Nanyang Junior College

JC1 H2 Mathematics 2019

Lecture Notes

Chapter	Topic
2	Functions
3	Advanced Graphing Techniques

Name:_____________________

CT: 19__

Chapter 2

Functions

At the end of this chapter, students should be able to

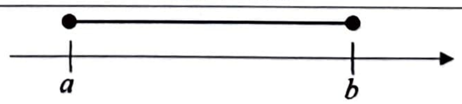
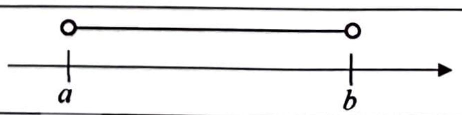
- understand the concepts of function, domain and range;
- use notations such as $f(x) = x^2 + 5$, $f : x \mapsto x^2 + 5$, $f^{-1}(x)$, $fg(x)$ and $f^2(x)$;
- find and define the inverse of a function and restrict the domain of a function to obtain the inverse function (if necessary)
- articulate the relationship between a function and its inverse as a reflection in the line $y = x$;
- obtain the composition of two functions and determine its domain and range;
- understand the conditions for the existence of inverse functions and composite functions;

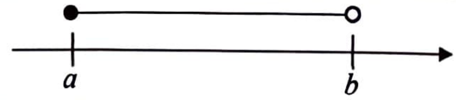
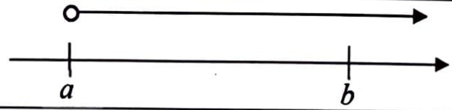
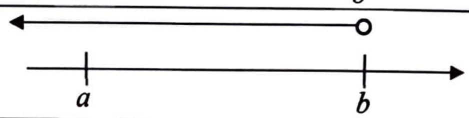
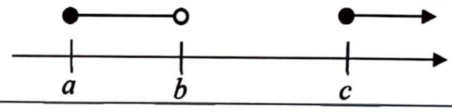
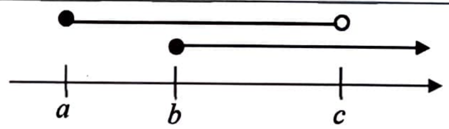
Functions arise when one quantity depends on one or more changing variables. For example, in mensuration, the area A of a circle depends on the radius r of the circle where the rule connecting A and r is given by $A = \pi r^2$. Many quantities in our daily lives such as human population, velocity and savings in a bank are also functions of time. In this chapter, we will study the different types of functions and the underlying concepts. We will begin by introducing set notations that are used throughout the course of study.

2.1 Set Notation

Symbol	Meaning	Example
$\mathbb{N}$	set of natural numbers, i.e., set of counting numbers	$\{1, 2, 3, 4, \cdots\}$
$\mathbb{Z}$	set of integers	$\{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
$\mathbb{Q}$	set of rational numbers	$\left\{\cdots, -\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{2}, \cdots\right\}$
$\mathbb{R}$	set of real numbers	$\left\{\cdots, -\pi, -3, -2.2, -\sqrt{2}, -1, 0, \frac{1}{3}, \sqrt{2}, 2, 3, \pi, \cdots\right\}$
$\varnothing$	empty set i.e., set with no elements	$\{ \}$

We often use the interval notation as an alternative to set notation for real numbers. Suppose a, b, c are real numbers such that $a < b < c$.

Interval notation	Set notation	Number line
$[a, b]$	$\{x \in \mathbb{R} : a \le x \le b\}$	
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$	

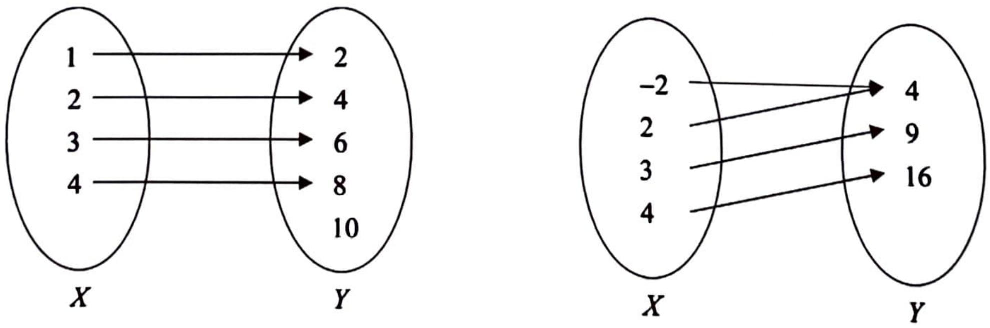
$[a,b)$	$\{x \in \mathbb{R} : a \le x < b\}$	
(a,∞)	$\{x \in \mathbb{R} : x > a\}$	
$(-\infty,b)$	$\{x \in \mathbb{R} : x < b\}$	
$[a,b)\cup[c,\infty)$ Union	$\{x \in \mathbb{R} : a \le x < b \text{ or } x \ge c\}$	
$[a,c)\cap[b,\infty)$ Intersection	$\{x \in \mathbb{R} : a \le x < c \text{ and } x \ge b\}$ (or simply $\{x \in \mathbb{R} : b \le x < c\}$)	

Note:

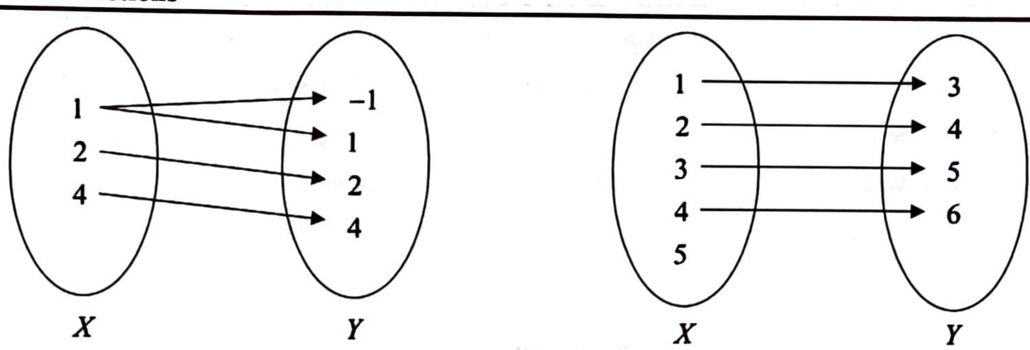
1. The use of square and round brackets in interval notation is used to include and exclude the end points of an interval respectively.
2. The use of shaded and unshaded circles on the number line is used to include and exclude the end points of an interval respectively.
3. The set notation $\{x \in \mathbb{R} : a \le x \le b\}$ means the set of all x such that x is an element of the real numbers and x is between a and b inclusive.
4. The variable x in the set notation is a dummy variable and can be replaced by any other letter such as s or y, e.g., $[a,b] = \{x \in \mathbb{R} : a \le x \le b\} = \{s \in \mathbb{R} : a \le s \le b\} = \{y \in \mathbb{R} : a \le y \le b\}$.
5. For the case of intersection, the expression should be simplified before stating it as the final answer. That is, if $b < c$, then $[a,c)\cap[b,\infty) = [b,c)$.

2.2 What is a function?

Suppose the elements of the set X are related to the elements of a second set Y, then this association between X and Y is called a *relation* from X to Y. Figure 1 gives some examples of relations between two sets X and Y:



(a) (b)



(c) (d)

Figure 1: Examples of relations between two sets X and Y

A rule or relation may also be represented algebraically as an equation, numerically as a table of values or graphically on the Cartesian plane. For example, the relations in Figures 1(a) and 1(b) can be represented in the form $y = 2x$ and $y = x^2$ respectively.

A **function (or mapping)** is a relation from X to Y where **every input** x from the set X gives a **unique** output y (belonging to the set Y). The relations in Figures 1(a) and 1(b) are functions.

We typically use the letters f, g, h to refer to (and distinguish between) different functions. For example, we may use f to denote the function in Figure 1(a) and g for the function in Figure 1(b).

The **set of all input values**, X, in which a function is defined on is known as the **domain** of the function. The notation D_f is used for the domain of the function f, thus, for Figure 1(a), $D_f = \{1,2,3,4\}$.

Also, for Figure 1(a), we say that 2, 4, 6 and 8 are **images** (or outputs) of 1, 2, 3 and 4 respectively. Thus, the **range** of f, denoted by R_f, is the **set of all images** of f, i.e., $R_f = \{2,4,6,8\}$.

Note: Domain, D_f and range, R_f are sets and must be presented in the proper set notation or interval form.

Not all relations are functions. For example, the relation in Figure 1(c) is not a function as the element 1 in set X has two images. Also, the relation in Figure 1(d) is not a function as the element 5 in set X has no image. We can use a graphical method show that a relation is **not** a function (see Figure 2).

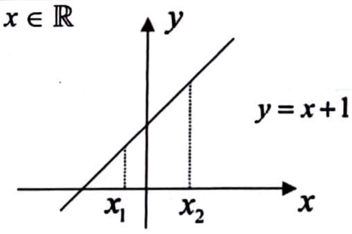
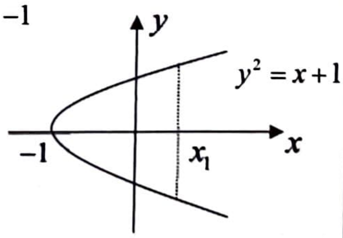
(a) $y = x+1,\ x \in \mathbb{R}$ 	(b) $y^2 = x+1,\ x \geq -1$ 
The relation is a function since each $x \in X$ is mapped onto one and only one $y \in Y$.	The relation is **not** a function since **there exists** a $x_1 \in X$ that is mapped onto two distinct y values (e.g. when $x = 3$, $y = -2$ or $y = 2$).

Figure 2

We can also consider the function f as a 'input-output machine' that takes in an element $x \in D_f$ and returns an element $y \in R_f$. For example, a machine which takes in a number and returns double that of the original number, as captured in Figure 3, can be expressed by equation $y = 2x$. We write $f(x) = 2x$ or $f : x \mapsto 2x$ (read f such that x is mapped to $2x$) to represent the **rule** for the function f.

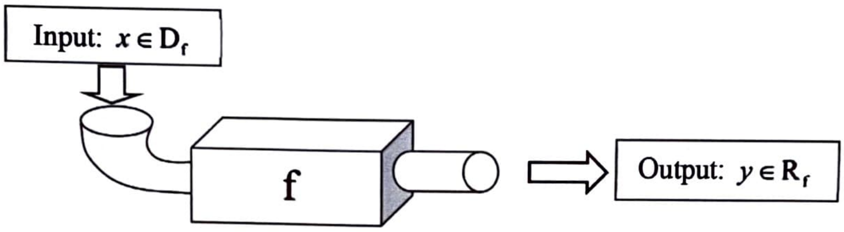


Figure 3: An 'input-output machine'

2.3 Graphical representation of functions

Consider two functions, f and g having the same rule, e.g., $f : x \mapsto x^2$ and $g : x \mapsto x^2$, but with different domains, e.g., $D_f = \mathbb{R}$ and $D_g = \mathbb{R}^+$. The functions f and g may be represented graphically on the Cartesian plane in Figure 4:

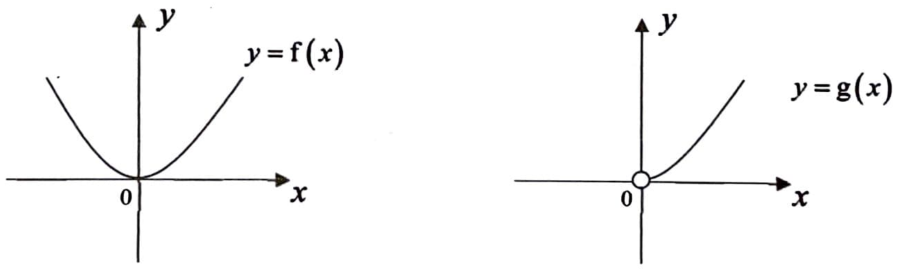


Figure 4

Even though both functions f and g have the same rule, there are distinct differences between them, as illustrated in the following table.

	Domain	Range
f	$\mathbb{R}$ or $(-\infty, \infty)$	$\mathbb{R}^+ \cup \{0\}$ or $[0, \infty)$
g	$\mathbb{R}^+$ or $(0, \infty)$	$\mathbb{R}^+$ or $(0, \infty)$

Thus, in order to **define a function, we need to provide <u>both</u> the rule and the domain,** i.e., $f : x \mapsto x^2$, $x \in \mathbb{R}$ and $g : x \mapsto x^2$, $x > 0$.

2.4 Using the Graphing Calculator to graph functions and finding the range

Example 1:

Use the G.C. to sketch the graph of the following functions and hence find the exact range for each of the functions:

(a) $\mathrm{f}: x \mapsto x^2 + 2x + 7,\ x > 0,$

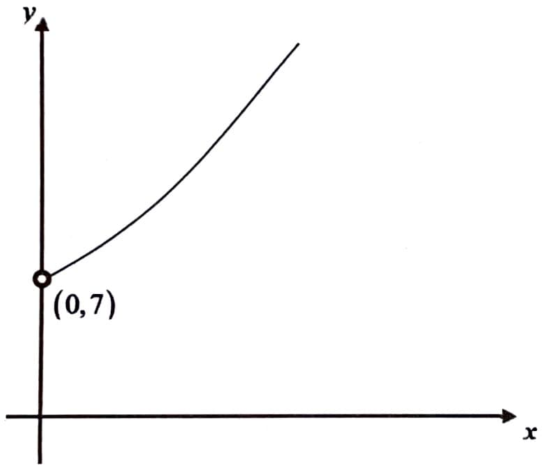
(b) $\mathrm{g}: x \mapsto x^2 + 2x + 7,\ -2 \le x \le 1,$

(c) $\mathrm{h}: x \mapsto \mathrm{e}^{x^2+2x+1},\ -2 \le x < \frac{1}{2},$

(d) $\mathrm{j}: x \mapsto -\ln\left(x^2 + 2x + 7\right) + 6,\ 1 \le x \le 3.$

Solution:

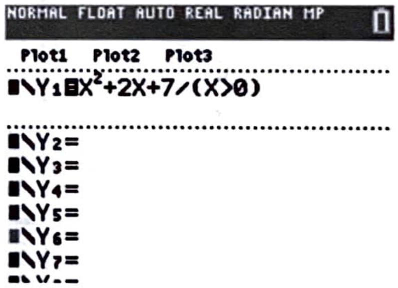
(a)



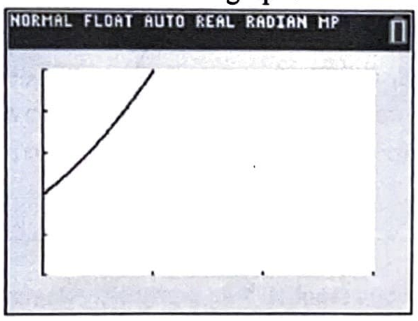
From the graph, $R_f = (7, \infty)$

ThinkZone:

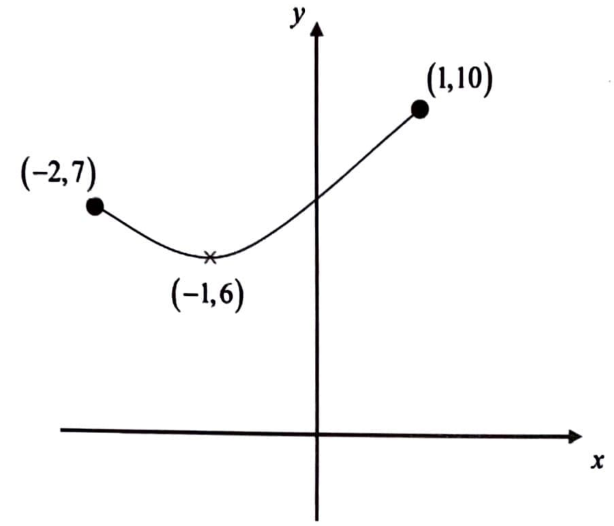
Press [Y=] and key in the function in Y_1. Specify the domain by pressing [÷] [(] $[X, T, \theta, n]$ and [2nd] [MATH] to access the Test Menu. Choose the appropriate inequality and close the domain with [)].



Use the zoom features to get a clearer view of the graph.



(b)

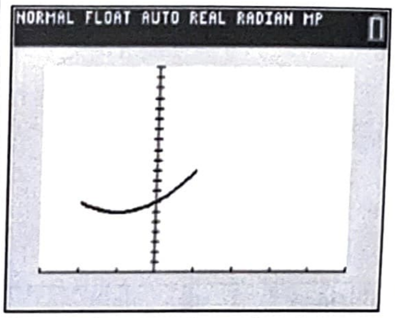


From the graph, $R_g = [6, 10]$

We have to key in the function and domain as

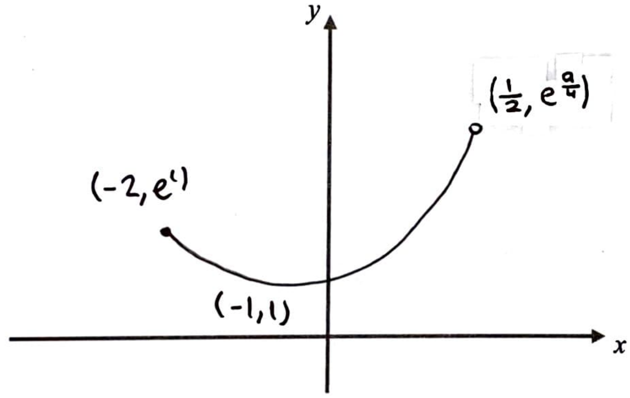
$Y_1 = x^2 + 2x + 7 / ((-2 \le x)(x \le 1))$.

GC screenshot:



Can we just use the end points to determine the range of the function? Under what circumstances can we rely solely on the end points?

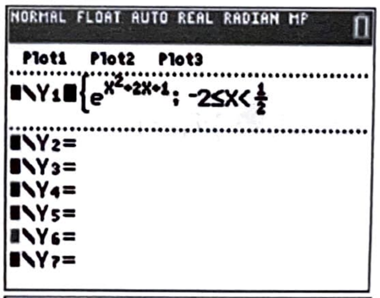
(c)

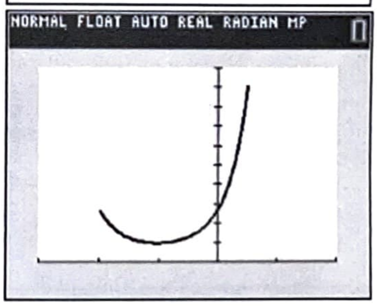


From the graph, $R_h = [1, e^{\frac{9}{4}})$

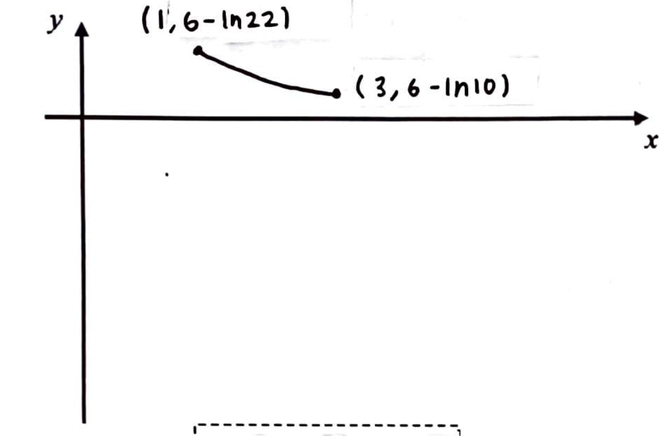
Instead of restricting the domain using the method in (b), we can also use the piecewise function command to plot the graph.
Press [Y=] [MATH] and select B:Piecewise(. Choose 1 piece.

GC screenshot:



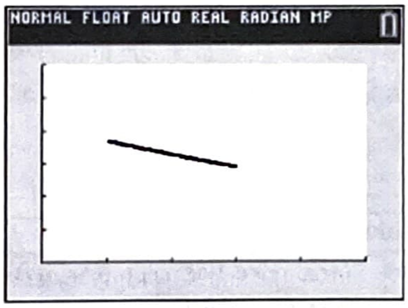


(d)



From the graph, $R_j = [6-\ln 10, 6-\ln 22]$

GC screenshot:



It looks like a straight line. Is it really a straight line?
Under what circumstances can we rely solely on the end points to get the range of the function?

2.5 One-One Functions

A function is said to be a **one-one function** if distinct elements in the domain have different images. Mathematically, a function $f : X \mapsto Y$ is one-one if and only if for all $x_1, x_2 \in X$, $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

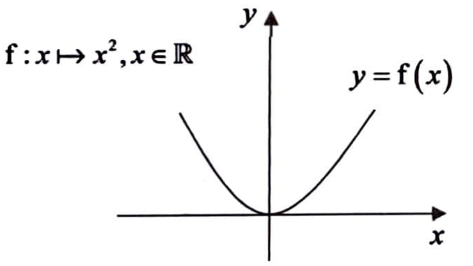


Figure 5(a)

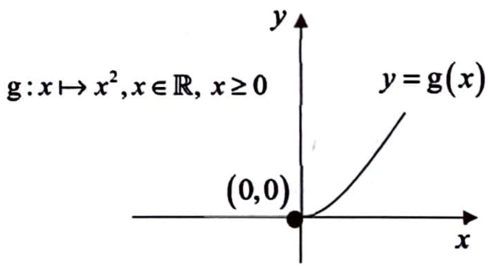


Figure 5(b)

We can see that for the function f, we have $-2 \neq 2$, but $f(-2) = f(2)$, however, for the function g, the above condition holds, i.e., different inputs for g gives rise to different images $y = g(x)$. Thus, we conclude that g is one-one while f is not one-one. Note that g is a **restriction** of f to the non-negative real numbers. It is also the **largest possible domain** of non-negative real numbers for which the function is one-one.

<u>**Horizontal line test**</u>

A function f is **one-one** if every horizontal line $y = k$, $k \in \mathbb{R}$, intersects the graph of f at most once.

Note:

A function f is **not one-one** if there exists a horizontal line $y = k$, $k \in \mathbb{R}$ (the value of k should be specified) which intersects the graph of f more than once.

The horizontal line test is a pictorial way to show that a given function is not one-one.

Example 2:

For each of the following functions, determine if it is one-one. If the function is not one-one, find a largest possible domain for which the function is one-one.

(a) $f: x \mapsto (x-1)^2$, $x \in \mathbb{R}$,

(b) $g: x \mapsto (x-1)^2$, $x \geq 1$,

(c) $h: x \mapsto \sin x$, $-2\pi \leq x \leq 2\pi$.

Solution:

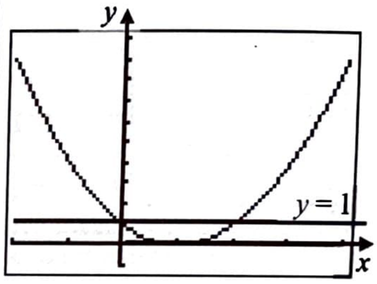
ThinkZone:

(a) Since the line y=1 intersects the curve y=f(x) more than once, f is not one-one

Alternatively, since $f(0) = f(2) = 1$, f is not one-one.

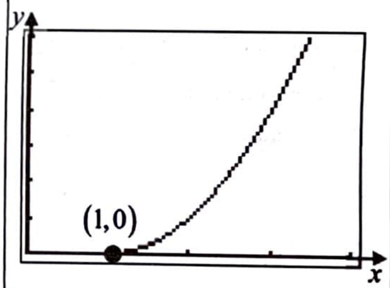
f is one-one is for either $\{x \in \mathbb{R} : x \leq 1\}$ or $\{x \in \mathbb{R} : x \geq 1\}$

i.e. The largest possible domain for which f is one-one is either $(-\infty, 1]$ or $[1, \infty)$



(b)

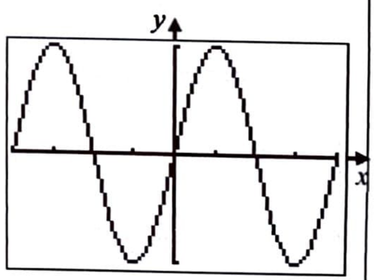
Since every horizontal line $y = k$, $k \in \mathbb{R}$ intersects the curve of $y = g(x)$ at most once, is one-one.



(c) Since the line y=0 intersects the curve $y = h(x)$ more than once, h is not one-one.

From the graph, h is one-one for $\left\{x \in \mathbb{R} : -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\right\}$.

Hence, a largest **possible** domain for which h is one-one is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



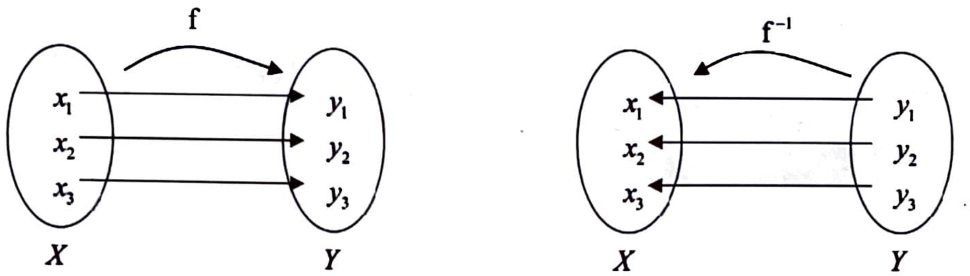
Remark: This is known as the **principal domain** for $\sin(x)$ to be a one-one function.

Can you identify other possible domain for h to be one-one?

$D_g = \mathbb{R} \setminus \{1\}$ or $D_g = (-\infty, 1) \cup (1, \infty)$

2.6 Inverse functions

Consider a **one-one function**, $f: X \mapsto Y$. Then, the function which maps Y back onto X is known as the inverse function of f, denoted by f^{-1} (see diagram below). The inverse function f^{-1} exists if and only if f is a one-one function.



If $y = f(x)$ is the rule of f, then we have $f^{-1}(y) = x$.
For example, if $f: x \mapsto 3x$, $x \in \mathbb{R}$, $x > 0$. To find the rule of the inverse function f^{-1}, let $y = f(x) = 3x$ and **make x the subject**, i.e. $x = f^{-1}(y) = \frac{y}{3}$.
Since we usually use x to represent any element in the domain of a function and x and y are just dummy variables, we interchange y with x to get $y = f^{-1}(x) = \frac{x}{3}$.
When $x = 2$, we have $f(2) = 3(2) = 6$. Since f^{-1} is the function which maps 6 onto 2, $f^{-1}(6) = \frac{6}{3} = 2$.

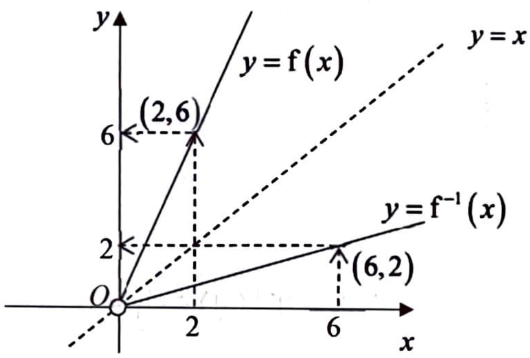


Figure 6

Figure 6 gives an illustration of the relationship between the graphs of $y = f(x) = 3x$ and $y = f^{-1}(x) = \frac{x}{3}$ on the same diagram. The graphs are reflections of each other in the line $y = x$.

<u>Important properties for inverse functions:</u>

1. The inverse function f^{-1} exists if and only if f is a one-one function.
2. If f^{-1} exists, then $D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$.
3. If f is a one-one function, so is f^{-1}. Furthermore, $\left(f^{-1}\right)^{-1} = f$.
4. The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ in the line $y = x$.

Note: The notation f^{-1} refers to the inverse of function f, and not $\frac{1}{f}$. $\left[\text{i.e. } f^{-1}(x) \neq [f(x)]^{-1} = \frac{1}{f(x)}\right]$

Example 3:

For each of the following functions, state whether the inverse function exists. If it exists, define the inverse function in a similar form.

(a) $f: x \mapsto x^2 - 1,\ -1 < x < 2$,

(b) $g: x \mapsto \dfrac{1}{x-1},\ x > 1$.

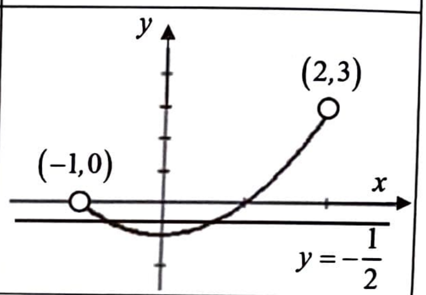
Solution:

(a)

Since the line $y = -\frac{1}{2}$ intersects the curve $y = f(x)$ more than once, f is not one-one.

Thus, the inverse function does not exist.

ThinkZone:



(b) Since every horizontal line $y = k$, $k \in \mathbb{R}$ intersects the curve of $y = g(x)$ at most once, g is one-one, i.e., g^{-1} exists.

Let $y = \dfrac{1}{x-1},\ x > 1$.

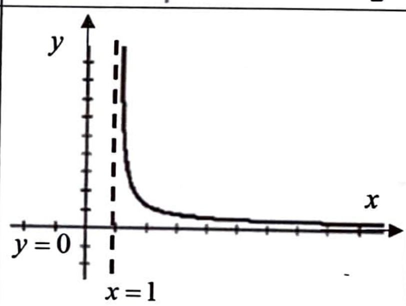
Making x the subject, we have $x = \frac{1}{y} + 1$

$\therefore x = g^{-1}(y) = \frac{1}{y} + 1$

and since $R_g = D_{g^{-1}} = (0, \infty)$

Interchanging y with x, we have $g^{-1}(x) = \frac{1}{x} + 1,\ x > 0$

ThinkZone:



Example 4:

Consider the function $f(x) = \frac{9}{5}x + 32$, where x is the temperature in degree Celsius and $f(x)$ is the temperature in degree Fahrenheit. Find a formula to convert degree Fahrenheit to degree Celsius.

Solution:	ThinkZone:
Let $y = \frac{9}{5}x + 32$ $\therefore x = \frac{5}{9}(y - 32)$ $f^{-1}(x) = \frac{5}{9}(x - 32)$	So $f^{-1}(x)$ is the formula to convert degree Fahrenheit to degree Celsius, where x is the degree Fahrenheit

Example 5:

The function f is defined by $f: x \mapsto (x+1)^2+2$, $x \geq a$, where a is a real constant.

(i) Find the smallest value of a such that f is a one-one function.

For the value of a found in (i),

(ii) find the inverse function f^{-1} ~~is~~ in a similar form.

(iii) sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on a single diagram, showing the relation between these two graphs clearly.

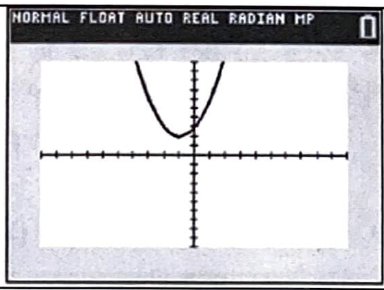
Solution:

(i) From the graph, the curve has a minimum point at $(-1, 2)$.

Thus f will be one-one for $a \geq -1$

Hence the smallest value of a is -1.

ThinkZone:



(ii) Let $y = (x+1)^2+2$, $x \geq -1$. Note that $R_f = [2, \infty)$.

Making x the subject, we have

$\pm\sqrt{y-2} - 1 = x$

Since $x \geq -1$, we must have $x = -1 + \sqrt{y-2} = f^{-1}(y)$

Thus, $f^{-1}: x \mapsto -1+\sqrt{x-2}$, $x \geq 2$

Note:

$y = f(x) \Leftrightarrow f^{-1}(y) = x$. Thus by replacing y with x, we have

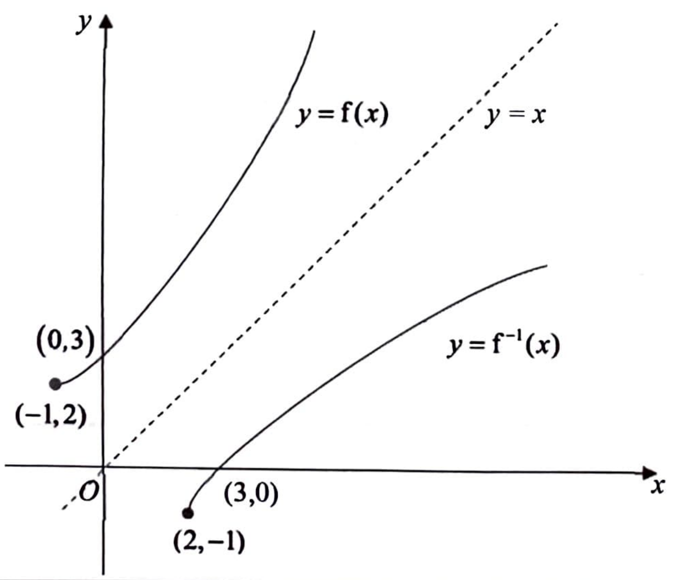
$f^{-1}(y) = -1+\sqrt{y-2}$

$\Rightarrow f^{-1}(x) = -1+\sqrt{x-2}$

Remember $D_{f^{-1}} = R_f$

(iii)

$y = f(x)$ and $y = f^{-1}(x)$ are reflections in the line $y = x$.



Plot $y = f(x)$ and $y = x$ into the GC. To draw the $y = f^{-1}(x)$, press [2nd] [prgm] and select 8: DrawInv. Then press [alpha] [trace], select Y1 and press [enter].

Is it necessary to use the same scale when sketching both $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram?

use similar scale for both y and x axis

Note that from the graph of $y=\mathrm{f}(x)$, the points on the graph of $y=\mathrm{f}^{-1}(x)$ can be obtained by swapping the x and y-coordinates.	We **must** also include the graph of $y=x$ when sketching the graphs of both $y=\mathrm{f}(x)$ and $y=\mathrm{f}^{-1}(x)$ on the same diagram. This is to illustrate the relationship between the two graphs.

Example 6:

The function f is defined by $\mathrm{f}:x\mapsto \ln(x+3),\ x>-3$.

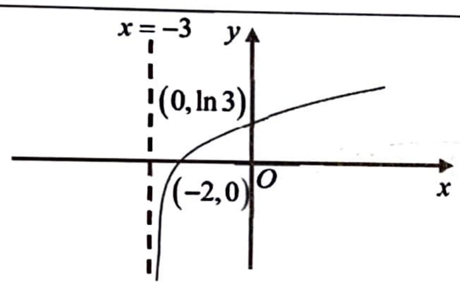
(i) Sketch the graph of $y=\mathrm{f}(x)$ and state its range.

(ii) Find the inverse function f^{-1} in similar form and state the range of f^{-1}.

(iii) Sketch the graph of $y=\mathrm{f}^{-1}(x)$ on the same axes as $y=\mathrm{f}(x)$ and state the relationship between the two curves.

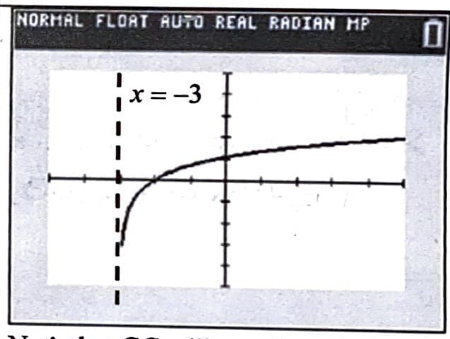
Solution:

(i)



From the graph, $R_f = (-\infty, \infty) = \mathbb{R}$

ThinkZone:



Note that GC will not show the asymptote. The use of grid line will help.

(ii) Let $y=\ln(x+3),\ x>-3$.

Making x the subject, we have $x+3=e^{y}$, i.e., $x=e^{y}-3=\mathrm{f}^{-1}(y)$

and since $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\mathbb{R}$

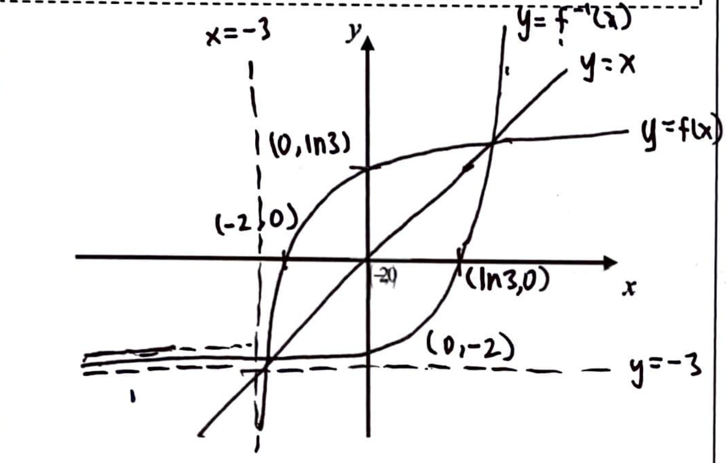
Thus, $\mathrm{f}^{-1}:x\mapsto e^{x}-3,\ x\in\mathbb{R}$ and $R_{\mathrm{f}^{-1}}=D_{\mathrm{f}}=(-3,\infty)$.

Note:

$y=\mathrm{f}(x)\Leftrightarrow \mathrm{f}^{-1}(y)=x$.

(iii)

$y=f(x)$ and $y=f^{-1}(x)$ are reflections in the line $y=x$.



Note that the vertical asymptote of the graph of $y=\mathrm{f}(x)$ is "mapped" to the horizontal asymptote of the graph of $y=\mathrm{f}^{-1}(x)$ via reflection in the line $y=x$.

Note that the intersection points of the graph of $y=\mathrm{f}(x)$ and the graph of $y=\mathrm{f}^{-1}(x)$ **must** lie on the line $y=x$. (WHY?)

Example 7:

The function f is defined by $\mathrm{f}: x \mapsto -x^2+6x-5,\ x \in \mathbb{R}$.

(i) Explain why the inverse function f^{-1} does not exist.

(ii) If the domain of f is restricted to a subset of $\mathbb{R}$ for which $x \le a$, find the largest value of a such that f^{-1} is defined. Find f^{-1} in this case, in similar form.

(iii) Using the restricted domain found in (ii), sketch the graphs of $y=\mathrm{f}^{-1}(x)$ on the same axes as $y=\mathrm{f}(x)$ and find the exact solution of $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$.

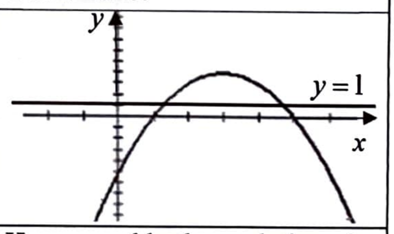
Solution:

(i)

Since the line $y=1$ intersects the curve $y=\mathrm{f}(x)$ more than once, f is not one-one.

Thus, f^{-1} does not exist.

ThinkZone:



(ii) From the graph, f is one-one for $x \le 3$.

Hence, the greatest $a=3$ and $D_{\mathrm{f}}=(-\infty,3]$.

Let $y=-x^2+6x-5,\ x \le 3$.

By completing the square, we have $y=-(x-3)^2+4$.

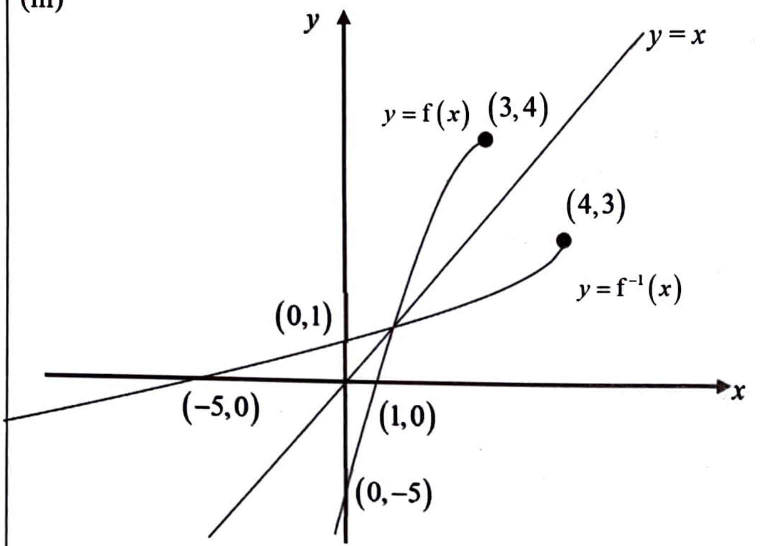
Making x the subject, we have $x=3\pm\sqrt{4-y}$.

Since $x \le 3$, $x=3-\sqrt{4-y}$.

Thus, $\mathrm{f}^{-1}: x \mapsto 3-\sqrt{4-x},\ x \le 4$.

How would the solution be different if the question wanted the domain of f to be restricted to a subset of $\mathbb{R}$ for which $x \ge a$ such that f^{-1} is defined?
Check the domain of the function to determine the expression to reject. State the reason for rejection.

(iii)



The intersection points of the graph of $y = f(x)$ and the graph of $y = f^{-1}(x)$ **must** lie on the line $y = x$.

Since the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect in the line $y = x$, we can consider $f(x) = x$, i.e.,

$-x^2 + 6x - 5 = x$

$-x^2 + 5x - 5 = 0$

$x = \dfrac{5 \pm \sqrt{5}}{2}$

Thus, $x = \dfrac{5-\sqrt{5}}{2}$ or $x = \dfrac{5+\sqrt{5}}{2}$ (reject since $x \le 3$).

Why do we consider $f(x) = x$ when solving for $f(x) = f^{-1}(x)$, instead of $f(x) = f^{-1}(x)$ or $f^{-1}(x) = x$?

Example 8 (Self-Learning):

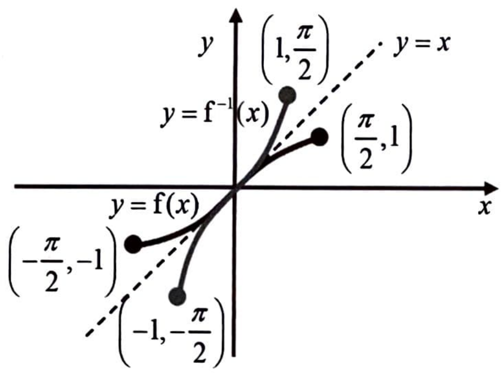
The function f is defined by $f : x \mapsto \sin x$, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$.

(i) Define $f^{-1}(x)$ and state its corresponding domain and range.

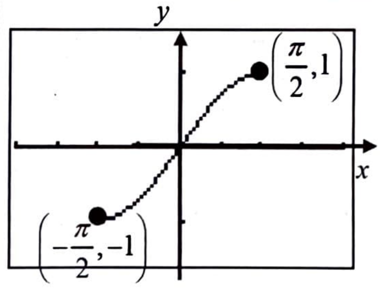
(ii) Sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same axes.

Solution:	**ThinkZone:**
(i) Let $y = \sin x$, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$. Making x the subject, we have $x = \sin^{-1} y$. Thus, $f^{-1}(x) = \sin^{-1} x$, $D_{f^{-1}} = R_f = [-1, 1]$ and $R_{f^{-1}} = D_f = [-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$.	

(ii)



Note: GC has its limitations. The two graphs actually intersect at the origin only (WHY?). Be careful when drawing the graphs.



Why is there a need to restrict the domain of f to $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ (which is also the principal angles given in the calculator)? Would f^{-1} exist is the domain is extended (say to $-\pi \le x \le \pi$)? How about the other inverse trigonometric functions for tangent and cosine?

Note: $\sin^{-1}(x) \neq \frac{1}{\sin x}$, $\cos^{-1}(x) \neq \frac{1}{\cos x}$, $\tan^{-1}(x) \neq \frac{1}{\tan x}$. $\left[\text{Recall that } f^{-1}(x) \neq [f(x)]^{-1} = \frac{1}{f(x)}\right]$

2.7 Composite Functions

A composite function is a combination of two or more functions taken as a single function. Suppose f and g are functions such that $R_g \subseteq D_f$. Then the composite function denoted by $f \circ g$ or fg, is defined as follows:

$$fg : x \mapsto f[g(x)],\ x \in D_g.$$

An illustration of the composite function fg as an 'input-output machine' is given in Figure 7.

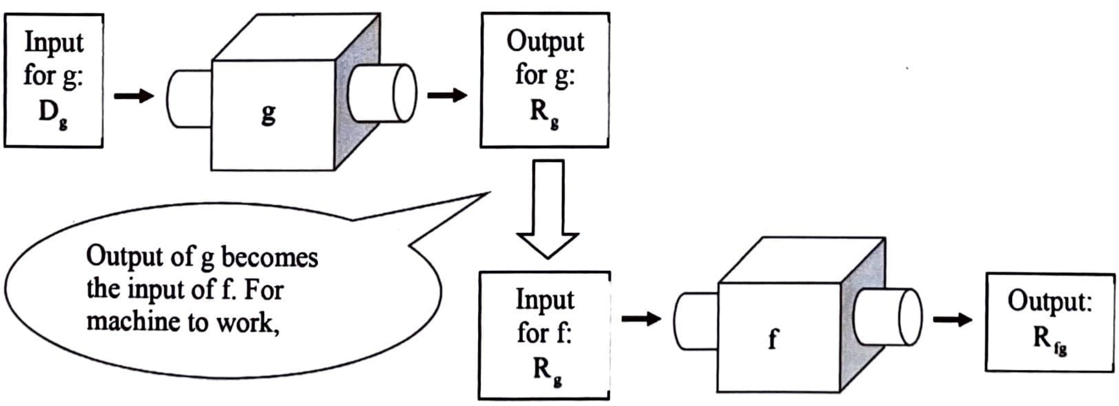


Figure 7

As an example of how we can interpret Figure 7, let the functions g and f be defined by $g: x \mapsto x^2,\ x \in \mathbb{R}$, and $f: x \mapsto 2x,\ x \in \mathbb{R}$. If $x = 2$, $g(2) = 2^2 = 4$ and thus we have $f(4) = 2(4) = 8$. Hence, $fg(2) = f(g(2)) = 8$.

Suppose now g and f are defined by $g: x \mapsto x^2,\ x \in \mathbb{R}$, and $f: x \mapsto 2x,\ x \in \mathbb{R},\ x \geq 5$. If $x = 2$, $g(2) = 4$. However, since $4 \notin D_f$, we cannot evaluate $f(4)$. Hence fg does not exist.

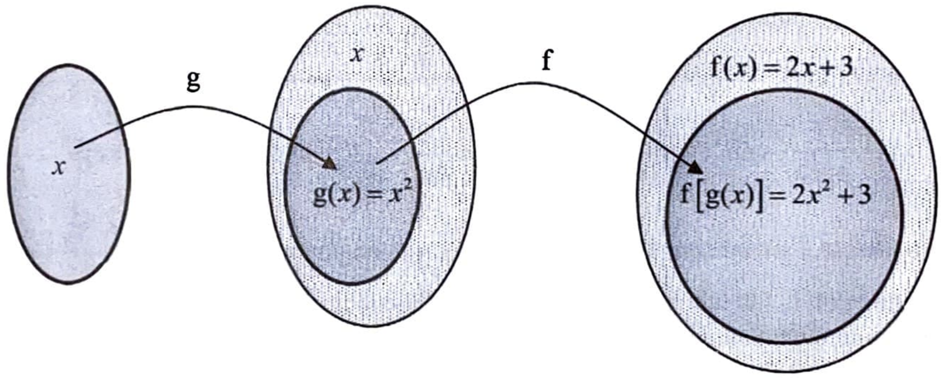
The composite function fg exists if and only if $R_g \subseteq D_f$.

Consider the following functions:

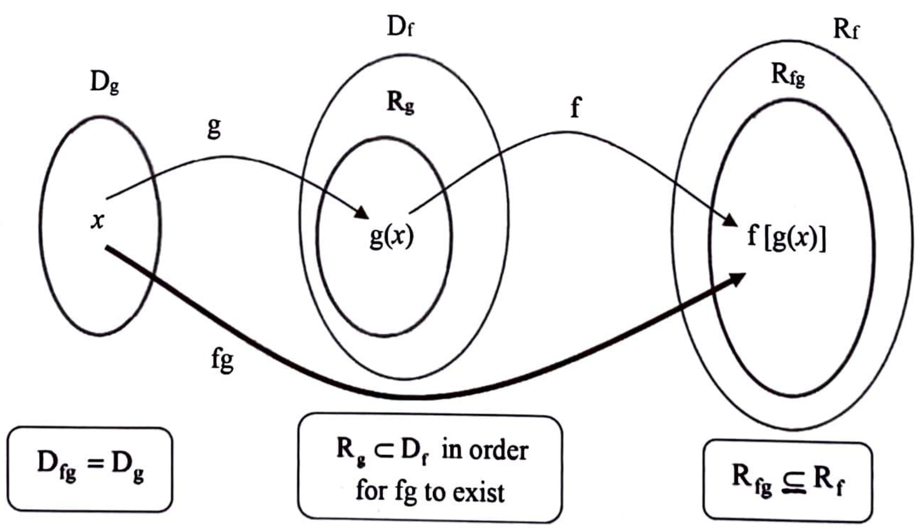
$$f: x \mapsto 2x + 3,\ x \in \mathbb{R},$$
$$g: x \mapsto x^2, x \in \mathbb{R}.$$

Note that $R_f = \mathbb{R}$ and $R_g = [0, \infty)$. Since $R_g \subset D_f$, thus fg exists.

Pictorially, we have



The following diagram illustrates the idea of taking composition for these functions.



Note that in the earlier example, $gf(x) = g(f(x)) = g(2x+3) = (2x+3)^2 \neq fg(x)$. Thus, in general, $gf(x) \neq fg(x)$.

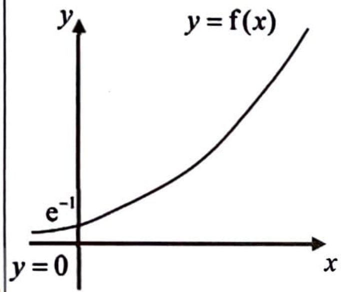
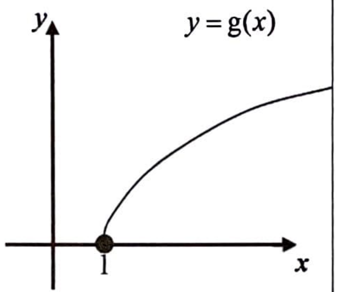
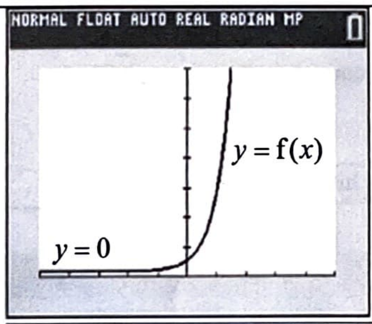
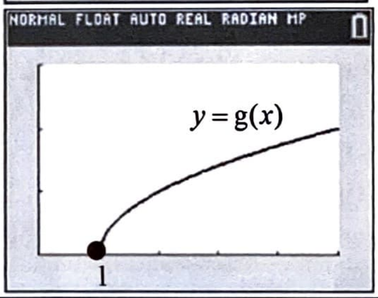
Example 9:

Functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \mathrm{e}^{2x-1},\ x \in \mathbb{R},$$
$$\mathrm{g}: x \mapsto \sqrt{x-1},\ x \geq 1.$$

Determine if each of the following composite functions exists. If it exists, define the composite function in a similar form.

(i) gf, (ii) fg.

Solution:	**ThinkZone:**
We first draw the graphs of $y=\mathrm{f}(x)$ and $y=\mathrm{g}(x)$ on separate diagrams, based on the given domains.   From the graphs, $\mathrm{R_f}=(0,\infty)$ and $\mathrm{R_g}=[0,\infty)$.	 
(i) Since $R_f = (0,\infty) \not\subseteq [1,\infty) = D_g$, gf does not exist.	
(ii) Since $\mathrm{R_g}=[0,\infty)\subseteq\mathbb{R}=\mathrm{D_f}$, fg exists. $\mathrm{fg}(x)=\mathrm{f}[\mathrm{g}(x)]$ $=\mathrm{f}[\sqrt{x-1}]$ $=\mathrm{e}^{2\sqrt{x-1}-1}$. Thus, $\mathrm{fg}: x \mapsto \mathrm{e}^{2\sqrt{x-1}-1}, x \geq 1$.	Note that the final answer must be in similar form.

Remark: For this example, we cannot construct gf as it does not exist. In general, even if both compositions exist, $\mathrm{fg} \neq \mathrm{gf}$. Note further that $\mathrm{D_{fg}} \neq \mathrm{D_{gf}}$ as well.

2.8 Finding the Range of a Composite Function

In this section, we will explore how to find the range of a composite function fg, using a graphical approach.

Example 10:
Functions f, g and h are defined as follows:

$$\mathrm{f}: x \mapsto x^2,\ x \geq 0,$$

$$\mathrm{g}: x \mapsto \frac{2x+3}{1-x},\ 0 \leq x < 1,$$

$$\mathrm{h}: x \mapsto \ln x,\ x > 0.$$

For each of the following composite functions, define the composite function in similar form and state its corresponding range.

(i) fg, (ii) hg.

Solution:

(i)

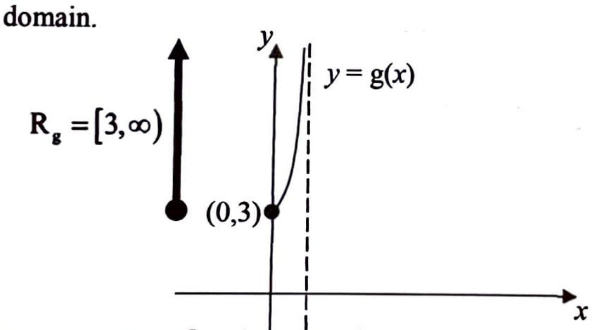
$$\mathrm{fg}(x) = \mathrm{f}\left[\mathrm{g}(x)\right] = \mathrm{f}\left[\frac{2x+3}{1-x}\right] = \left(\frac{2x+3}{1-x}\right)^2.$$

Thus, $\mathrm{fg}: x \mapsto \left(\frac{2x+3}{1-x}\right)^2,\ 0 \leq x < 1.$

To find the range of fg:

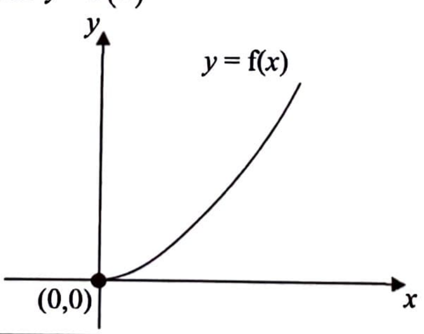
Method 1: Mapping Method

Step 1: We first draw the graph of $y = \mathrm{g}(x)$, based on the given domain.



From the graph, $R_g = [3, \infty)$.

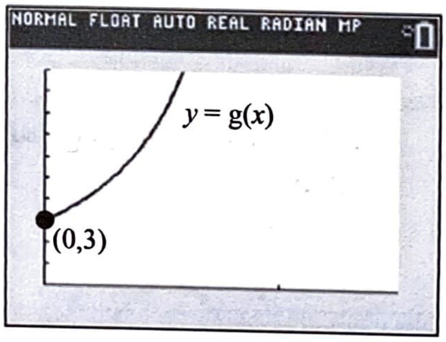
Step 2: Get a sketch of $y = \mathrm{f}(x)$



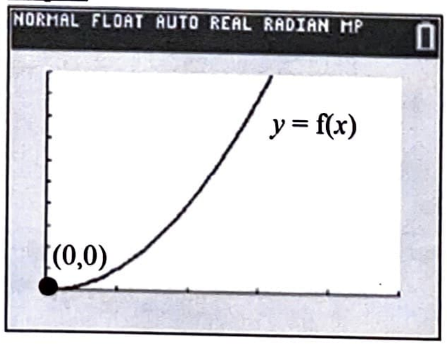
ThinkZone:

We do not need to check for existence of the composite funtion if the question did not specify.

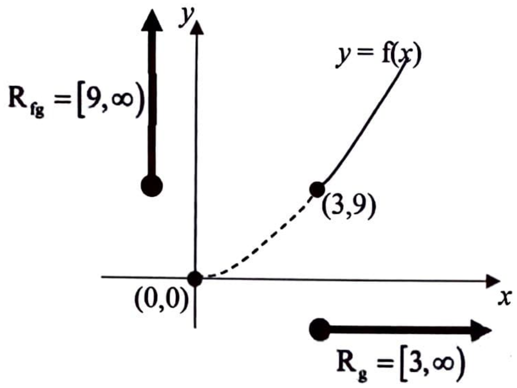
Step 1:



Step 2:



Step 3: Apply the range of g onto the graph of $y=\mathrm{f}(x)$ as the 'restricted domain'.



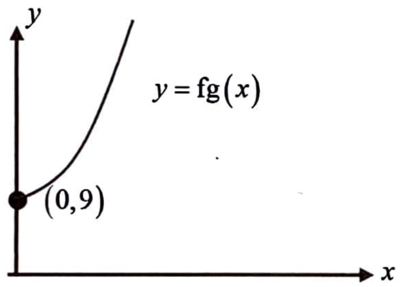
$D_{fg}=[0,1)\xrightarrow{g}[3,\infty)\xrightarrow{f}[9,\infty)=R_{fg}$

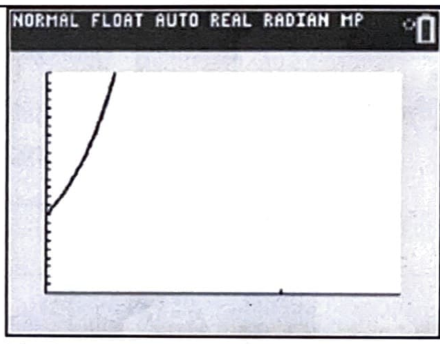
Thus, from the graph, $R_{fg}=[9,\infty)$.

Step 3:
We restrict the domain of f to be that of the range of g.

Method 2: Graphing composite function using GC

From the graph of $y=\mathrm{fg}(x)$, $R_{fg}=[9,\infty)$.





If the question requires you give answers in exact form, which method would you use? Why? Under which scenario would you use the other method?

(ii)

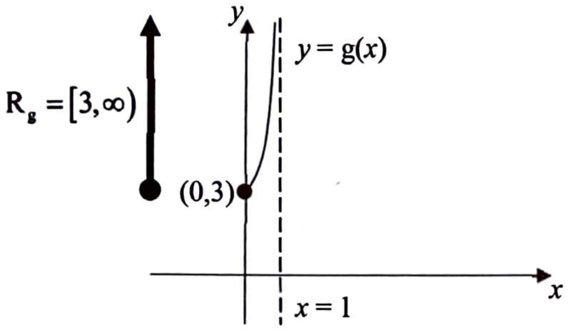
$$\mathrm{hg}(x)=\mathrm{h}[\mathrm{g}(x)]=\mathrm{h}\left[\frac{2x+3}{1-x}\right]=\ln\left(\frac{2x+3}{1-x}\right).$$

Thus, $\mathrm{hg}:x\mapsto\ln\left(\frac{2x+3}{1-x}\right)$, $0\le x<1$

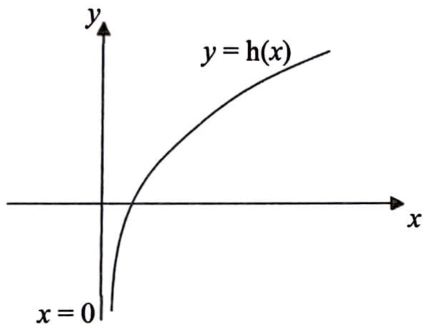
To find the range of hg:

Method 1: Mapping Method

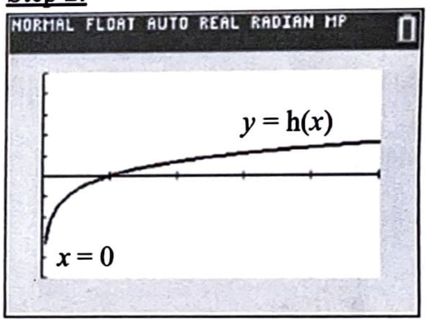
Step 1: We first draw the graph of $y = \mathrm{g}(x)$, based on the given domain.



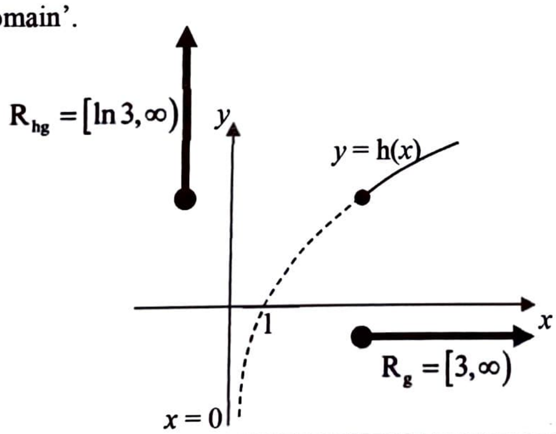
Step 2: Get a sketch of $y = \mathrm{h}(x)$



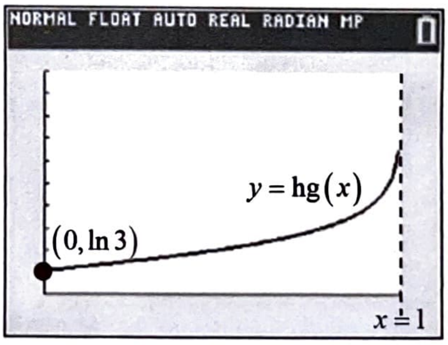
Step 2:



Step 3: Apply the range of g onto the graph of $y = \mathrm{h}(x)$ as the 'domain'.



Step 3:
We restrict the domain of h to be that of the range of g.



Thus, from the graph,

Method 2: Graphing composite function using GC

From the graph of $y = \mathrm{hg}(x)$, $R_{hg} = [\ln 3, \infty)$.

Is it easy to obtain this graph?

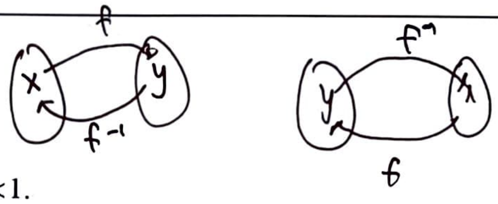
$f(x) = f^{-1}(x)$

$ff(x) = x$ or $f(x) = x$

Important properties for composite functions:

1. Composition of functions is not commutative, i.e., $\mathrm{fg}(x) \neq \mathrm{gf}(x)$ in general. The order of composition of the functions is important.
2. If fg is defined, then $D_{fg} = D_g$.
3. If f^{-1} exists, then $f^{-1}f$ and ff^{-1} exists. Furthermore, $f^{-1}f(x) = x$, $x \in D_f$ and $ff^{-1}(x) = x$, $x \in D_{f^{-1}}$, i.e., $ff^{-1}(x) \neq f^{-1}f(x)$ in general. $ff^{-1}(x) = f^{-1}f(x)$ for $x \in D_f \cap D_{f^{-1}}$.
4. f^2 refers to $f \circ f$ or ff. It only exists when $R_f \subseteq D_f$.

2.9 More Examples



Example 11:

The function f is defined by $f : x \mapsto (x-1)^2 + 2,\ x < 1$.

(i) Define f^{-1} in a similar form and state its corresponding range.

(ii) Sketch on the same axes, the graphs of f, f^{-1} and hence show that $f^{-1}f$ exists. State clearly the relationship between the graphs of f and f^{-1}.

(iii) Find $f^{-1}f$ and ff^{-1} in similar form and illustrate the difference between $f^{-1}f$ and ff^{-1} by drawing the graphs of $y = f^{-1}f(x)$ and $y = ff^{-1}(x)$ on the same diagram.

(iv) Find the set of values of x such that $f^{-1}f(x) = ff^{-1}(x)$.

Solution:

(i)

Let $y = (x-1)^2 + 2,\ x < 1$.

Making x the subject, we have $x = 1 \pm \sqrt{y-2}$.

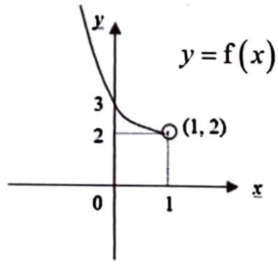
Since $x < 1$, $x = 1 - \sqrt{y-2}$.

Thus, $f^{-1} : x \mapsto 1 - \sqrt{x-2},\ x > 2$.

Also, $R_{f^{-1}} = (-\infty, 1)$.

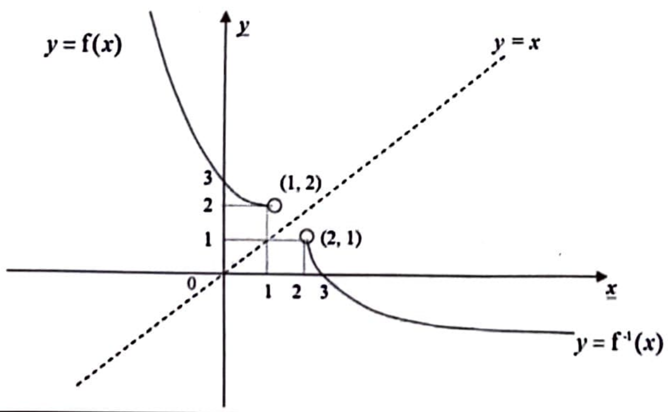
ThinkZone:

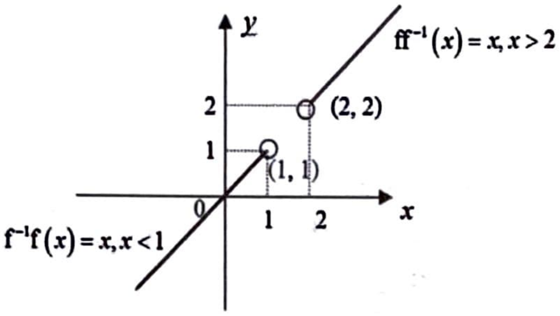
Which rule should we select?



(ii)

Since $R_f = (2, \infty) = D_{f^{-1}}$, $f^{-1}f$ exists.



By adding the line $y = x$, we can see that the graphs of f and f^{-1} are reflections of each other in the line $y = x$.	
(iii) $\mathrm{f}^{-1}\mathrm{f} : x \mapsto x,\quad x < 1$ $\mathrm{ff}^{-1} : x \mapsto x,\quad x > 2$ 	
(iv) Since the graphs do not intersect, the solution set for $\mathrm{f}^{-1}\mathrm{f}(x) = \mathrm{ff}^{-1}(x)$ is $\varnothing$.	

Example 12 (N2006/I/3 modified):
The function f is defined by

$$\mathrm{f} : x \mapsto \frac{3}{x},\quad x > 0.$$

Find in a similar form f^2 and f^{35}, and hence evaluate $\mathrm{f}^{2014}(13)$.
(**Note:** f^2 denotes ff.)

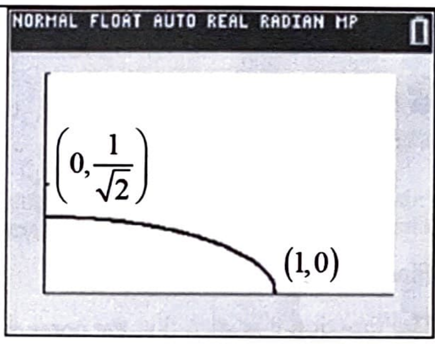
Solution:	**ThinkZone:**
(i) $\mathrm{f}^2(x) = \mathrm{f}\left[\mathrm{f}(x)\right] = \mathrm{f}\left[\frac{3}{x}\right] = x.$ $\mathrm{f}^3(x) = \mathrm{f}\left[\mathrm{f}^2(x)\right] = \mathrm{f}[x] = \frac{3}{x}.$ $\mathrm{f}^4(x) = \mathrm{f}\left[\mathrm{f}^3(x)\right] = \mathrm{f}\left[\frac{3}{x}\right] = x.$ By observation, $\mathrm{f}^{35}(x) = \frac{3}{x}$. Thus, $\mathrm{f}^2 : x \mapsto x,\ x > 0$ and $\mathrm{f}^{35} : x \mapsto \frac{3}{x},\ x > 0$. Since $\mathrm{f}^{2014}(x) = x$, $\mathrm{f}^{2014}(13) = 13$.	Are you able to see a pattern? Is $\mathrm{f}^{35}(x) = \mathrm{f}^{\frac{35}{2}}\left[\mathrm{f}^2(x)\right]$ or $\mathrm{f}^{33}\left[\mathrm{f}^2(x)\right]$?

Example 13:

The function f is defined by $\mathrm{f}: x \mapsto \sqrt{\frac{1-x^2}{2}},\ x \in \mathbb{R},\ 0 \le x \le 1$

(i) Determine whether the composite function ff exists. If it exists, find the composite function ff(*x*).

(ii) Hence solve $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$.

Solution:	ThinkZone
(i) Since $R_f = [0, \frac{1}{\sqrt{2}}] \subset D_f = [0, 1]$, ff exists. $\mathrm{ff}(x)=\sqrt{\frac{1-\left(\sqrt{\frac{1-x^2}{2}}\right)^2}{2}}=\sqrt{\frac{1-\frac{1-x^2}{2}}{2}}=\sqrt{\frac{1+x^2}{4}}$ $\mathrm{ff}(x)=\sqrt{\frac{1+x^2}{4}}$ (ii) $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ $\mathrm{ff}(x)=x$ $\sqrt{\frac{1+x^2}{4}}=x$ $1+x^2=4x^2$ $x^2=\frac{1}{3}$ $x=\frac{1}{\sqrt{3}}$ or $x=-\frac{1}{\sqrt{3}}$ (reject since $0 \le x \le 1$)	 Note: GC has its limitations. GC would not be able to find the exact form (if the question requires) of the coordinates. To do it, we have to do it algebraically.

Example 14 (Piecewise Periodic Functions):
It is given that

$$\mathrm{f}(x)=\begin{cases} 4x, & \text{for } 0 \le x < 1 \\ (x-3)^2, & \text{for } 1 \le x \le 3, \end{cases}$$

It is also known that $\mathrm{f}(x)=\mathrm{f}(x+3)$ for all real values of x.

(i) Show that $\mathrm{f}(5)=1$.

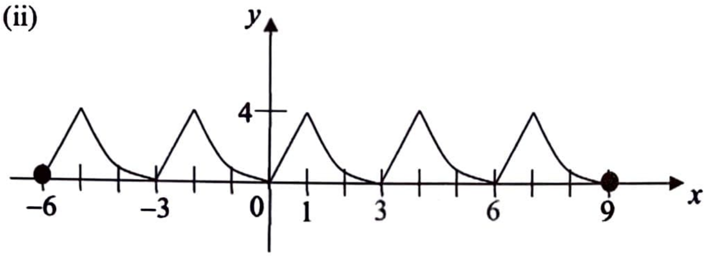
(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-6 \le x \le 9$.

Solution:

(i) We first observe that $f(5)=f(2+3)=f(2)$,

i.e., $f(5)=f(2)=(2-3)^2=(-1)^2=1$.

(ii)



ThinkZone:

What does the condition $f(x)=f(x+3)$ mean and how does it affect the graph? (Recall Chapter 1)

Example 15:

The function h is defined by

$$h: x \mapsto x^2+2, \qquad x\in\mathbb{R},\ x>0.$$

(i) The function g is such that the composite function gh exists and $gh(x)=x^4-2x^2+5$, $x\in\mathbb{R}, x>0$. Find $g(x)$.

(ii) The function g is such that the composite function hg exists and $hg(x)=x^2+10x+27$, $x\in\mathbb{R}$. Find $g(x)$.

Solution:

(i)

$h(x)=x^2+2$

$gh(x)=x^4-2x^2+5$

$=(x^2+2)^2-4x^2-4-2x^2+5$

$=(x^2+2)^2-6(x^2+2)+13$

$\Rightarrow g(x)=x^2-6x+13$

ThinkZone:

$gh(x)=g[h(x)]=g[x^2+2]$

(ii)

Let $y=h(x)=x^2+2$

$x=\pm\sqrt{y-2}$

Since $x>0$, $x=\sqrt{y-2}$.

Hence $h^{-1}(x)=\sqrt{x-2}, x>2$.

$hg(x)=x^2+10x+27$

$g(x)=h^{-1}(x^2+10x+27)$

$=\sqrt{(x^2+10x+27)-2}$

$=\sqrt{x^2+10x+25}$

$=\sqrt{(x+5)^2} = |x+5|$

$=x+5$

ThinkZone:

Do you know why h^{-1} exists?

Note:

$g(x)=h^{-1}[hg(x)]$

2.10 Special Functions

Even Functions

A function f is said to be *even* if $f(-x)=f(x)$ for all $x \in D_f$.

An example will be $f(x)=x^2$. In fact, a polynomial that comprises of terms with only even powers of x is an even function.

The graph of an even function is symmetric about the y-axis as illustrated in Chapter 1.

Odd Functions

A function f is said to be *odd* if $f(-x)=-f(x)$ for all $x \in D_f$.

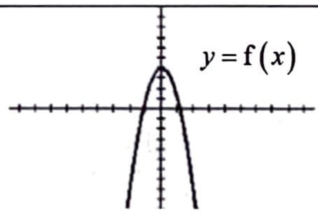
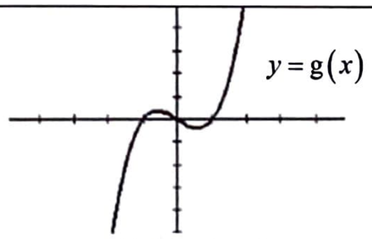
An example will be $f(x)=x^3$. In fact, a polynomial that comprises of terms with only odd powers of x is an odd function.

The graph of an odd function is symmetric about the origin.

Example 16:

Determine if the following functions are odd or even.

(i) $f(x)=-3x^2+4$ (ii) $g(x)=x^3-x$

Solution:	**ThinkZone:**
$f(-x)=-3(-x)^2+4$ $=-3x^2+4$ $=f(x)$ Thus f is an even function.	 Notice that, even function is symmetric about the y-axis.
$g(-x)=(-x)^3+(-x)$ $=-x^3-x$ $=-(x^3+x)$ $=-g(x)$ Thus g is an odd function.	 Notice that, odd function is symmetric about the origin.

Monotone Functions

A function f is said to be *increasing* if $x > y \Rightarrow f(x) > f(y)$ for all $x, y \in D_f$.

Similarly, a function f is said to be *decreasing* if $x > y \Rightarrow f(x) < f(y)$ for all $x, y \in D_f$.

A function f is said to be *monotone* if it is either increasing or decreasing.

If $D_f = [a,b]$, then $R_f = [f(a), f(b)]$ if f is increasing; $R_f = [f(b), f(a)]$ if f is decreasing.

If f is monotone, then f is one-one and hence, f^{-1} exists.

Example 17 (N02/II/5):

The function f is defined for $x \geq 1$ by $f(x) = x + \frac{1}{x}$.

(i) Show that $f(x)$ increases as x increases.

(ii) State the range of f.

(iii) Find an expression for $f^{-1}(x)$.

Solution:	**ThinkZone:**
(i) To show that $f(x)$ increases as x increases, it is similar to showing when $x > y, f(x) \geq f(y) \Rightarrow f(x) - f(y) \geq 0$. Let $x > y$. Then $f(x) - f(y) = x + \frac{1}{x} - \left(y + \frac{1}{y}\right)$ $= x - y + \frac{y - x}{xy}$ $= \frac{(x-y)(xy-1)}{xy}$ Since $x, y \geq 1$, thus $xy - 1 \geq 0 \Rightarrow xy \geq 1$. Furthermore, $x > y \Rightarrow x - y > 0$. Thus $f(x) - f(y) \geq 0 \Rightarrow f(x) \geq f(y)$. Hence, $f(x)$ increases as x increases. (ii) Since f is increasing, thus $f(x) \geq f(1) = 2$. Thus $R_f = [2, \infty)$. (iii) Let $y = x + \frac{1}{x}$. Thus $xy = x^2 + 1$ $\Rightarrow x^2 - xy + 1 = 0$ $\Rightarrow x = \frac{y \pm \sqrt{y^2 - 4}}{2}$ Since $x \geq 1$, we have $x = \frac{y + \sqrt{y^2 - 4}}{2}$. Thus $f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$.	The illustrated method is an algebraic approach. Is there another way to show that f is an increasing function? Hint: What can be said about the derivative of an increasing or decreasing function? (To be learnt in Chapter 7)

Continuous and Discontinuous Functions

Loosely speaking, a function is continuous if there is no "break" in the graph of the function. Otherwise, the function is said to be discontinuous.

More specifically, a function f is said to be continuous at $x = a$ if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = f(a)$.

Note:

$\lim_{x \to a^+} f(x)$ is called the **right limit** of f as x tends to a from the right and $\lim_{x \to a^-} f(x)$ is called the **left limit** of f as x tends to a from the left.

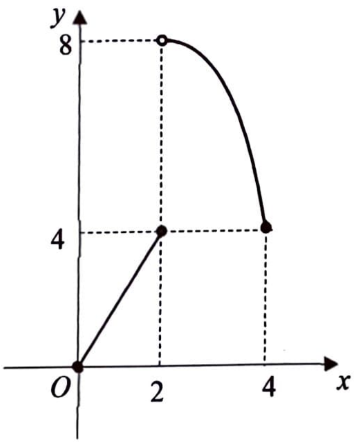
Example 18 (Self Reading)

The function f is defined on the interval $[0, 4]$ as follows:

$$f(x) = \begin{cases} 2x, & 0 \le x \le 2, \\ 8-(x-2)^2, & 2 < x \le 4. \end{cases}$$

(i) Sketch the graph of f.

(ii) Explain whether f is continuous at $x = 2$.

Solution:	**ThinkZone:**
(i) Note that this is a piecewise function.  (ii) From the diagram, $\lim_{x \to 2^+} f(x) = 8$, $\lim_{x \to 2^-} f(x) = f(2) = 4$. Thus f is not continuous at $x = 2$.	Compare this example to Example 14. Is the function in Example 14 a continuous function?

Remark: Are you able to find any discontinuous function in your daily life?

Summary of key concepts

1. A function is made up of two parts, the **domain** (the set of input) and the **rule** (what to do to the input). The **range** of a function is the set of all output (graphically, it is the y-coordinates of the graph of the function $y = \mathrm{f}(x)$).
2. To show that a function is not **one-one**, state and provide a horizontal line $y = k$, where $k \in \mathbb{R}$, that intersects the graph more than once. Use a specific value of k.
3. For f^{-1} to exist, f has to be a one-one function. When f^{-1} exists,
   (i) $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f}$, $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_\mathrm{f}$,
   (ii) The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.
   (iii) $\mathrm{f}^{-1}\mathrm{f}(x) = x$, $x \in \mathrm{D}_\mathrm{f}$ and $\mathrm{f}\mathrm{f}^{-1}(x) = x$, $x \in \mathrm{D}_{\mathrm{f}^{-1}}$.
4. For the composite function fg to exist, $\mathrm{R}_\mathrm{g} \subseteq \mathrm{D}_\mathrm{f}$. In this case, $\mathrm{D}_{\mathrm{fg}} = \mathrm{D}_\mathrm{g}$.

Chapter 3

Advanced Graphing Techniques

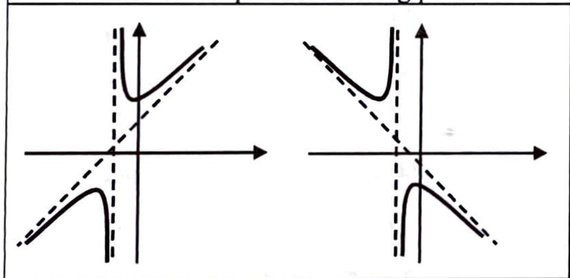
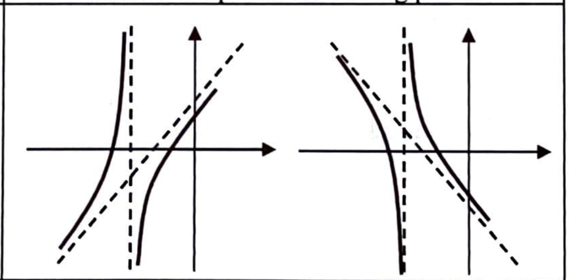
3.1 Graph of $y=\dfrac{ax^2+bx+c}{dx+e}$ where $a\neq 0, d\neq 0, x\neq -\dfrac{e}{d}$

Suppose y is an improper rational function. We will first perform long division to check for any asymptotes. Doing so, we have

$$y=px+q+\frac{r}{dx+e},\quad x\neq -\frac{e}{d} \text{ for some constants } p, q, \text{ and, } r.$$

In general, the graph of $y=\dfrac{ax^2+bx+c}{dx+e}=px+q+\dfrac{r}{dx+e}$ has two asymptotes: 1 vertical asymptote $x=-\dfrac{e}{d}$ and 1 oblique asymptote $y=px+q$.

There are two possible sketches of the graph as shown below:

Case 1: Graph has 2 turning points	Case 2: Graph has no turning points
	

may cut x-axis

Example 1

The curve has equation $y=\dfrac{x^2+a}{4-x}$, where a is a constant and $a\neq -16$.

(i) Find the equations of the asymptotes.

(ii) Find the range of values of a such that the curve has 2 distinct real roots.

(iii) Find the range of values of a such that the curve has two turning points.

(iv) Sketch the graph of C for $a<-16$, indicating clearly all equations of asymptotes and coordinates of the axial intercepts.

Hence, deduce the number of real roots for the equation $x^4+ax^2+bx-4b=0$ for $a<-16$ and $b>0$.

Solution	Think Zone
(i) Rewriting, $y=\frac{x^2+a}{4-x}=-x-4+\frac{a+16}{4-x}$ Vertical asymptote is $x=4$ Oblique asymptote is $y=-x-4$	
(ii) Let $y=0$. Then $\frac{x^2+a}{4-x}=0$ $x^2+a=0$ $x^2=-a$ For 2 distinct real roots, $-a>0$ $a<0$	Alternatively, use discriminant. $\frac{x^2+a}{4-x}=0$ $x^2+a=0$ $b^2-4ac>0$ $(0)^2-4(1)(a)>0$ $4a<0 \Rightarrow a<0$
(iii) $\frac{dy}{dx}=-1+\frac{a+16}{(4-x)^2}$ For turning points, let $\frac{dy}{dx}=0$ $\frac{a+16}{(4-x)^2}=1$ $(4-x)^2=a+16$ For real and distinct roots, $a+16>0$ $\therefore a>-16$	Alternatively, For turning points, let $\frac{dy}{dx}=0$ $\frac{a+16}{(4-x)^2}=1$ $\Rightarrow a+16=16-8x+x^2$ $\Rightarrow x^2-8x-a=0$ Since the curve has two turning points, Discriminant > 0 $\Rightarrow 64-4(1)(-a)>0$ $4a>-64$ $\therefore a>-16$

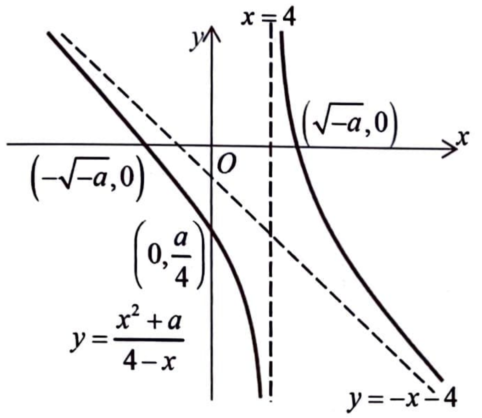
$$\frac{dy}{dx}=-1+\frac{(4-x)(0)-(-1)(a+16)}{(4-x)^2}$$

$$=-1+\frac{a+16}{(4-x)^2}$$

(iv) when $x=0,\ y=\frac{a}{4}$

when $y=0,\ x^2=-a \Rightarrow x=\pm\sqrt{-a}$

If $a>-16$, the graph has two turning points, what happen if $a<-16$? What about for $a=-16$?



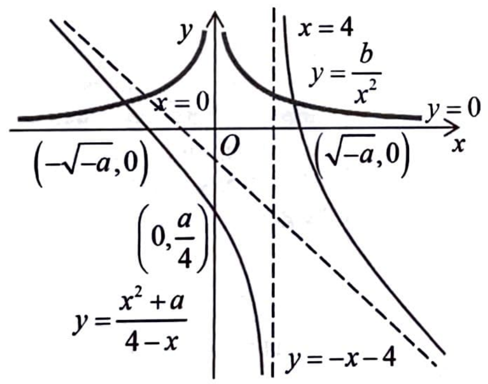
$x^4+ax^2+bx-4b=0$

$\Rightarrow x^4+ax^2=4b-bx$

$\Rightarrow x^2(x^2+a)=b(4-x)$

$\Rightarrow \frac{x^2+a}{4-x}=\frac{b}{x^2}$

Rearrange the given equation into the form f(x) = g(x) such that one side of the equation (either f(x) or g(x)) is the equation of the curve that you have sketched earlier.



Since we have sketched the graph of $y=\frac{x^2+a}{4-x}$ above, we will simply sketch the graph of $y=\frac{b}{x^2}$ on the same diagram.

The graphs intersect at 2 points $\Rightarrow$ the equation has 2 real roots.

Example 2

Consider the curve $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x^2 - 1}{x+2}$.

(i) State the coordinates of any points of intersection with the axes.

(ii) Find the equations of the asymptotes.

(iii) Show algebraically, that the curve cannot lie between two values which are to be determined.

(iv) Sketch $y = \mathrm{f}(x)$, indicating clearly coordinates of any axial intercept(s), turning point(s) and equations of asymptotes.

Solution

(i) From GC, coordinates are $(0,-0.5)$, $(1,0)$ and $(-1,0)$

(ii) Vertical asymptote is $x = -2$

Performing long division, $y = \dfrac{x^2-1}{x+2} = x - 2 + \dfrac{3}{x+2}$

$\therefore$ oblique asymptote $y = x - 2$

(iii) Let $y = \dfrac{x^2-1}{x+2}$,

$\Rightarrow y(x+2) = x^2 - 1$

$yx + 2y = x^2 - 1$

$x^2 - yx - (1+2y) = 0$

For real values of x,

$(-y)^2 - 4(-1)(1+2y) \geq 0$

$y^2 + 8y + 4 \geq 0$

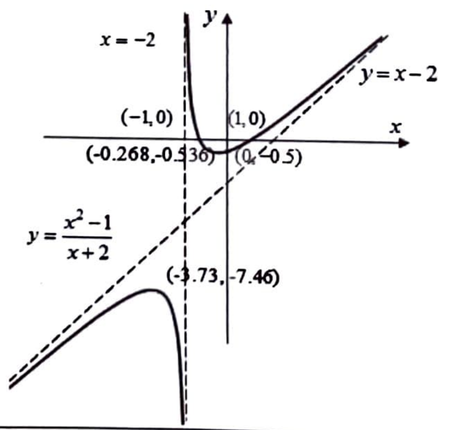
$(y+4)^2 - 12 \geq 0$

$\left[(y+4) - \sqrt{12}\right]\left[(y+4) + \sqrt{12}\right] \geq 0$

$y \leq -4 - 2\sqrt{3}$ or $y \geq -4 + 2\sqrt{3}$

The curve cannot lie between $-4-2\sqrt{3}$ and $-4+2\sqrt{3}$.

(iv)



Think Zone

'State' questions do not require workings. Just check the coordinates using GC functions.

As $x \to \pm\infty$, $\dfrac{3}{x+2} \to 0$

$y \to x - 2$

This procedure is equivalent to finding the intersection between *any* horizontal line $y = k$ with the graph.

$\left.\begin{array}{l} y = k \\ y = \dfrac{x^2-1}{x+2} \end{array}\right\} k = \dfrac{x^2-1}{x+2}$ and find range of k

Since we are looking for the two values for which the curve cannot lie between, there would be no point of intersection between any horizontal line and the graph. Hence, Discriminant < 0.

You can verify your answer using the sketch in (iv).

Use GC to obtain the shape of the graph and to find the coordinates of the axial intercepts and turning points.

Self-Review 1:

The curve has equation $y = \dfrac{x^2 - 2x + 2}{x - 1}$.

(i) Find the coordinates of any points of intersection with the axes.
(ii) Find the equations of the asymptotes.
(iii) Show algebraically, the region that the curve must lie.
(iv) Sketch the curve, indicating clearly coordinates of any axial intercept(s), turning point(s) and equations of asymptotes.

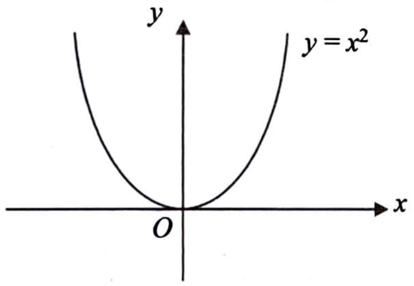
[(i) (0,–2), (ii) $y = x - 1$, (iii) $y \le -2$ or $y \ge 2$, (iv) $y = x - 1$, $x = 1$, (2, 2), (0, –2)]

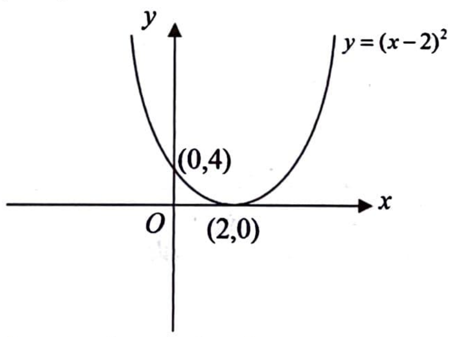
3.2 Transformations

In this section, we will investigate the effect of 3 types of linear transformation for graphs (translation, scaling, and reflection) and how we can represent them. We will also investigate the graphs of curves involving the modulus function. We will see how these transformations affect the graph with respect to the x-axis and y-axis.

3.2.1 Translation

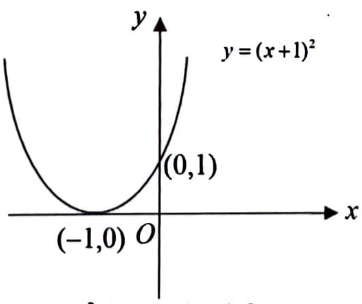
Consider the graphs of the function $y = x^2$ and $y = (x-2)^2$:





How are these two graphs related? From the illustration, we observe that the graph of $y = (x-2)^2$ can be obtained from **translating** the graph of $y = x^2$ by 2 units in the direction of the x-axis.

How about the graph of the function $y = (x+1)^2$?



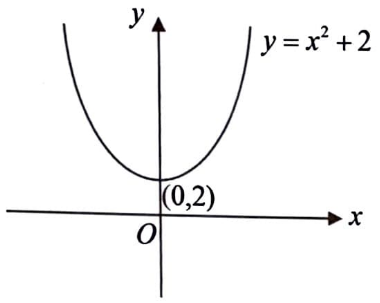
Also, we can see that the graph of $y = (x+1)^2$ is obtained from **translating** the graph of $y = x^2$ by -1 unit in the direction of the x-axis.

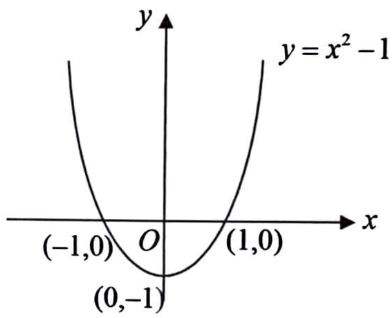
In general, where $a > 0$,

(i) $y = \text{f}(x-a)$ is obtained by **translating the graph of y = f(x) by a units in the direction of the x–axis.**

(ii) $y = \text{f}(x+a)$ is obtained by **translating the graph of y = f(x) by $-a$ units in the direction of the x–axis.**

Now, let's consider the graphs of the function $y = x^2 + 2$ and $y = x^2 - 1$. How are these graphs related to that of $y = x^2$?





We see that the graph of $y = x^2 + 2$ can be obtained from that of $y = x^2$ by translating 2 units in the direction of the y-axis. As for the graph of $y = x^2 - 1$, it can be obtained from translating the graph of $y = x^2$ by -1 unit in the direction of the y-axis.

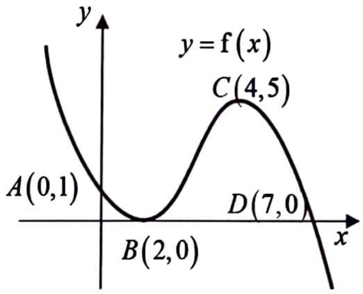
In general, where $a > 0$,

(i) $y = \text{f}(x) + a$ is obtained by **translating the graph of y = f(x) by a units in the direction of the y–axis.**

(ii) $y = \text{f}(x) - a$ is obtained by **translating the graph of y = f(x) by $-a$ units in the direction of the y–axis.**

Example 3

The following diagram shows the graph of $y = f(x)$.



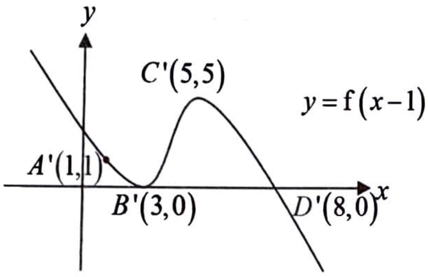
Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to *A*, *B*, *C*, and *D*.

(a) $y = f(x-1)$,

(b) $y = f(x)+3$.

Solution

(a)

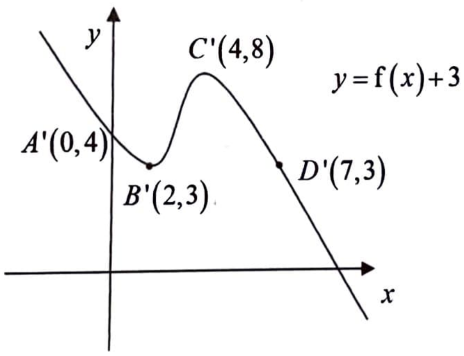


ThinkZone

What do you observe about the *x*-coordinate of points *A*, *B*, C, and *D*?

What about the *y*-coordinate?

(b)



What do you observe about the *y*-coordinate of points *A*, *B*, *C*, and *D*?

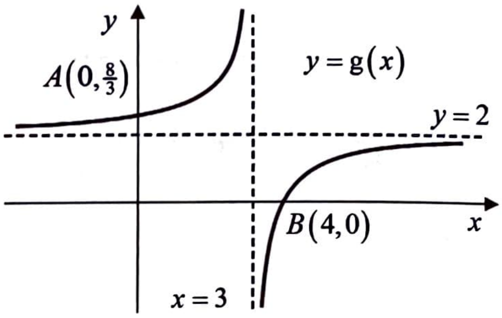
What about the *x*-coordinate?

$2+\frac{-2}{x-3}=g(x)$

$2+\frac{-2}{x-1}=g(x+2)$

Example 4

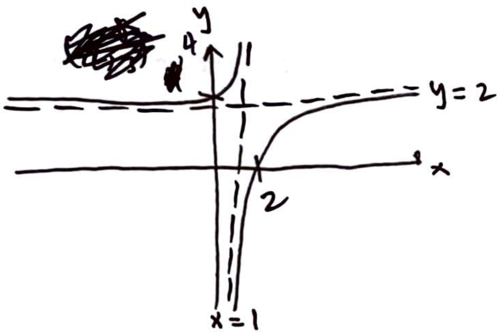
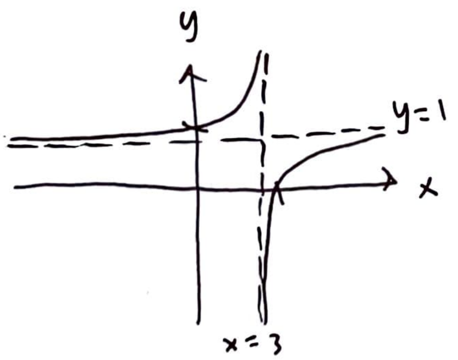
The following diagram shows the graph of $y=g(x)$.



Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to A and B and the equations of asymptotes.

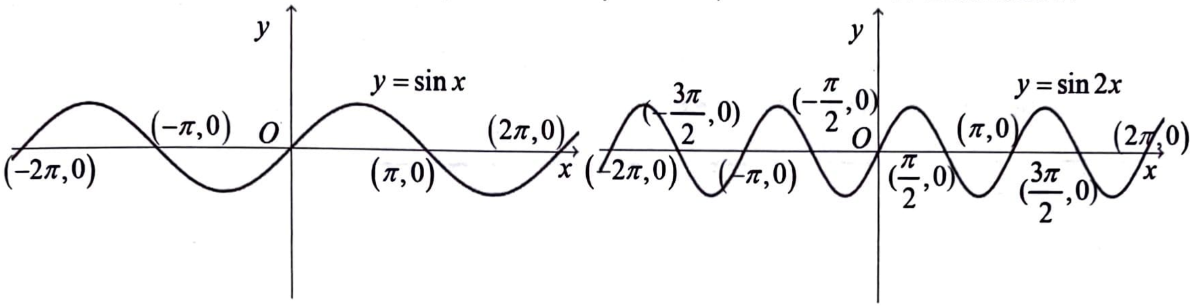
(a) $y=g(x+2)$,

(b) $y=g(x)-1$.

Solution	ThinkZone
(a) 	What can you observe about the equations of the asymptotes?
(b) 	What can you observe about the equations of the asymptotes?

3.2.2 Scaling

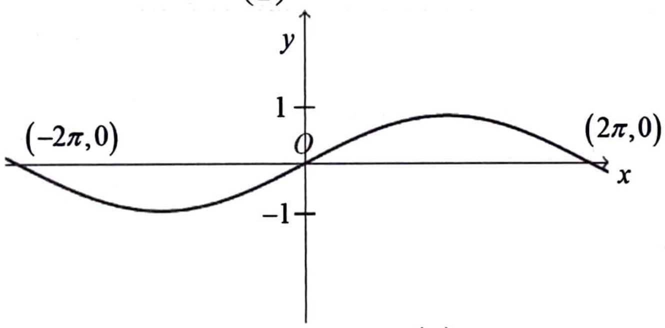
Consider the graphs of the functions $y=\sin x$ and $y=\sin 2x$, $-2\pi \le x \le 2\pi$ as shown below:



How are these two graphs related?

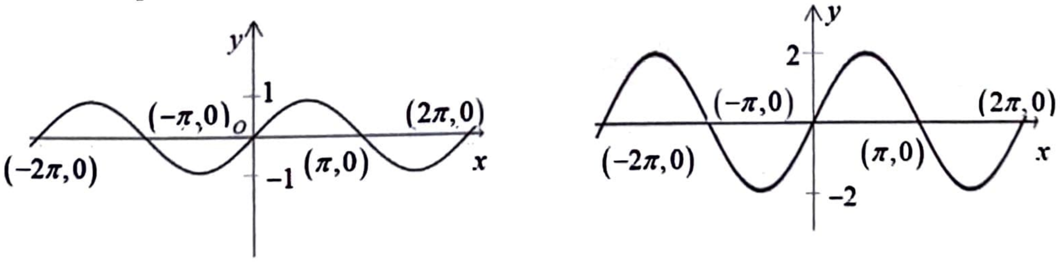
From the illustration, we observe that the graph of $y=\sin x$ has a period of 2π while the graph of $y=\sin 2x$ has a period of π. Based on the observation, we notice that $y=\sin 2x$ is obtained from **scaling** the graph of $y=\sin x$ by a factor of $\frac{1}{2}$ parallel to the x-axis.

How about the graph of the function $y=\sin\left(\frac{x}{2}\right)$?



From the illustration, we observe that the graph of $y=\sin\left(\frac{x}{2}\right)$ has a period of 4π. As such, we can see that $y=\sin\left(\frac{x}{2}\right)$ is obtained from **scaling** the graph of $y=\sin x$ by a factor of 2 parallel to the x-axis.

What is the required transformation to obtain the graph of $y=2\sin x$ from the graph of $y=\sin x$?



Using your GC, you will observe that $y=2\sin x$ has an amplitude of 2 while the graph of $y=\sin x$ has an amplitude of 1. From this observation, we can see that $y=2\sin x$ is obtained from **scaling** the graph of $y=\sin x$ by a factor of 2 parallel to the y-axis.

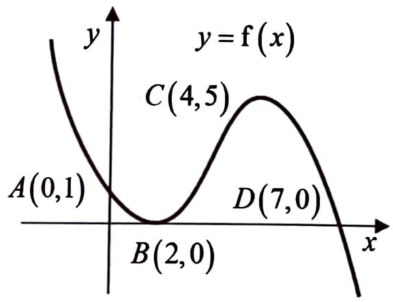
In general, where $a > 0$,

(i) $y = \mathrm{f}(ax)$ is obtained by **scaling the graph of $y = \mathrm{f}(x)$ by a factor of $\frac{1}{a}$ parallel to the x-axis.**

(ii) $y = a\mathrm{f}(x)$ is obtained by **scaling the graph of $y = \mathrm{f}(x)$ by a factor of a parallel to the y–axis.**

Example 5

The following diagram show the graph of $y = \mathrm{f}(x)$.



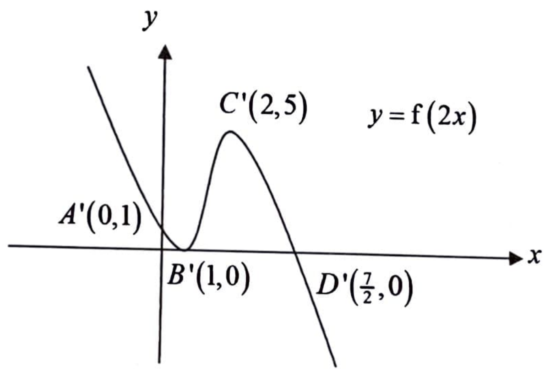
Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to A, B, C, and D.

(a) $y = \mathrm{f}(2x)$,

(b) $y = \frac{1}{2}\mathrm{f}(x)$.

Solution

(a)

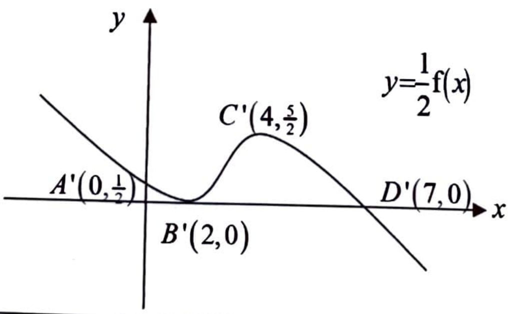


ThinkZone

What do you observe about the x-coordinate of points A, B, C, and D?

What about the y-coordinate?

(b)

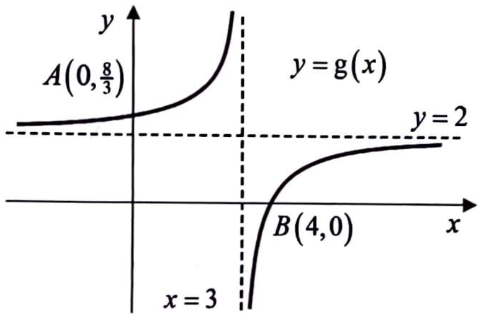


What do you observe about the y-coordinate of points A, B, C, and D?

What about the x-coordinate?

Example 6

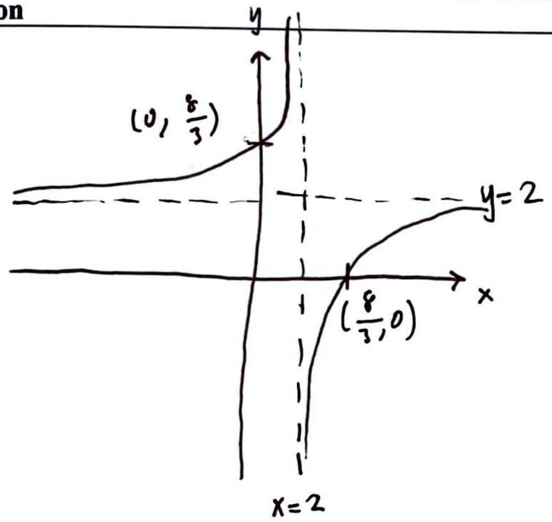
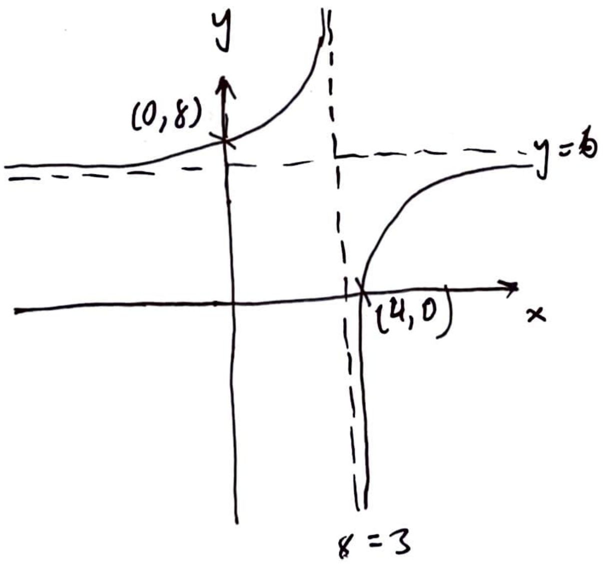
The following diagram shows the graph of $y = g(x)$.



Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to A and B, and the equations of asymptotes.

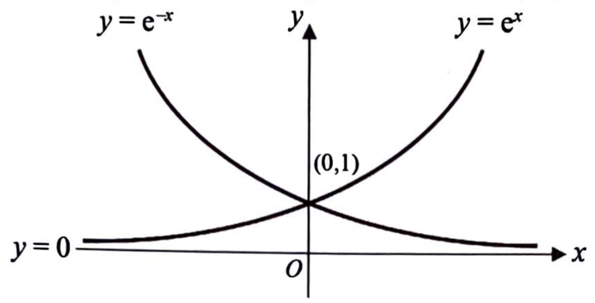
(a) $y = g\left(\frac{3x}{2}\right)$,

(b) $y = 3g(x)$.

<table>
<tr><th>Solution</th><th>ThinkZone</th></tr>
<tr><td>(a)

</td><td>What can you observe about the equations of the asymptotes?</td></tr>
<tr><td>(b)

</td><td>What can you observe about the equations of the asymptotes?</td></tr>
</table>

3.2.3 Reflection

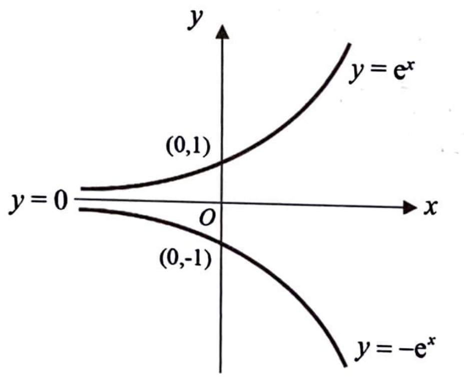
Consider the graphs $y = e^x$ and $y = e^{-x}$. How do you think the two graphs are related?



We can see that the graph of $y = e^{-x}$ is the **reflection** of the graph of $y = e^x$ in the ***y*-axis**. If we let $y = f(x) = e^x$, then $y = e^{-x}$ will be denoted as $y = f(-x)$.

In general, **the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the *y*-axis.**

Now, let's consider the graphs of the functions $y = e^x$ and $y = -e^x$, what can you observe?

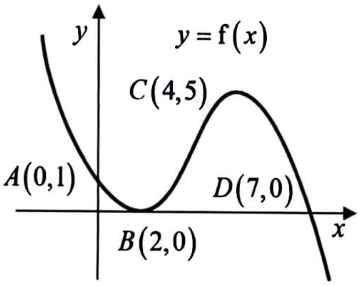


For this case, the graph of $y = -e^x$ is the **reflection** of the graph of $y = e^x$ in the ***x*-axis**. Similarly, $y = -e^x$ will be denoted as $y = -f(x)$.

In general, **the graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the *x*-axis.**

Example 7

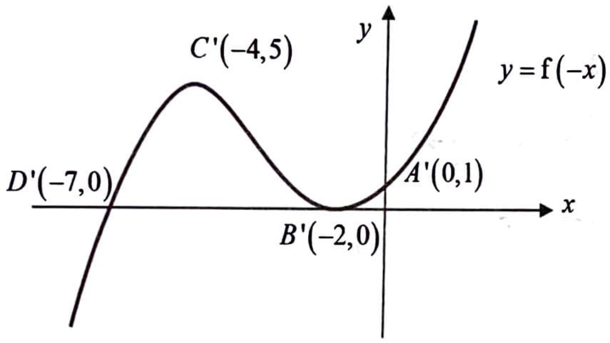
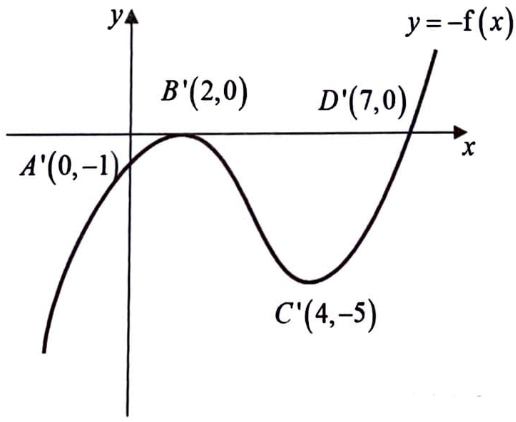
The following diagrams shows the graphs of $y=\mathrm{f}(x)$.



Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to *A*, *B*, *C*, and *D*.

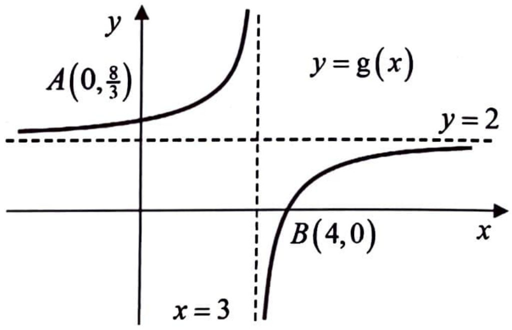
(a) $y=\mathrm{f}(-x)$,

(b) $y=-\mathrm{f}(x)$.

Solution	**ThinkZone**
(a) 	
(b) 	

Example 8:

The following diagram shows the graph of $y = g(x)$.



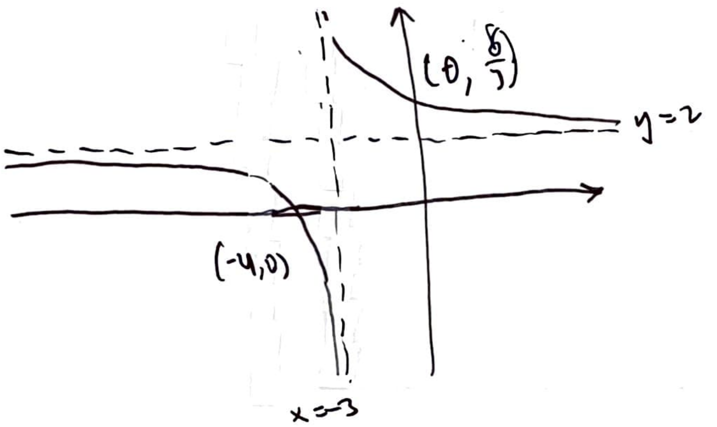
Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to A and B, and the equations of asymptotes.

(a) $y = g(-x)$,

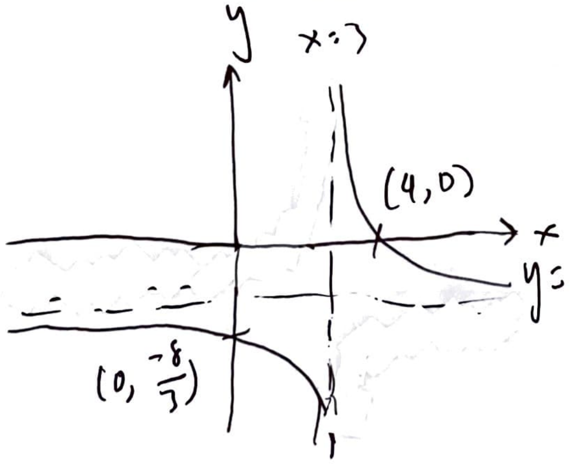
(b) $y = -g(x)$.

Solution

(a)



(b)



ThinkZone

Summary of linear transformations

Given $y=\text{f}(x)$ and $a>0$, you need to be able to do and describe the following linear transformations.

Type	Description	In the sketch of the curve
a) $y=\text{f}(x-a)$	Translation of graph of $y=\text{f}(x)$ by a units in the direction of the x-axis.	Add a to all x-coordinates
$y=\text{f}(x+a)$	Translation of graph of $y=\text{f}(x)$ by $-a$ units in the direction of the x-axis.	Subtract a from all x-coordinates
b) $y=\text{f}(x)+a$	Translation of graph of $y=\text{f}(x)$ by a units in the direction of the y-axis.	Add a to all y-coordinates
$y=\text{f}(x)-a$	Translation of graph of $y=\text{f}(x)$ by $-a$ units in the direction of the y-axis.	Subtract a from all y-coordinates
c) $y=\text{f}(ax)$	Scaling of graph of $y=\text{f}(x)$ by factor $\frac{1}{a}$ parallel to the x-axis.	Multiply $\frac{1}{a}$ to all x-coordinates
d) $y=a\text{f}(x)$	Scaling of graph of $y=\text{f}(x)$ by factor a parallel to the y-axis.	Multiply a to all y-coordinates
e) $y=\text{f}(-x)$	Reflection of graph of $y=\text{f}(x)$ in the y-axis.	Multiply -1 to all x-coordinates
f) $y=-\text{f}(x)$	Reflection of graph of $y=\text{f}(x)$ in the x-axis.	Multiply -1 to all y-coordinates

3.2.4 Transformations involving the modulus function

Recall that the modulus function is defined by

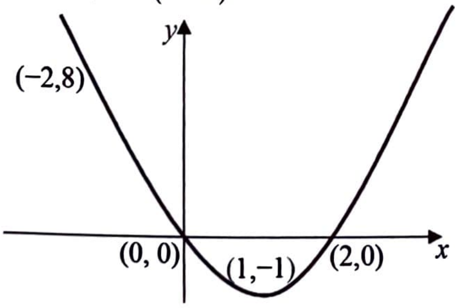
$$|x|=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

The modulus function is obtained by keying [2nd] followed by [0] and select abs(in the graphing calculator.

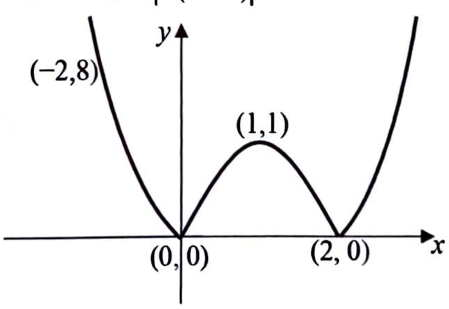
(a) Graph of $y=|\text{f}(x)|$ ('O' level knowledge)

Graph the following functions

(i) $y=x(x-2)$



(ii) $y=|x(x-2)|$



$y=|\text{f}(x)|$ is obtained from the graph of f(x) as follows:

Step 1: Keep the portion of the graph of f(x) on and above the x-axis (i.e. for $y\geq 0$).

Step 2: Reflect the portion of the graph of f(x) below the x-axis in the x-axis.

(b) Graph of $y = f(|x|)$

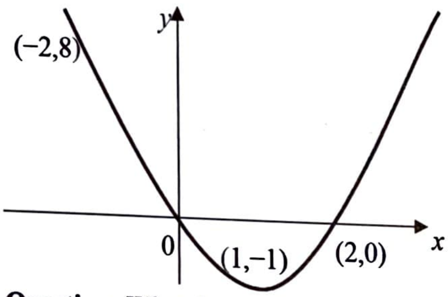
Given that $f(x) = x(x-2)$, fill in the table of values below:

x	−3	−2	−1	0	1	2	3
$f(x)$	15	8	3	0	−1	0	3
$\lvert x \rvert$	3	2	1	0	1	2	3
$f(\lvert x \rvert)$	3	0	−1	0	−1	0	3

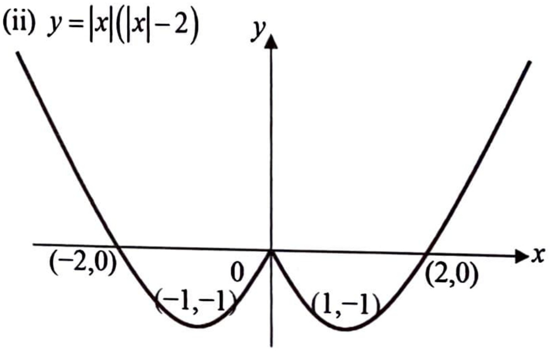
Q: From the table of values, where is the line of symmetry of the graph $y = f(|x|)$?

Graph the following functions

(i) $y = x(x-2)$



(ii) $y = |x|(|x|-2)$



Question: What do you think is the relationship between the graph of $y = f(x)$ and $y = f(|x|)$?

$y = f(|x|)$ is obtained from the graph of $f(x)$ as follows:

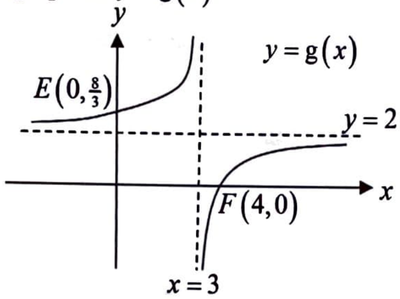
Step 1: Erase the portion of the graph of $f(x)$ on the left side of the y-axis (i.e. for $x < 0$).

Step 2: Keep the portion of the graph of $f(x)$ on the right of the y-axis (i.e. for $x \geq 0$) and reflect it about the y-axis to get a resulting graph that is symmetrical about the y-axis.

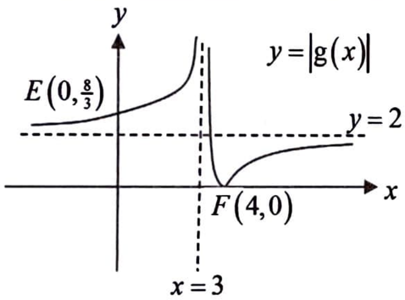
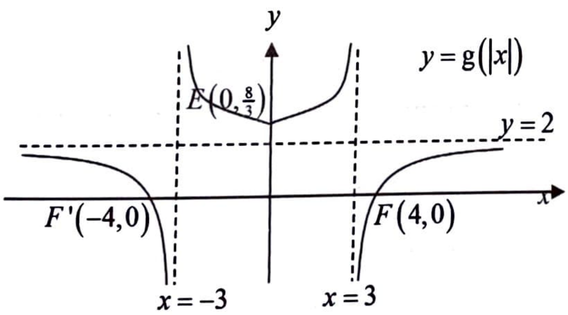
(Tip: A way to remember the transformation is **"Remove left, Reflect right"**)

Example 9

The following diagram show the graph of $y = g(x)$.



Sketch on separate diagrams, the graph of (a) $y = |g(x)|$, and (b) $y = g(|x|)$, showing clearly the coordinates of the points corresponding to E and F and the equations of asymptotes.

Solution	**ThinkZone**
(a) 	
(b) 	Is point E a stationary point?

3.2.5 Sequence of Transformations

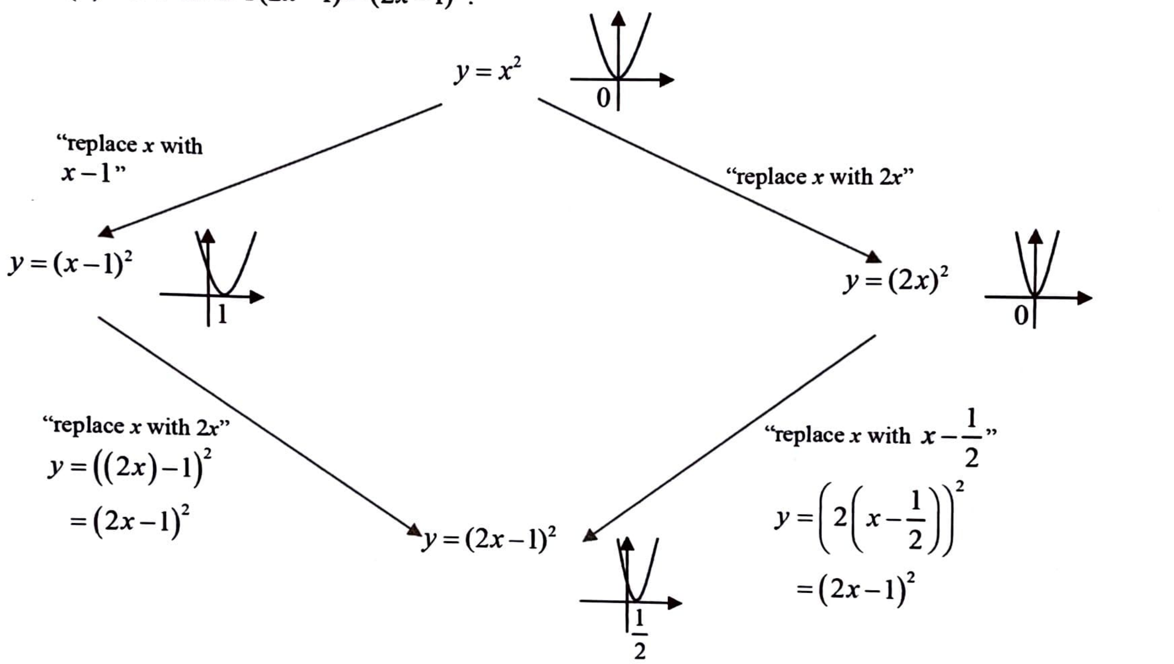
Most of the time we need to apply a sequence of two or more of the transformations that we have learnt earlier. We will determine some possible sequences of transformations by *successively* **replacing** the x or y values in the equation $y = f(x)$ so as to obtain the desired equation.

Example 10

Describe a sequence of transformations to obtain the graph of $y = f(2x - 1)$ from $y = f(x)$.

Solution

Let $f(x) = x^2$. Then $f(2x-1) = (2x-1)^2$.



1) Translation of 1 unit in the direction of the x-axis. 2) Scaling by factor $\frac{1}{2}$ parallel to the x-axis.	1) Scaling by factor $\frac{1}{2}$ parallel to the x-axis. 2) Translation of $\frac{1}{2}$ units in the direction of the x-axis.

There is usually more than one possible sequence of transformations to obtain the final graph. In general, we can following "rule" to show the **recommended** sequence as follows:

$$T_xS_xS_yT_y$$

where T_x : translation in the direction of the x-axis,
S_x : scaling parallel to the x-axis,
S_y : scaling parallel to the y-axis,
T_y : translation in the direction of the y-axis.

By using this rule, we can easily identify the required transformation.

$$y = a\,f(bx + c) + d$$

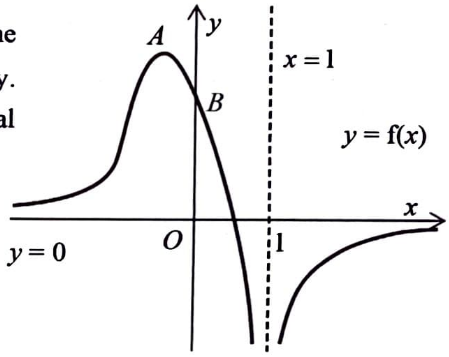
$a \uparrow S_y$, $b \uparrow S_x$, $c \uparrow T_x$, $d \uparrow T_y$

If $a = -1$, it is a reflection about the x-axis – S_y

If $b = -1$, it is a reflection about the y-axis – S_x

Example 11

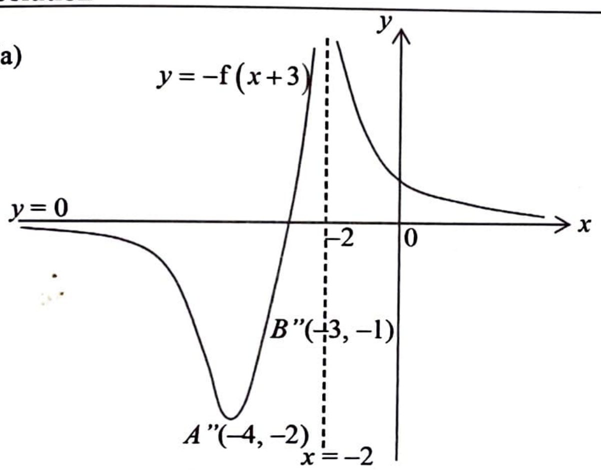
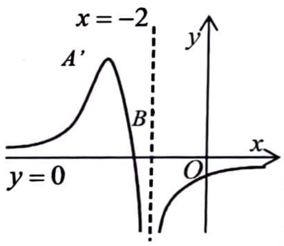
The diagram on the right shows the graph of $y = \mathrm{f}(x)$. The coordinates of point A and B are $(-1, 2)$ and $(0, 1)$ respectively. The graph has a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 1$.



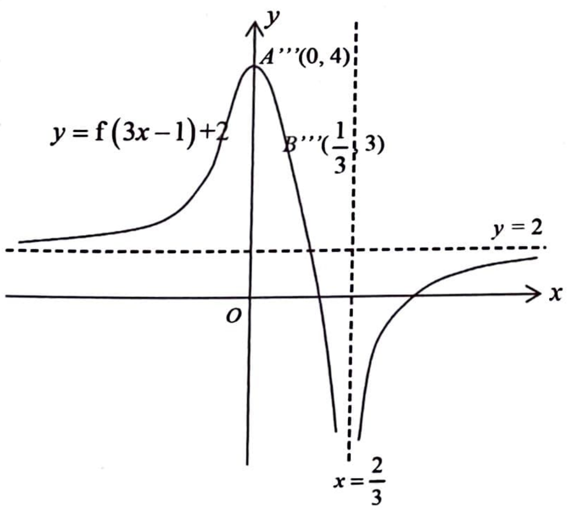
Sketch on separate diagrams, the graphs of

(a) $y = -\mathrm{f}(x+3)$,

(b) $y = \mathrm{f}(3x-1)+2$, giving the coordinates of the corresponding points of A and B.

<table>
<tr><th>Solution</th><th>Sequence of transformations</th></tr>
<tr><td>(a)

</td><td>(1) Translation of -3 units in the direction of the x-axis.


Vertical asymptote: $x = 1 \rightarrow x = -2$
A(−1, 2) → A'(−4, 2)
B(0, 1) → B'(−3, 1)

(2) Reflection in the x-axis.
'No change' in horizontal asymptote.
A'(−4, 2) → A''(−4, −2)
B'(−3, 1) → B''(−3, −1)</td></tr>
<tr><td>(b)

</td><td>(1) Translation of 1 unit in the direction of the x-axis.
Vertical asymptote: $x = 1 \rightarrow x = 2$
A(−1, 2) → A'(0, 2)
B(0, 1) → B'(1, 1)

(2) Scaling by scale factor $\frac{1}{3}$ parallel to the x-axis.
Vertical asymptote: $x = 2 \rightarrow x = \frac{2}{3}$
A'(0, 2) → A''(0, 2)
B'(1, 1) → B''($\frac{1}{3}$, 1)

(3) Translation by 2 units in the direction of the y-axis.
Horizontal asymptote: $y = 0 \rightarrow y = 2$
A''(0, 2) → A'''(0, 4)
B''($\frac{1}{3}$, 1) → B'''($\frac{1}{3}$, 3)</td></tr>
</table>

Example 12

Describe the sequence of transformations that maps $y = f(x)$ to $y = 1 + f(2 - x)$.

Solution	ThinkZone
1) Translate the graph of $y = f(x)$ by -2 units in the direction of the x-axis. 2) Reflect in the y-axis. 3) Translate by 1 unit in the direction of the y-axis.	$y = f(-x+2)+1$ ↑ ↑ ↑ S_x T_x T_y ② ① ③

Example 13

The graph of $y = x(x+2)$ is scaled by factor $\frac{1}{2}$ parallel to the x-axis, followed by a translation of 2 units in the direction of the x-axis and a reflection in the y-axis.

Determine the equation of the resulting graph.

Solution	ThinkZone
Scaling by factor $\frac{1}{2}$ parallel to the x-axis $y = x(x+2) \longrightarrow y = (2x)(2x+2) = 4x(x+1)$	Replace x with $2x$.
Translation of 2 units in the direction of the x-axis $y = 4x(x+1) \longrightarrow y = 4(x-2)(x-2+1)$ $= 4(x-2)(x-1)$	Replace x with $x-2$.
Reflection in the y-axis. $y = 4(x-2)(x-1) \longrightarrow y = 4(-x-2)(-x-1)$ $= 4(x+2)(x+1)$	Replace x with $-x$.
Equation of the resulting graph is $y = 4(x+2)(x+1)$.	

Self-Review 2:

The following diagram shows the graph of $y = f(x)$.

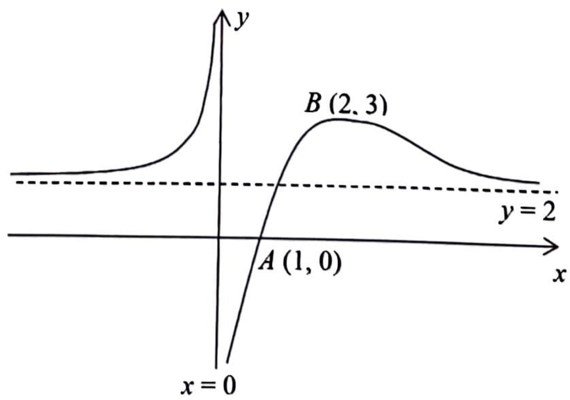
Sketch on separate diagrams, the following graphs, showing clearly the coordinates of the points corresponding to A and B, and the equations of asymptotes.

(a) $y = f(1-2x)$,

(b) $y = 2f(x)+1$.

[(a) A'(0, 0), B'$(-\frac{1}{2}, 3)$, $x = -\frac{1}{2}, y = 2$

(b) A'(1, 1), B'(2, 7), $x = 0, y = 5$]



Example 14: A curve $y = \mathrm{f}(x)$ undergoes in succession, the following transformations:

A: A reflection in the y-axis

B: A translation of 2 units in the direction of the x-axis

C: A translation of -1 unit in the direction of the y-axis

The equation of the resulting curve is $y = \mathrm{g}(x)$, where $\mathrm{g}(x) = \dfrac{-3}{-x+5}$. Determine the equation of the curve $y = \mathrm{f}(x)$, in terms of x.

Solution

$$y = \frac{-3}{-x+5}$$

Reverse C: $y = \dfrac{-3}{-x+5} + 1$

Reverse B: $y = \dfrac{-3}{-(x+2)+5} + 1 = \dfrac{-3}{-x+3} + 1$

Reverse A: $y = \dfrac{-3}{-(-x)+3} + 1 = \dfrac{-3}{x+3} + 1 = \dfrac{-3+x+3}{x+3} = \dfrac{x}{x+3}$

The original equation is $y = \dfrac{x}{x+3}$.

ThinkZone

To determine the equation of the curve $y = \mathrm{f}(x)$, we have to "undo" the transformation in the reverse direction.

Reverse C: A translation of 1 unit in the direction of the y-axis

Reverse B: A translation of -2 units in the direction of the x-axis

Reverse A: A reflection in the y-axis

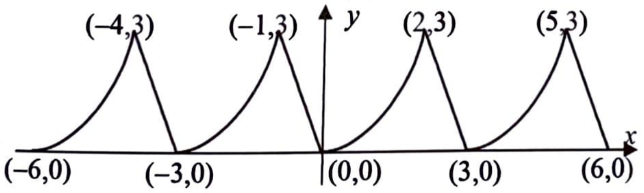
Example 15

Let
$$\mathrm{f}(x) = \begin{cases} \frac{3}{4}x^2, & 0 < x \le 2 \\ 9 - 3x, & 2 < x \le 3 \end{cases}$$

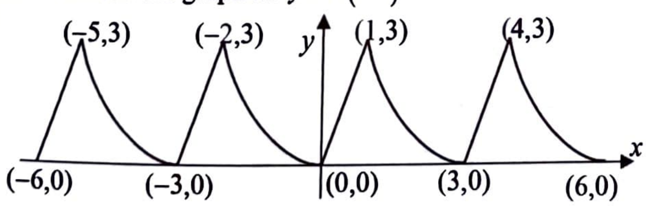
It is given that $\mathrm{f}(x) = \mathrm{f}(x+3)$ for all values of x. Sketch the graph of $y = -\mathrm{f}(-x)$ for $-6 \le x \le 6$.

Solution

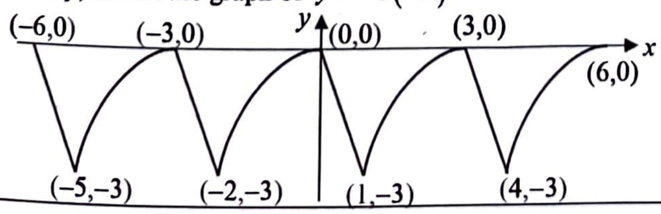
First sketch the graph of $y = \mathrm{f}(x)$.



Then sketch the graph of $y = \mathrm{f}(-x)$



Finally, sketch the graph of $y = -\mathrm{f}(-x)$



ThinkZone

Using TSST, we get 2 transformations:

1) Reflection in the y-axis
2) Reflection in the x-axis

In the context of this question, does the order of transformation matter?

How is the graph of $y = -\mathrm{f}(-x)$ related to the graph of $y = \mathrm{f}(x)$?