TEMASEK JUNIOR COLLEGE

JC2 Preliminary Examination

Higher 2

FURTHER MATHEMATICS **9649/01**

Paper 1 **12 September 2023**

3 hours

Additional Materials: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the answer booklet. If you need additional answer paper ask the invigilator for a continuation booklet.

Write your Civics Group and name on all the work that you hand in.

Write in dark blue or black pen on both sides of the paper.

You may use a soft pencil for any diagrams or graphs.

Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.

Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

You are expected to use a graphic calculator.

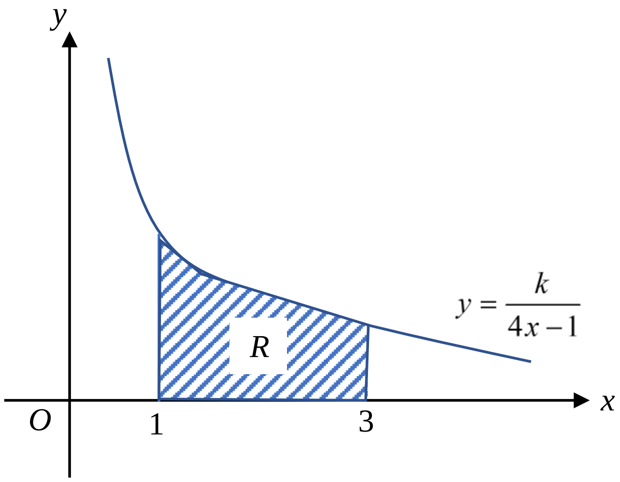
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.

Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

1



The diagram above shows part of the curve C with equation $y = \dfrac{k}{4x-1}$, where k is a positive constant. The region R is bounded by C, the x-axis, and the lines $x = 1$ and $x = 3$.

(a) Given that the area of R is approximately $\dfrac{3386}{31185}$ when Simpson's Rule with 5 ordinates is used, find the value of k. [4]

(b) Hence find the approximate value of the volume of solid formed when the region bounded by the curve with equation $y = \dfrac{k}{4x^2 - x}$, the x-axis, and the lines $x = 1$ and $x = 3$, is rotated completely about the y-axis. [2]

2 Prove by mathematical induction $\sqrt[n]{n} < 2 - \dfrac{1}{n}$, for $n \geq 2$. [6]

3 A curve C has polar equation $r = 1 + \cos 3\theta$, $\quad 0 \leq \theta \leq 2\pi$.

(a) Sketch C and determine the cartesian equation of the tangent lines to C at the pole. [3]

(b) Find in terms of θ, an integral for the area A enclosed by the graph and evaluate A to 3 decimal places. [2]

(c) Write down the polar coordinates of the points where r is maximum and show that the distance between any two points where r is maximum is $2\sqrt{3}$. [2]

4 The Happy Planet Index (HPI) is an index of human well-being and environmental impact that was introduced by the New Economics Foundation in 2006. A group of graduate students are interested to model the HPI for country A. It is estimated that at the start of 2010, the HPI for country A was 50 and that at the start of 2011, the HPI was recorded to be 54.9.

From previous studies, due to the efforts of the government, the graduate students found that the increase in HPI for country A from year n to $(n+1)$ is 1.025 times the increase from year $(n-1)$ to year n.

(a) Let H_n denote the HPI n years after the start of 2010. Find an expression for H_n in terms of n. [5]

Unfortunately, due to the presence of disease X that became prevalent in 2020, the measured HPI from 2021 onwards was found to have decreased by half of the HPI at the start of the previous year, undoing the previous efforts from the government of country A.

(b) Find the year in which the HPI for country A is first lowered to less than 10. [3]

5 By considering $\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k$, show that

$$\sum_{k=0}^{n-1}\cos k\theta\sec^k\theta = \cot\theta\sin n\theta\sec^n\theta,$$

provided θ is not an integer multiple of $\frac{\pi}{2}$. [5]

Hence or otherwise, show that

$$\sum_{k=0}^{n-1}2^k\cos\left(\frac{k\pi}{3}\right) = \frac{2^n}{\sqrt{3}}\sin\left(\frac{n\pi}{3}\right). \quad [2]$$

Given that $0<x<1$, show that

$$\sum_{k=0}^{n-1}\frac{\cos(k\cos^{-1}x)}{x^k} = \frac{\sin\left(n\cos^{-1}x\right)}{x^{n-1}\sqrt{1-x^2}}. \quad [3]$$

6 The point A represents a fixed complex number a where $\frac{\pi}{4} < \arg(a) < \frac{\pi}{2}$. The complex numbers $-\mathrm{i}a$ and $-a$ are represented by the points B and C respectively.

On a single clearly labelled Argand diagram, show the points A, B, C and the set of points representing all complex numbers z satisfying both relations

$$|z| \le |-\mathrm{i}a| \text{ and } \arg(z+a) = \arg(a) - \frac{\pi}{4}.$$ [4]

Find

(a) the minimum value of $|z|$ in terms of $|a|$, [2]

(b) the range of values of $\arg(z)$ in terms of $\arg(a)$. [2]

Hence or otherwise, determine if it is possible for $\mathrm{Re}\left(z^2\right) < 0$. [3]

7 The function f is defined by $\mathrm{f}(x) = k(x+1)^2 - (3+x)\mathrm{e}^{-x}$, where $k \in \mathbb{R}$.

(a) By sketching suitable graphs on a single diagram, find the range of values of k such that the equation $\mathrm{f}(x) = 0$ has

(i) two negative roots,

(ii) one negative and one positive real root. [3]

Let $k = 1$ for the rest of the question.

(b) Use linear interpolation to find a_1, the first approximation to the positive root α, correct to 2 decimal places. By considering the values of $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$ in the interval $[n, n+1]$, determine if a_1 is an under-estimate or over-estimate of α. [4]

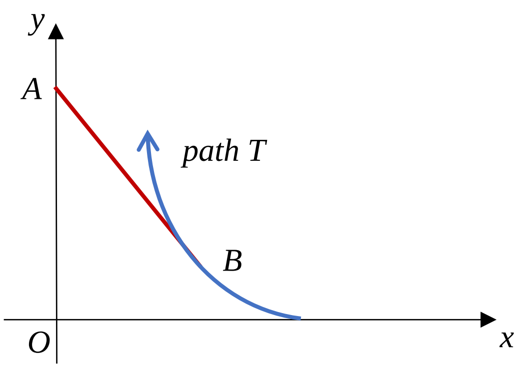
(c) Use one iteration of the Newton Raphson Method with a_1 as the first approximation to estimate the value of a_2, correct to 2 decimal places. [2]

(d) Prove that the following iterative formula

$$x_{n+1} = (1+x_n)^2 \mathrm{e}^{x_n} - 3$$

can be derived from the equation $(x+1)^2 - (3+x)\mathrm{e}^{-x} = 0$. Given that the iteration $x_{n+1} = \mathrm{F}(x_n)$ is convergent if $|\mathrm{F}'(x)| < 1$ around the neighbourhood of the root α, determine if the sequence is convergent. [3]

8 A thin straight rod AB of fixed length a has one end A located at the y-axis of a cartesian coordinate system with origin O. Initially, the rod lies horizontally with one end A at O and the other end B on the x-axis. The end A of the rod moves upwards along the y-axis and the other end B correspondingly moves in such a way that the rod AB is always tangential to the path T traced out by B as shown in the figure below.



(a) Letting the coordinates of B be (x, y), show that the equation of the path T obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt{a^2 - x^2}}{x}.$$ [1]

(b) By solving the differential equation in **(a)** using the substitution $x = a\sin\theta$, show that the cartesian equation of the path T can be written in the form

$$y = a\ln\left(\frac{a + \sqrt{a^2 - x^2}}{x}\right) - \mathrm{f}(x)$$

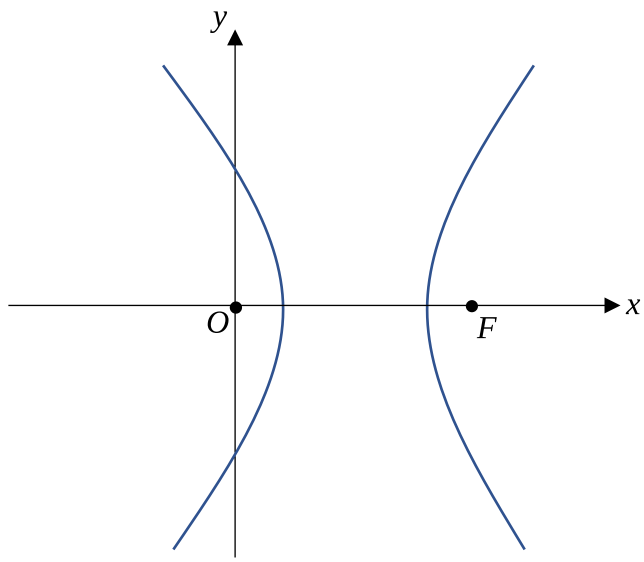
where $\mathrm{f}(x)$ is a function of x to be determined. [6]

The end B of the rod moves on T from the point $(a, 0)$ to the point where the x-coordinate is b. Find in terms of a and b, an expression for

(c) the height of the end A of the rod above O at this instant, [2]

(d) the length of the arc of T traversed by B. [3]

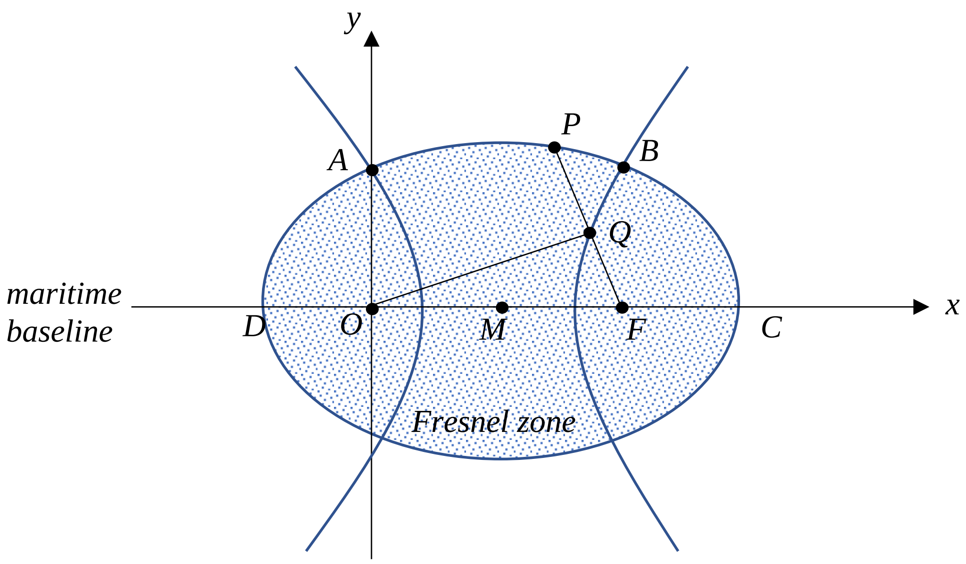
9 The diagram below shows a hyperbola H defined parametrically by $x = 3 + \sec\theta$ and $y = 2\sqrt{2}\tan\theta$. The origin O and the point $F(\alpha, 0)$ are the foci of H.



(a) Find the cartesian equation of H and deduce the value of α. [3]

(b) State the eccentricity of H. [1]

A *Fresnel zone* is an ellipsoidal region of space around a transmitting and receiving system. As shown in the diagram below (not drawn to scale) where 1 unit on the coordinate axis represents 1 km in actual distance, a *Fresnel zone* bordered by the ellipse E is mapped onto a maritime region with shipping routes travelling along the hyperbola H, with E meeting H at the points A and B, where A is on the y-axis. Two transmitters are placed at the foci of the ellipse E located also at the origin O and the point F, along a straight maritime baseline with the *Fresnel zone* terminating at 9 km due east and west of a beacon located at the point M, as indicated by the points C and D respectively. A receiver is placed on the point P at the edge of the *Fresnel zone*.



(c) Find the polar equation of E. [3]

(d) Boats travelling between points A and B are not allowed to sail within the *Fresnel zone*. By finding $\angle BOF$ or otherwise, find the distance the boats can travel along the arc AB. [4]

(e) A boat travelling at the right branch of H seeks to intercept some signals transmitting from F to the receiver at P. When the boat is at point Q, some of the signals will be reflected and travel towards O. Show that as P moves along E, the perimeter of the triangle OPQ remains a constant. [2]

10 The non-singular matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ d & 0 & 2 \end{pmatrix}$.

(a) **(i)** Find the possible values of d. [2]

(ii) By performing row operations on $(\mathbf{A}|\mathbf{I})$, where $\mathbf{I}$ is the 3×3 identity matrix, determine $\mathbf{A}^{-1}$. [4]

(b) The transformation T, from $\mathbb{R}^3 \to \mathbb{R}^3$, is defined by

$$\mathrm{T} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \mathbf{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

T maps the plane π_1 with cartesian equation $2x + y + 4z = 0$ to another plane π_2.

(i) Show that π_1 is a subspace of $\mathbb{R}^3$. [3]

(ii) Using $\mathbf{A}^{-1}$ found in part **(a)(ii),** or otherwise**,** find the cartesian equation of π_2. [3]

(c) The matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ with corresponding eigenvalues, 0, 1 and 2 respectively. Find $\mathbf{M}^n$ in terms of n, where n is a positive integer. [3]

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