A*, has a non-zero first term *a*. The first three terms of *G* are equal to the seventh, third and first term of *A* respectively.	
	(i) Show that $2r^2 - 3r + 1 = 0$.	**[3]**
	(ii) Deduce that *G* is convergent.	**[1]**
	(iii) Find the sum to infinity of the even-numbered terms of *G* in terms of *a*.	**[3]**
	Solution	
	(i) Let *b* be the first term of the G and *d* and *b* be the common difference of the AP.	
	$b = a + 6d$ $br = a + 2d$ $br^2 = a \cdots\cdots (1)$	
	$br - br^2 = 2d \cdots\cdots (2)$	
	$b - br = 4d \cdots\cdots (3)$	
	$\frac{(3)}{(2)}$ gives, $\frac{b - br}{br - br^2} = 2$	
	$1 - r = 2(r - r^2)$	
	$2r^2 - 3r + 1 = 0$	
	(ii) $(2r-1)(r-1) = 0$	
	$\therefore r = \frac{1}{2}$ or 1 (rejected $\because r \neq 1$)	
	Since $\lvert r \rvert = \left\lvert \frac{1}{2} \right\rvert < 1$, $\therefore G$ is convergent.	
	(iii) From (1) , $b = 4a$	
	Sum to infinity of the even-numbered terms $= \frac{br}{1 - r^2} = \frac{4a\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2}$	

	$= \dfrac{8a}{3}$	
4	Given that $\ln y = \sqrt{1-x}$, show that $$2\ln y \frac{d^2y}{dx^2} + \frac{2}{y}\left(\frac{dy}{dx}\right)^2 = -\frac{dy}{dx}.$$	**[2]**
	(i) By further differentiation of this result, or otherwise, find the Maclaurin's series of y up to and including the term in x^3.	**[3]**
	(ii) Deduce the series expansion of $y = \dfrac{e^{\sqrt{1-x}}}{\sqrt{1-x}}$ up to and including the term in x^2.	**[2]**
	Solution	
	$\ln y = \sqrt{1-x}$ $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1)$ $\frac{1}{y}\frac{dy}{dx} = -\frac{1}{2}\frac{1}{\sqrt{1-x}}$ $2\ln y \frac{dy}{dx} = -y$ $2\ln y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = -\frac{dy}{dx}$ $2\ln y \frac{d^2y}{dx^2} + \frac{2}{y}\left(\frac{dy}{dx}\right)^2 = -\frac{dy}{dx}$ **(i)** $2\ln y \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right)$ $+\frac{4}{y}\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2\left(-\frac{1}{y^2}\right)\left(\frac{dy}{dx}\right) = -\frac{d^2y}{dx^2}$ When $x = 0$, $y = e$ $\frac{dy}{dx} = -\frac{e}{2}$ $\frac{d^2y}{dx^2} = 0$ $\frac{d^3y}{dx^3} = -\frac{e}{8}$ Maclaurin's series of y is	

	$y = \mathrm{e} + \left(-\frac{\mathrm{e}}{2}\right)(x) + (0)\frac{x^2}{2!} + \left(-\frac{\mathrm{e}}{8}\right)\frac{x^3}{3!} + \dots$ $y = \mathrm{e} - \frac{\mathrm{e}}{2}x - \frac{\mathrm{e}}{48}x^3 + \dots$	
	(ii) $\ln y = \sqrt{1-x}$	
	$y = e^{\sqrt{1-x}}$	
	$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{e^{\sqrt{1-x}}}{2\sqrt{1-x}} = -\frac{\mathrm{e}}{2} - \frac{\mathrm{e}}{16}x^2 + \dots$	
	$\therefore \frac{\mathrm{e}^{\sqrt{1-x}}}{\sqrt{1-x}} = \mathrm{e} + \frac{\mathrm{e}}{8}x^2 + \dots$	
5	Paul, a life guard standing at point A along a straight stretch of the beach, looks through his binoculars and sees a boy clinging on to his overturned canoe and struggling to keep afloat at point B in the sea. P is the point on the straight stretch of the beach nearest to B such that $BP = 1$ km and $PA = 2$ km. To reach the boy, Paul first runs to Q and then swims in a straight line to B.  When Paul runs, he covers 1 km in 4 minutes. When he swims, he covers 1 km in 10 minutes.	
	(i) If $PQ = x$ km, $0 \le x \le 2$, show that the time T minutes taken by Paul to reach B is given by $T = 8 - 4x + 10\sqrt{1+x^2}$.	**[1]**
	(ii) Find the exact value of x such that he would take the shortest time to reach the boy.	**[4]**
	(iii) Hence, find the shortest time he would take to reach the boy, leaving your answers in exact form.	**[2]**
	Solution	
	(i) Total time, $T = T_{\text{run}} + T_{\text{swim}}$ $= 4\times(2-x) + 10\times\sqrt{1+x^2} = 8 - 4x + 10\sqrt{1+x^2}$	
	(ii) $\frac{\mathrm{d}T}{\mathrm{d}x} = -4 + 10\left(\frac{1}{2}\right)\left(1+x^2\right)^{-\frac{1}{2}}(2x)$	

$$= -4 + \frac{10x}{\sqrt{1+x^2}}$$

For shortest time,

$$\frac{dT}{dx} = 0 \Rightarrow 4 = \frac{10x}{\sqrt{1+x^2}}$$

$$\Rightarrow 2\sqrt{1+x^2} = 5x$$
$$\Rightarrow 4\left(1+x^2\right) = (5x)^2$$
$$\Rightarrow 21x^2 = 4$$
$$\Rightarrow x^2 = \frac{4}{21}$$

$$\Rightarrow x = \frac{2}{\sqrt{21}} \text{ since } x \geq 0$$

Method 1

x	$\left(\frac{2}{\sqrt{21}}\right)^-$	$\frac{2}{\sqrt{21}}$	$\left(\frac{2}{\sqrt{21}}\right)^+$
$\frac{dT}{dx}$	$-$	0	$+$
sketch	\	—	/

Method 2

$$\frac{d^2T}{dx^2} = \frac{10\sqrt{1+x^2} - 10x\left(\frac{2x}{2\sqrt{1+x^2}}\right)}{1+x^2}$$

$$= \frac{10(1+x^2) - 10x^2}{\left(1+x^2\right)^{3/2}}$$

$$= \frac{10}{\left(1+x^2\right)^{3/2}} > 0 \Rightarrow T \text{ is a minimum}$$

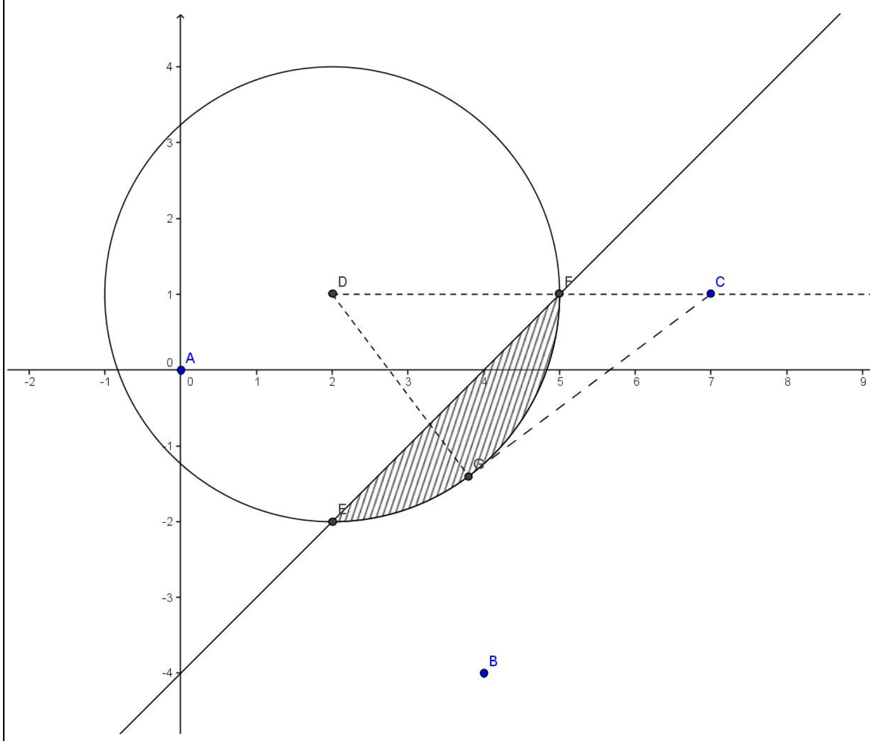
Hence, when $x = \frac{2}{\sqrt{21}}$, Paul would take the shortest time.

(iii) When $x = \frac{2}{\sqrt{21}}$, $T = 8 - 4\left(\frac{2}{\sqrt{21}}\right) + 10\sqrt{1+\frac{4}{21}}$

$$= 8 + 2\sqrt{21} \text{ minutes}$$

Shortest time taken by Paul $8 + 2\sqrt{21}$ minutes

6	A water tank has a horizontal base with a fixed cross sectional area A m^2. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds the depth of the water in the tank is x metres. If the depth is 1 m, it remains at this constant value. Show that	
	$\frac{dx}{dt}=k(1-x)$,	**[3]**
	where k is a constant.	
	Initially, the depth of the water is 2 m and is decreasing at a rate of 0.01ms^{-1}.	
	Find the exact time taken at which the depth of the water is 1.5m.	**[4]**
	Solution	
	$\frac{dV}{dt}=\frac{dV_{in}}{dt}-\frac{dV_{out}}{dt}$	
	$A\frac{dx}{dt}=a-bx$	
	$\frac{dx}{dt}=\frac{a-bx}{A}$	
	When $\frac{dx}{dt}=0$, $x=1$.	
	$a=b$	
	$\therefore \frac{dx}{dt}=\frac{a-ax}{A}=k(1-x), \quad k=\frac{a}{A}\cdots\cdots(1)$	
	$\int\frac{1}{(1-x)}dx=\int k\ dt$	
	$-\ln\lvert 1-x\rvert=kt+c$	
	When t = 0, x = 2	
	So c = 0	
	When x = 2, $\frac{dx}{dt}=-\frac{1}{100}$	
	So from (1), $k=\frac{1}{100}$	
	$-\ln\lvert 1-x\rvert=\frac{t}{100}$	
	When x = 1.5,	
	$t=100\ln 2$ s	
7	Draw on an Argand diagram, the loci with equations, $\lvert z-2-i\rvert=3$ and $\lvert z\rvert=\lvert z-4+4i\rvert$.	**[3]**
	(i) Given that the complex number z_1 is in the first quadrant and lies on both the loci. Find $\lvert z_1-i\rvert$.	**[1]**
	(ii) The complex number w lies in the common region determined by the inequalities, $\lvert z-2-i\rvert\le 3$ and $\lvert z\rvert\ge\lvert z-4+4i\rvert$.	
	In your diagram , shade the region which represents the possible values of w.	**[1]**

	(iii) Hence find the range of values of arg $(w - 7 - \mathrm{i})$ in exact form.	**[3]**
	Solution	
		
	(i) 5 units	
	(iii) From the diagram, $CD = 5$ and $CG = 3$	
	arg $(w-7-i) = \pi$ or $-\pi <$ arg $(w-7-i) \le \sin^{-1}(3/5) - \pi$	
8	**(i)** Verify that $\ln\left(\dfrac{r^2-1}{r+2}\right) = \ln(r+1) - \ln(r+2) + \ln(r-1)$ for $r \ge 2$.	**[1]**
	(ii) Prove by the method of differences that $$\sum_{r=2}^{n}\left[\ln\left(\frac{r^2-1}{r+2}\right)\right] = \ln 3 - \ln(n+2) + \ln((n-1)!)\ .$$	**[3]**
	(iii) Hence, find $\displaystyle\sum_{r=1}^{n}\left[\ln\left(\frac{r^2+2r}{r+3}\right)\right]$.	**[2]**
	(iv) Deduce that $\displaystyle\sum_{r=2}^{n}\left[\ln\left(\frac{(r-1)^2}{r+2}\right)\right] - \ln((n-1)!) < \ln 3$	**[2]**
	Solution	
	(i) $\ln\left(\dfrac{r^2-1}{r+2}\right) = \ln\left(\dfrac{(r+1)(r-1)}{r+2}\right) = \ln(r+1) - \ln(r+2) + \ln(r-1)$	

	(ii) $\sum_{r=2}^{n}\left[\ln\left(\frac{r^2-1}{r+2}\right)\right]=\sum_{r=2}^{n}\left[\ln(r+1)-\ln(r+2)+\ln(r-1)\right]$ $=\begin{bmatrix}\ln 3-\ln 4+\ln 1\\ +\ln 4-\ln 5+\ln 2\\ +...\\ +\ln(n)-\ln(n+1)+\ln(n-2)\\ +\ln(n+1)-\ln(n+2)+\ln(n-1)\end{bmatrix}$	
	$=\ln 3-\ln(n+2)+\ln 1+\ln 2+...+\ln(n-2)+\ln(n-1)$	
	$=\ln 3-\ln(n+2)+\ln((1)(2)...(n-1))$	
	$=\ln 3-\ln(n+2)+\ln((n-1)!)$, shown	
	(iii) $\sum_{r=1}^{n}\left[\ln\left(\frac{r^2+2r}{r+3}\right)\right]=\sum_{r=1}^{n}\left[\ln\left(\frac{(r+1)^2-1}{r+3}\right)\right]$ Replace r by $r-1$ $=\sum_{r-1=1}^{r-1=n}\left[\ln\left(\frac{((r-1)+1)^2-1}{(r-1)+3}\right)\right]$ $=\sum_{r=2}^{r=n+1}\left[\ln\left(\frac{r^2-1}{r+2}\right)\right]$	
	$=\ln 3-\ln(n+2)+\ln((n-1)!)+\ln\left(\frac{(n+1)^2-1}{(n+1)+2}\right)$ $=\ln 3-\ln(n+2)+\ln((n-1)!)+\ln\left(\frac{n(n+2)}{n+3}\right)$ $=\ln 3-\ln(n+3)+\ln(n!)$	
	(iv) $\frac{(r-1)^2}{r+2}<\frac{(r-1)(r+1)}{r+2}$ for $r\geq 2$ $\ln\left(\frac{(r-1)^2}{r+2}\right)<\ln\left(\frac{r^2-1}{r+2}\right)$ $\sum_{r=2}^{n}\ln\left(\frac{(r-1)^2}{r+2}\right)<\sum_{r=2}^{n}\ln\left(\frac{r^2-1}{r+2}\right)$	

$$\sum_{r=2}^{n}\ln\left(\frac{(r-1)^2}{r+2}\right)-\ln\left((n-1)!\right)<\sum_{r=2}^{n}\ln\left(\frac{r^2-1}{r+2}\right)-\ln\left((n-1)!\right)$$

$$\sum_{r=2}^{n}\ln\left(\frac{(r-1)^2}{r+2}\right)-\ln\left((n-1)!\right)<\ln 3-\ln(n+2)+\ln\left((n-1)!\right)-\ln\left((n-1)!\right)$$

$$\sum_{r=2}^{n}\ln\left(\frac{(r-1)^2}{r+2}\right)-\ln\left((n-1)!\right)<\ln 3-\ln(n+2)$$

Since $1<n+2\Rightarrow\ln(n+2)>0$

$\Rightarrow\sum_{r=2}^{n}\ln\left(\frac{(r-1)^2}{r+2}\right)-\left(\ln(n-1)!\right)<\ln 3$, deduced.

9 The hyperbola C has equation given by $2(x-h)^2-b(y+k)^2=18$, where b and h are positive real numbers and $0<k<\sqrt{2}$. An asymptote of C is given as $y=\sqrt{\frac{2}{b}}(x-h)-k$.

(i) Write down the equation of the other asymptote in terms of b, h and k. **[1]**

It is given that b = 9 and h = 3.

(ii) Sketch the graph of C, clearly indicating the asymptotes and the coordinates of the centre of the hyperbola. **[3]**

(iii) Find the range of values of m such that the line $y=mx-\sqrt{2}-k$ intersects the curve C at exactly two points. **[1]**

(iv) By drawing an additional graph on the diagram drawn in (ii), state the number of real roots of the equation $9(y+k)^2=y+k-18$. **[3]**

Solution

(i) $\frac{(x-h)^2}{9}-\frac{(y+k)^2}{18/b}=1$

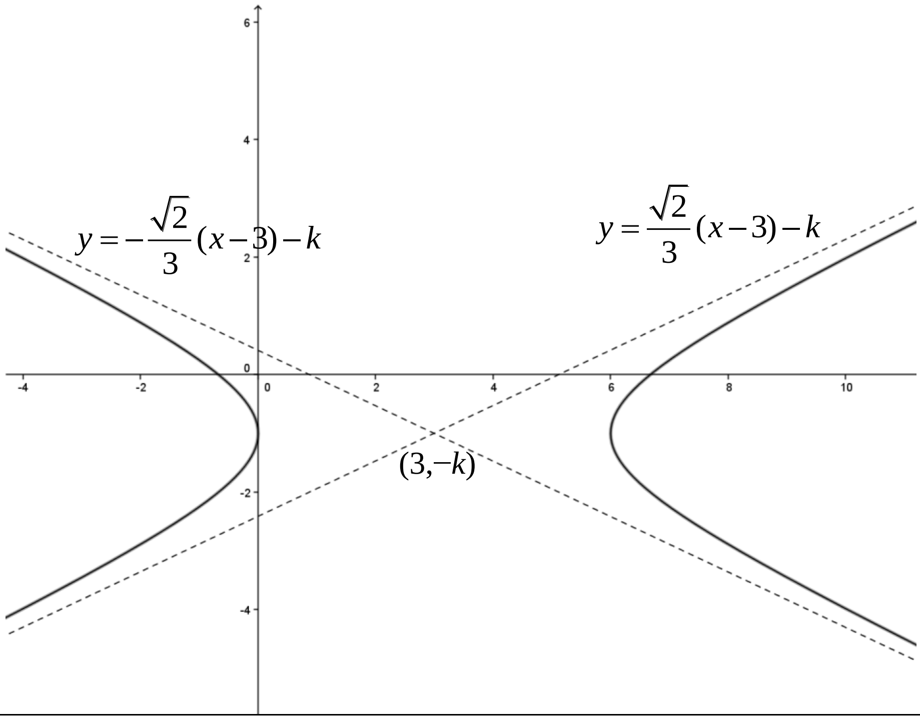
The asymptotes are $y+k=\pm\sqrt{\frac{2}{b}}(x-h)$

Therefore the second asymptote is

$y=-\sqrt{\frac{2}{b}}(x-h)-k$

(ii)

$$\frac{(x-3)^2}{9}-\frac{(y+k)^2}{2}=1\cdots\cdots(1)$$



(iii)

$$m<\frac{\sqrt{2}}{3},\ m\neq-\frac{\sqrt{2}}{3}$$

(iv)

$$9(y+k)^2=y+k-18$$

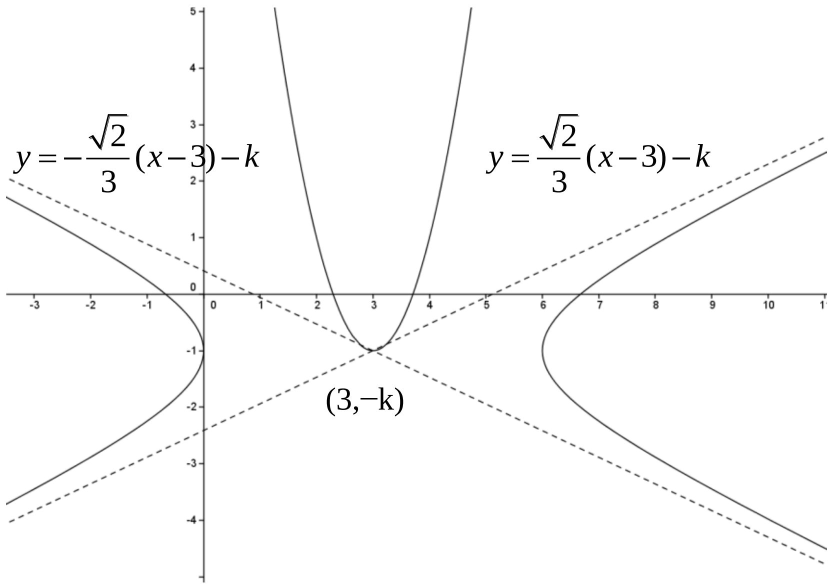
$$\frac{(y+k)^2}{2}=\frac{y+k-18}{18}$$

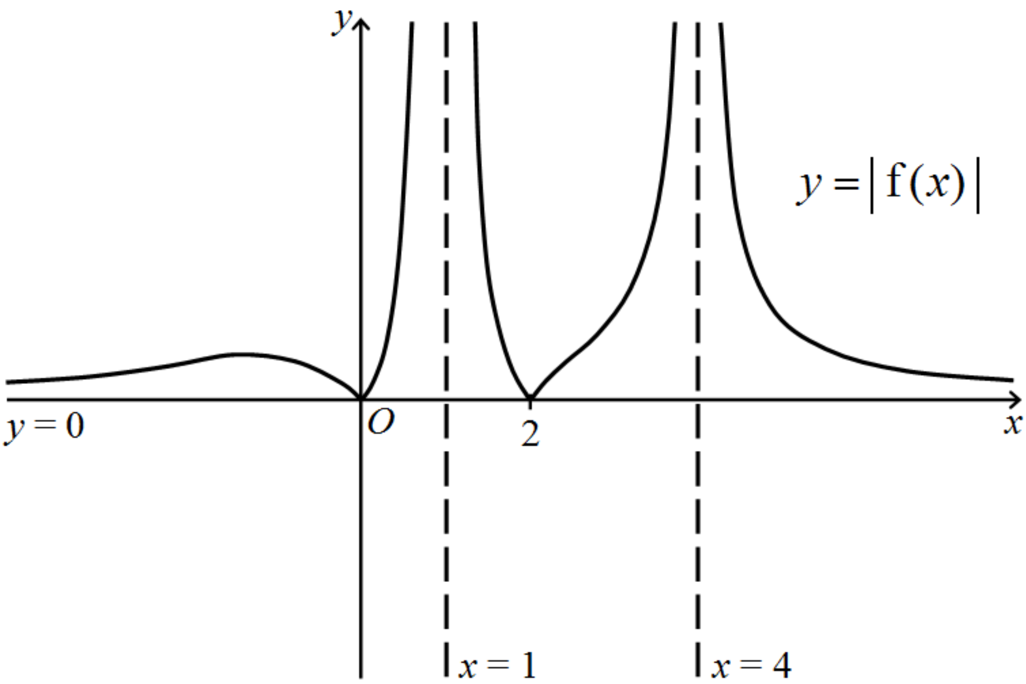
$$\frac{y+k}{18}-\frac{(y+k)^2}{2}=1\cdots\cdots(2)$$

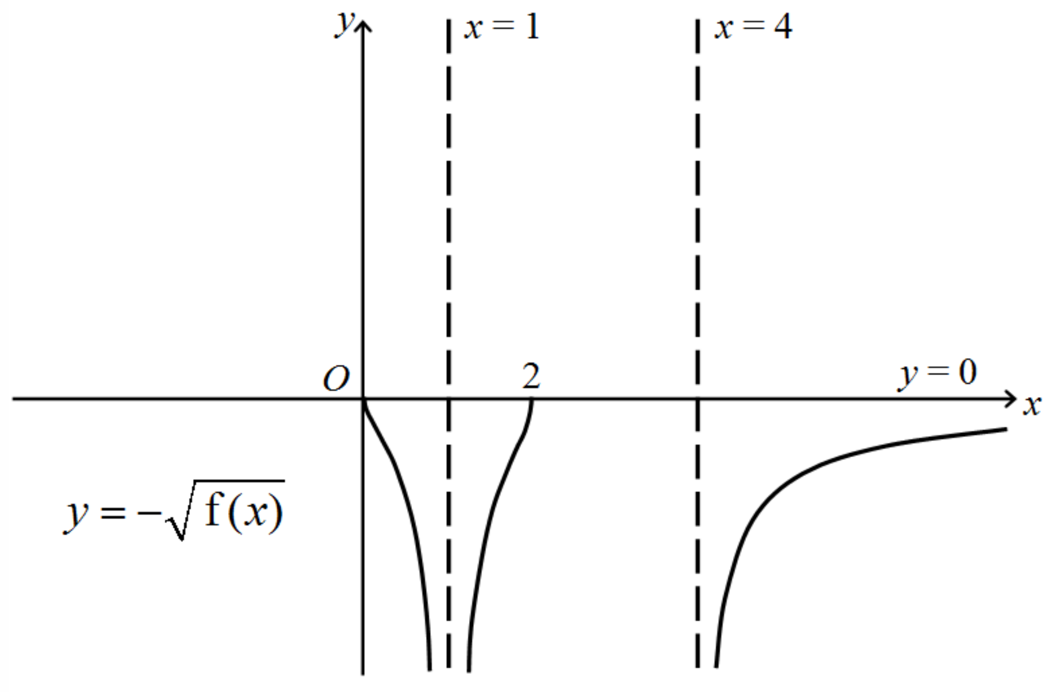
Comparing (1) and (2),

We let $\frac{y+k}{18}=\frac{(x-3)^2}{9}$

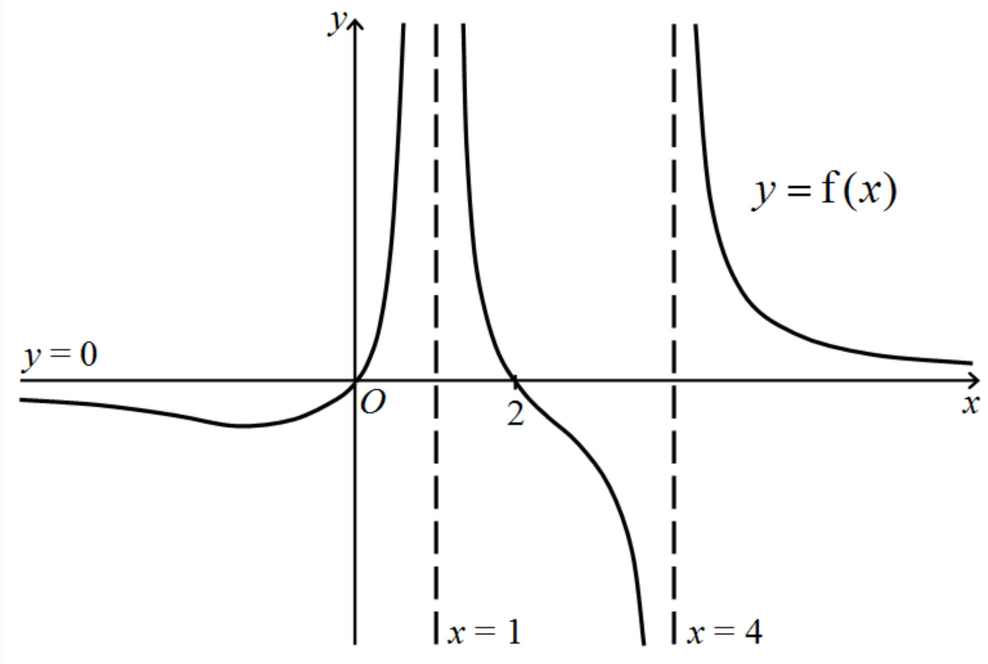
$\therefore y=2(x-3)^2-k$ is the additional curve to draw



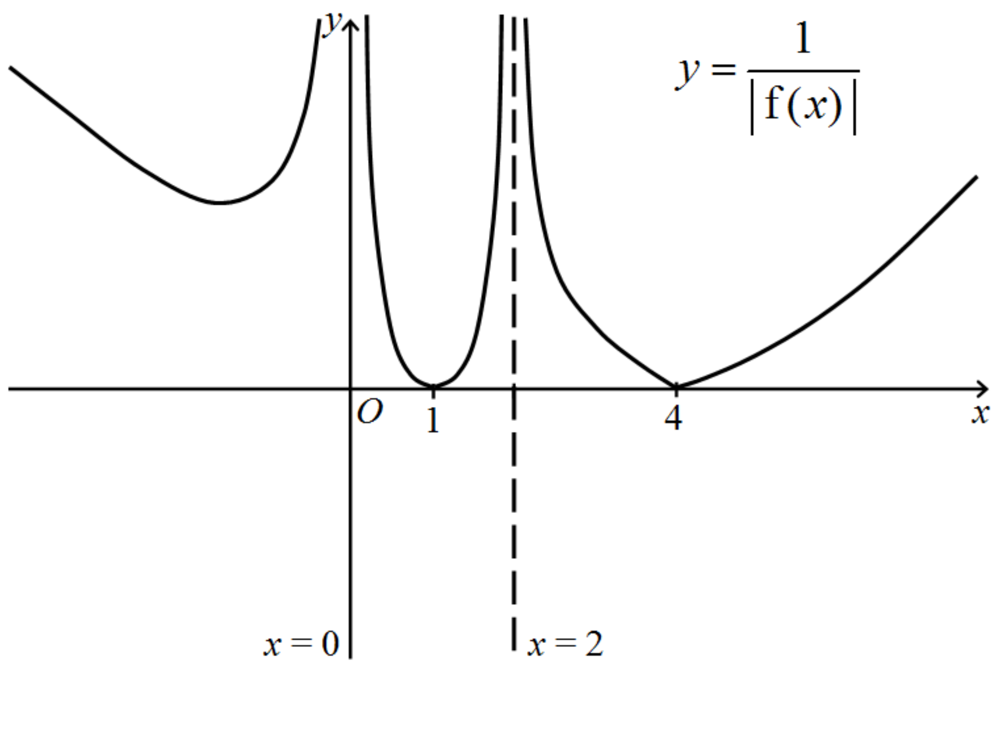
10	**(a)** The graphs of $y=\lvert \mathrm{f}(x)\rvert$ and $y=-\sqrt{\mathrm{f}(x)}$ are given below. 	



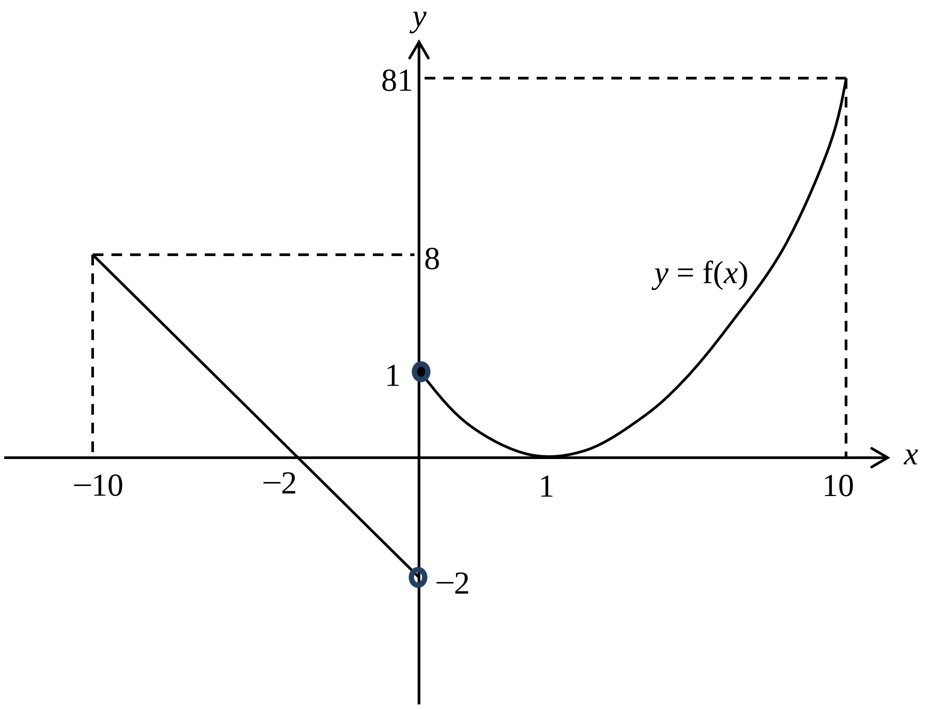
	Sketch separately the graph of	
	(i) $y = \mathrm{f}(x)$	**[3]**
	(ii) $y = \dfrac{1}{\lvert \mathrm{f}(x) \rvert}$	**[3]**
	stating clearly any asymptotes, turning points and axial intercepts.	
	(b) A graph with equation $y = \mathrm{f}(x)$ undergoes in succession, the following transformations: A: Scaling parallel to the y-axis by a factor of $\frac{1}{\mathrm{e}}$ B: Translation of 1 unit in the direction of the negative x-axis C: Scaling parallel to the x-axis by a factor of $\frac{1}{2}$ The equation of the resulting curve is given by $y = \mathrm{e}^{-2(x+1)}$. Find the equation $y = \mathrm{f}(x)$.	**[3]**
	Solution	
	(a) **(i)**	

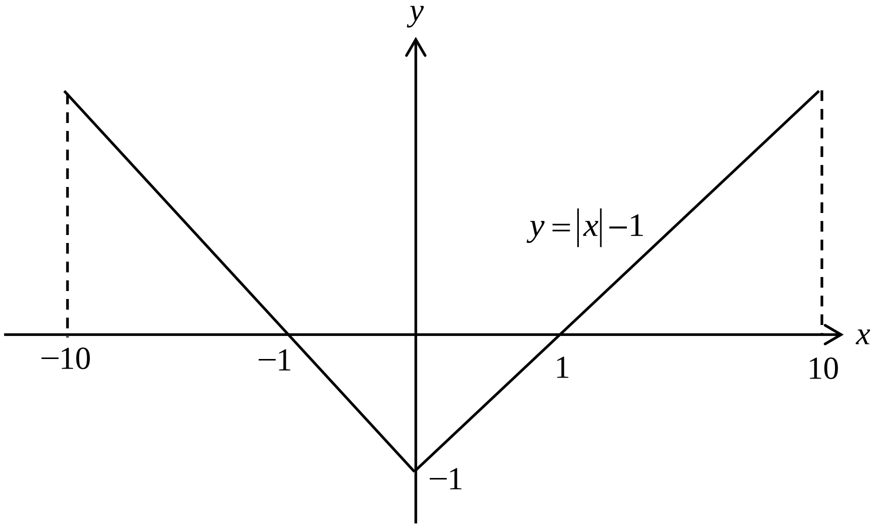


(ii)

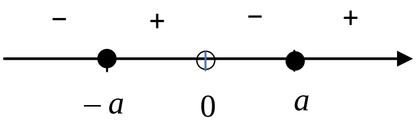


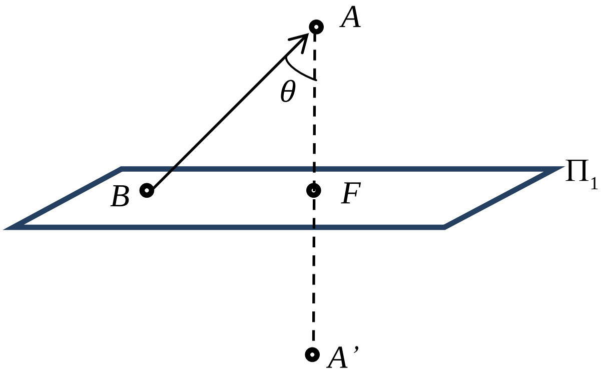
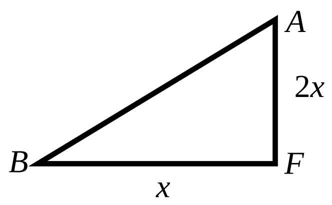
	(b) C': $y = e^{-2(\frac{1}{2}x+1)} = e^{-x-2}$		
	B': $y = e^{-(x-1)-2} = e^{-x-1}$		
	A': $y = e(e^{-x-1}) = e^{-x}$		

11	The functions f and g are defined by	
	$\mathrm{f}: x \mapsto \begin{cases} (x-1)^2, & 0 \le x \le 10, \\ -x-2, & -10 \le x < 0, \end{cases}$ $\mathrm{g}: x \mapsto \lvert x \rvert - 1, \quad -10 \le x \le 10.$	
	(i) Sketch the graph of f, indicating all axial intercepts. Hence explain why f^{-1} does not exist.	**[3]**
	(ii) Find the largest domain of f such that f^{-1} exists.	**[1]**
	(iii) With the domain found in (ii), explain how many real roots there will be for the equation $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$.	**[1]**
	(iv) Explain why fg exists. Hence find fg in similar form and its exact range.	**[4]**
	Solution	
	(i) 	
	Since the line $y = 0.5$ cuts the curve thrice, f is not a one-one function and so f^{-1} does not exist.	
	(ii) $[-10, 0) \cup (1+2\sqrt{2}, 10]$ or $(-10, 0) \cup [1+2\sqrt{2}, 10]$	
	(iii) Solving $\mathrm{f}^{-1}(x) = \mathrm{f}(x)$ is the same as solving $\mathrm{f}(x) = x$ and so finding the number of intersection points between the graph of f and the line $y = x$ in the domain in (ii) give rise to the number of real solution. So from the graph above, there will be 2 distinct real roots.	
	(iv) $\mathrm{R}_{\mathrm{g}} = [-1, 9]$ $\mathrm{D}_{\mathrm{f}} = [-10.10]$	
	Since $\mathrm{R}_{\mathrm{g}} \subseteq \mathrm{D}_{\mathrm{f}}$ so fg(x) exist.	

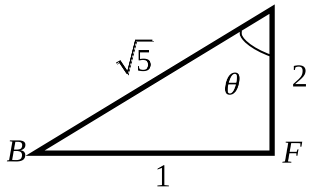
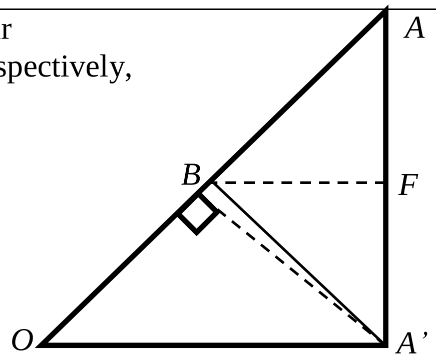
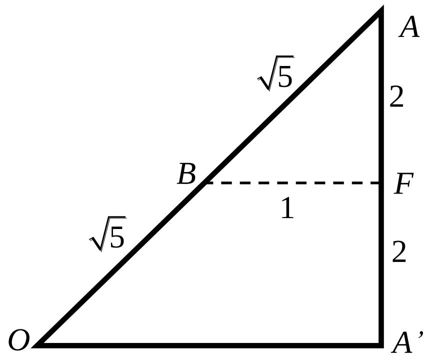
	$\text{fg}: x \mapsto \begin{cases} (\lvert x\rvert - 2)^2, & x \in [-10,-1] \cup [1,10] \\ -\lvert x\rvert - 1, & -1 < x < 1 \end{cases}$ 	
	$[-10,10] \xrightarrow[g]{} [-1,9] \xrightarrow[f]{} (-2,-1] \cup [0,64]$ $R_{fg} = (-2,-1] \cup [0,64]$	
12	(a) Find the integral $\int (\ln \frac{x}{2})^2 \, dx$.	**[3]**
	(b) Solve the inequality $$x^3 \geq \frac{a^4}{x},$$ leaving your answers in terms of *a*, where $1 < a < 3$. Hence find $\int_1^3 \left\lvert x^3 - \frac{a^4}{x} \right\rvert dx$, in terms of *a*.	**[6]**
	Solution	
	(a) $\int (\ln \frac{x}{2})^2 \, dx = [x(\ln \frac{x}{2})^2] - \int x[2(\ln \frac{x}{2})(\frac{2}{x})(\frac{1}{2})] dx$ $= [x(\ln \frac{x}{2})^2] - 2\int (\ln \frac{x}{2}) dx$	
	$= [x(\ln \frac{x}{2})^2] - 2[x(\ln \frac{x}{2}) - \int x(\frac{2}{x})(\frac{1}{2}) \, dx]$	
	$= x(\ln \frac{x}{2})^2 - 2x(\ln \frac{x}{2}) + 2x + c$	
	(b) $x^3 \geq \frac{a^4}{x}$ $\frac{x^4 - a^4}{x} \geq 0$	

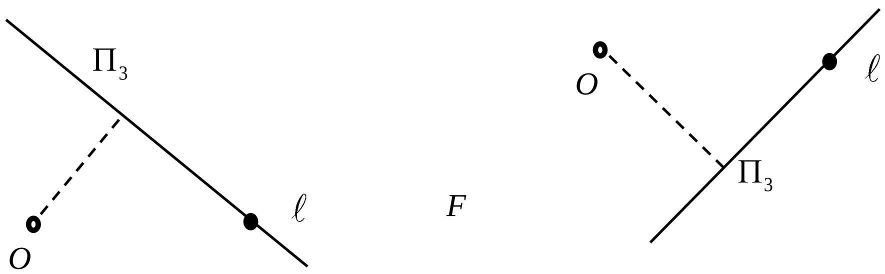
[TURN OVER]

	$\dfrac{(x^2-a^2)(x^2+a^2)}{x} \geq 0$ $\dfrac{(x-a)(x+a)(x^2+a^2)}{x} \geq 0$	
	Since $x^2+a^2>0$, $\therefore$ $\dfrac{(x+a)(x-a)}{x} \geq 0$ $-a \leq x < 0$ or $x \geq a$  	
	For $1 < x < a$, $x^3 - \dfrac{a^4}{x} < 0$ For $a < x < 3$, $x^3 - \dfrac{a^4}{x} > 0$ $\int_1^3 \left\| x^3 - \dfrac{a^4}{x} \right\| \mathrm{d}x$ $= \int_1^a -(x^3 - \dfrac{a^4}{x})\,\mathrm{d}x + \int_a^3 (x^3 - \dfrac{a^4}{x})\,\mathrm{d}x$	
	$= -[\dfrac{x^4}{4} - a^4 \ln\|x\|]_1^a + [\dfrac{x^4}{4} - a^4 \ln\|x\|]_a^3$	
	$= -[\dfrac{a^4}{4} - a^4 \ln a - (\dfrac{1}{4})] + [\dfrac{81}{4} - a^4 \ln 3 - (\dfrac{a^4}{4} - a^4 \ln a)]$ $= 2a^4 \ln a - a^4 \ln 3 - \dfrac{a^4}{2} + \dfrac{41}{2}$ $= a^4 \ln\left(\dfrac{a^2}{3}\right) - \dfrac{a^4}{2} + \dfrac{41}{2}$	
13	Referred to the origin *O*, the position vectors of the points *A* and *B* are	
	$4\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ respectively.	
	The plane Π_1 contains the points *B* and *F*, where *F* is the foot of perpendicular from *A* to the plane Π_1.	
	(i) Given that *AF* : *FB* = 2 : 1, find the exact area of triangle *AA'B*, where *A'* is the image of *A* about Π_1.	**[4]**
	(ii) Deduce the exact area of triangle *OAA'*.	**[2]**
	The equation of line ℓ is $\mathbf{r} = (2+3\lambda)\mathbf{i} + (-2-3\lambda)\mathbf{j} + (1+2\lambda)\mathbf{k}, \quad \lambda \in \mathbb{R}$.	
	(iii) Find the position vector of the point *C* on ℓ such that *OC* is perpendicular to ℓ.	**[3]**
	The plane Π_2 has equation $\mathbf{r} \cdot \begin{pmatrix}1\\1\\0\end{pmatrix} = 0$.	

	(iv) The 3 planes Π_1, Π_2 and Π_3 intersect along the line ℓ. If Π_3 is the plane that has the greatest possible distance from the origin, find its equation. Explain your answers clearly.	**[2]**
	Solution	
	(i) $\mathbf{a}=\begin{pmatrix}4\\-4\\2\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$	
		
	$\overrightarrow{BA}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$	
	Method 1	
	$\overrightarrow{BA}=\begin{pmatrix}2\\-2\\1\end{pmatrix}$  So $\left\lvert\overrightarrow{BA}\right\rvert=\left\lvert\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\rvert=3$ Thus $3^2=x^2+(2x)^2$	
	$x=\frac{3}{\sqrt{5}}$	
	Thus area of triangle $\Delta BAA'=2\times\frac{1}{2}\times\left(\frac{3}{\sqrt{5}}\right)\left(\frac{6}{\sqrt{5}}\right)$	
	$=\frac{18}{5}$	
	Method 2	
	Area of $\Delta BAA'=2\times\frac{1}{2}\times\left\lvert\overrightarrow{BF}\right\rvert\times\left\lvert\overrightarrow{AF}\right\rvert$	

A

	$\Delta BAA' = \left\|\overrightarrow{BA}\times\hat{\mathbf{n}}_1\right\|\left\|\overrightarrow{BA}\bullet\hat{\mathbf{n}}_1\right\|$ $= \left\|\overrightarrow{BA}\right\|\left\|\sin\theta\right\|\left\|\overrightarrow{BA}\right\|\left\|\cos\theta\right\|$  	
	$= \left\|\overrightarrow{BA}\right\|^2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$	
	$= \frac{2}{5}\left\|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\|^2$	
	$= \frac{18}{5}$	
	(ii) Method 1	
	Since $\Delta BAA'$ shared the same perpendicular height h as $\Delta OAA'$ with base BA and OA respectively, so area of $\Delta OAA' = 2\left(\frac{18}{5}\right) = \frac{36}{5}$  	
	Method 2	
	 Since ΔBAF is similar to $\Delta OAA'$	
	So $\frac{\text{Area of }\Delta OAA'}{\text{Area of }\Delta BAF} = \left(\frac{4}{2}\right)^2$	
	$\text{Area of }\Delta OAA' = 4\times\frac{1}{2}\text{Area of }\Delta BAA'$ $= \frac{36}{5}$	

	(iii) $\mathbf{c}=\begin{pmatrix}2\\-2\\1\end{pmatrix}+\lambda\begin{pmatrix}3\\-3\\2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$	
	$\therefore\begin{pmatrix}2+3\lambda\\-2-3\lambda\\1+2\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\-3\\2\end{pmatrix}=0$	
	$14+22\lambda=0$	
	$\lambda=-\frac{7}{11}$	
	So $\mathbf{c}=\frac{1}{11}\begin{pmatrix}1\\-1\\-3\end{pmatrix}$	
	(iv) Since ℓ lies on Π_3, B will then lie on Π_3.	
	To have Π_3 to be furthest away from the origin, **c** must be perpendicular to the plane because rotating Π_3 about ℓ will cause the distance from O to Π_3 to be shorter than the distance from O to the line ℓ if **c** is not perpendicular to the plane.  Cross sectional view of the plane and the line	
	$\therefore\Pi_3:\mathbf{r}\bullet\begin{pmatrix}1\\-1\\-3\end{pmatrix}=\begin{pmatrix}2\\-2\\1\end{pmatrix}\bullet\begin{pmatrix}1\\-1\\-3\end{pmatrix}=1$	

END OF PAPER