(ANSWERS) THREE MUSKETEERS EXAM 😈

Time: 2 hours 15 minutes

Name: ……………………………….

Marks: **90**

Topics: the triple threat
(*trigonometry*, *differentiation*, *integration*)

good luck cus the paper isnt very lucky

☆ = 40m
(tougher questions)

difficulty rating /10

90

Formula List:

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

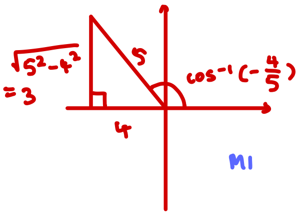
Answer **all** questions :)

Do not use a calculator for the whole of this question.

1 **(a)** Find the value of $\cot\left[\,2\cos^{-1}(-\frac{4}{5})\,\right]$. [2]

Since the principal values of $\cos^{-1}(-\frac{4}{5})$ are $0° \leq \cos^{-1}(-\frac{4}{5}) \leq 180°$, and $\cos(\cos^{-1}(-\frac{4}{5})) < 0$,

$\cos^{-1}(-\frac{4}{5})$ lies in the 2nd quadrant.



M1

$$\cot\left[2\cos^{-1}(-\tfrac{4}{5})\right] = \frac{1}{\tan(2\cos^{-1}(-\frac{4}{5}))}$$

$$= \frac{1-\tan^2(\cos^{-1}(-\frac{4}{5}))}{2\tan(\cos^{-1}(-\frac{4}{5}))}$$ flipped

$$= \frac{1-(-\frac{3}{4})^2}{2(-\frac{3}{4})}$$

$$= \frac{1-\frac{9}{16}}{-\frac{3}{2}}$$

$$= \frac{7}{16}\times(-\frac{2}{3})$$

$$= -\frac{7}{24}$$ A1

7/10

using $0° \leq \frac{B}{2} \leq 180°$ is ×
as 0° & 180° do not make $\sin\frac{B}{2} > 0$

(b) It is given that $\cot(-B) < 0$ and $\sec(-B) < 0$, where $0 \leq B \leq 2\pi$.

Explain why $\sin\frac{B}{2} > 0$. [1]

7/10

Since $-\cot B < 0$,
$\cot B > 0$
$\frac{1}{\tan B} > 0$
$\tan B > 0$

$\sec(-B) < 0$
$\frac{1}{\cos(-B)} < 0$
$\cos(-B) < 0$
$\cos B < 0$

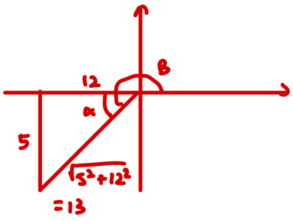
Since $\tan B > 0$ and $\cos B < 0$, B lies in the 3rd quadrant. $(180° < B < 270°)$ / $(\pi < B < \frac{3\pi}{2})$
Hence, $\frac{B}{2}$ lies in the 2nd quadrant,
and in the 2nd quadrant, sine is positive. $(90° < \frac{B}{2} < 135°)$ / $(\frac{\pi}{2} < \frac{B}{2} < \frac{3\pi}{4})$
Hence, $\sin\frac{B}{2} > 0$

(ii) Given that $\cot B = \frac{12}{5}$, find the exact value of $\cos\left(\frac{B-\pi}{2}\right)$. [2]

$\cot B = \frac{12}{5}$
$\tan B = \frac{5}{12}$
$\cos B = -\frac{12}{13}$

$\cos(\frac{B}{2} - \frac{\pi}{2})$
$= \cos(-(\frac{\pi}{2} - \frac{B}{2}))$
$= \sin\frac{B}{2}$

1. $\cos(-\theta) = \cos\theta$
2. $\cos(\frac{\pi}{2} - \theta) = \sin\theta$



$\cos[2(\frac{B}{2})] = 1 - 2\sin^2\frac{B}{2}$
$-\frac{12}{13} = 1 - 2\sin^2\frac{B}{2}$ M1
$\frac{12}{13} = 2\sin^2\frac{B}{2} - 1$
$2\sin^2\frac{B}{2} = \frac{25}{13}$
$\sin^2\frac{B}{2} = \frac{25}{26}$
$\sin\frac{B}{2} = \sqrt{\frac{25}{26}}$ $(\sin\frac{B}{2}$ belongs in the 2nd quadrant) OR $(\sin\frac{B}{2} > 0)$
$= \frac{5}{\sqrt{26}}$ → A1 OR $\frac{5\sqrt{26}}{26}$

(c) By finding the exact value of $\operatorname{cosec}^2(\frac{7\pi}{12})$, find $\sec^2(\frac{7\pi}{12})$. [3]

6/10

$$\operatorname{cosec}^2 \frac{7\pi}{12} = \frac{1}{\sin^2 \frac{7\pi}{12}}$$

$$= \frac{1}{\left[\sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right]^2}$$

$$= \frac{1}{\left(\sin \frac{\pi}{4} \cos \frac{\pi}{3} + \cos \frac{\pi}{4} \sin \frac{\pi}{3}\right)^2}$$

$$= \frac{1}{\left(\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)\right)^2}$$

$$= \frac{1}{\left(\frac{\sqrt{2}+\sqrt{6}}{4}\right)^2}$$

$$= \frac{1}{\frac{2+4\sqrt{3}+6}{16}}$$

$$= \frac{16}{8+4\sqrt{3}} \times \frac{8-4\sqrt{3}}{8-4\sqrt{3}} \quad \text{M1}$$

$$= \frac{128-64\sqrt{3}}{64-48}$$

$$= \frac{128-64\sqrt{3}}{16}$$

$$= \underline{8-4\sqrt{3}} \quad \text{A1}$$

$$\cot^2 \frac{7\pi}{12} + 1 = \operatorname{cosec}^2 \frac{7\pi}{12}$$

$$\frac{1}{\tan^2 \frac{7\pi}{12}} = 8-4\sqrt{3}-1$$

$$\frac{1}{\tan^2 \frac{7\pi}{12}} = 7-4\sqrt{3}$$

$$\tan^2 \frac{7\pi}{12} = \frac{1}{7-4\sqrt{3}} \times \frac{7+4\sqrt{3}}{7+4\sqrt{3}}$$

$$= 7+4\sqrt{3}$$

$$\sec^2\left(\frac{7\pi}{12}\right) = 1+\tan^2\left(\frac{7\pi}{12}\right)$$

$$= 1+(7+4\sqrt{3})$$

$$= \underline{\underline{8+4\sqrt{3}}} \quad \text{A1}$$

9.5/10

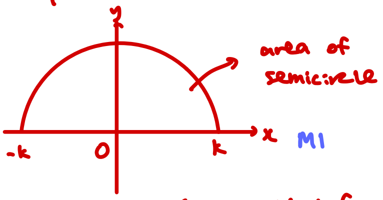
2 **(a)** Show, using the aid of a diagram, that $\int_{-k}^{k} 3\sqrt{k^2-x^2}\ dx = \frac{3\pi k^2}{2}$. [2]

$y = \sqrt{k^2-x^2}$

$y^2 = k^2 - x^2$

$x^2 + y^2 = k^2$

↳ equation of a circle above x-axis



M1

$\int_{-k}^{k} 3\sqrt{k^2-x^2}\ dx = 3\int_{-k}^{k}\sqrt{k^2-x^2}\ dx$

$= 3 \times$ (area of semicircle with radius k)

$= 3 \times \frac{\pi}{2} k^2$ A1

$= \frac{3\pi k^2}{2}$

(shown)

→ 1st part ✗ → cap at 1 for (i) & (ii) only if trigo part ✓

(b) Hence, find in terms of k and/or π, the value of:

(i) $\int_{k}^{0} \sqrt{4k^2-4x^2}\ -2\pi\cos\left(\frac{\pi x}{2k}\right) dx.$ [2]

$\int -2\pi\cos\frac{\pi x}{2k}\ dx$ correctly → M1

$\int_{k}^{0} 2\sqrt{k^2-x^2} - 2\pi\cos\left(\frac{\pi x}{2k}\right) dx$

$= 2\int_{k}^{0}\sqrt{k^2-x^2}\ dx \quad - \int_{k}^{0} 2\pi\cos\left(\frac{\pi x}{2k}\right) dx$

$= 2\left(-\frac{\pi k^2}{4}\right) - \left[2\pi\left(\frac{2k}{\pi}\right)\sin\frac{\pi x}{2k}\right]_{k}^{0}$

$= 2\left(-\frac{\pi k^2}{4}\right) - \left[4k\sin\frac{\pi}{2k}x\right]_{k}^{0}$

$= 2\left(-\frac{\pi k^2}{4}\right) - \left[(4k\sin 0) - (4k\sin\frac{\pi}{2})\right]$

$= -\frac{\pi k^2}{2} - (0 - 4k)$

$= \underbrace{-\frac{\pi}{2}k^2}_{A1} + \underbrace{4k}_{A1}$

(w/ steps)

(ii) $\int_{0}^{-k} \sqrt{k^2-x^2}\ + \pi\tan^2\left(\frac{\pi x}{3k}\right) dx.$ [2]

$\int \pi\tan^2\left(\frac{\pi x}{3k}\right) dx$ correctly → M1

$\int_{0}^{-k}\sqrt{k^2-x^2}\ dx + \int_{0}^{-k}\pi\tan^2\left(\frac{\pi x}{3k}\right) dx$

$= \left(-\frac{\pi k^2}{4}\right) - \int_{-k}^{0}\pi\left[\sec^2\left(\frac{\pi}{3k}x\right) - 1\right] dx$

$= -\frac{\pi k^2}{4} - \int_{-k}^{0}\pi\sec^2\frac{\pi}{3k}x - \pi\ dx$

$= -\frac{\pi k^2}{4} - \left[3k\tan\frac{\pi x}{3k} - \pi x\right]_{-k}^{0}$

$= -\frac{\pi k^2}{4} - \left[(3k\tan 0 - 0) - (3k\tan(-\frac{\pi}{3}) + k\pi)\right]$

$= -\frac{\pi k^2}{4} + (3k(-\sqrt{3}) + k\pi)$

$= \underbrace{-\frac{\pi k^2}{4}}_{A1}\underbrace{- 3\sqrt{3}k + k\pi}_{A1}$ //

(w/ steps)

3 **(a)** Prove that $\frac{4\sin\theta+4\sin^2\theta}{\sec\theta+\tan\theta}=2\sin 2\theta$. [2]

6/10

$$\text{LHS}=\frac{4\sin\theta+4\sin^2\theta}{\sec\theta+\tan\theta}$$

$$=(4\sin\theta+4\sin^2\theta)\div\left(\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}\right)$$

$$=(4\sin\theta+4\sin^2\theta)\div\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$=4\sin\theta(1+\sin\theta)\times\frac{\cos\theta}{1+\sin\theta}\quad \text{M1}$$

$$=4\sin\theta\cos\theta$$

$$=2(2\sin\theta\cos\theta)$$

$$=2\sin2\theta\quad \text{M1}$$

$$=\text{RHS}$$

(b) Hence, solve $\frac{\sin3\theta+\sin^2 3\theta}{\sec3\theta+\tan3\theta}=\cot6\theta$, for ~~$-\pi\le\theta\le\pi$.~~ $-1\le\theta\le1$ [3]

$-6\le 6\theta\le 6$

☆ 9/10

$$\frac{1}{4}\left(\frac{4\sin3\theta+4\sin^2 3\theta}{\sec3\theta+\tan3\theta}\right)=\cot6\theta$$

$$\frac{1}{4}(2\sin6\theta)=\cot6\theta$$

$$\frac{1}{2}\sin6\theta=\cot6\theta$$

$$\frac{1}{2}\sin6\theta=\frac{\cos6\theta}{\sin6\theta}$$

$$\sin^2 6\theta=2\cos6\theta$$

$$1-\cos^2 6\theta=2\cos6\theta$$

$$0=\cos^2 6\theta+2\cos6\theta-1\quad \text{M1}$$

$$\cos6\theta=\frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)}$$

$$=\frac{-2\pm\sqrt{8}}{2}$$

$$=-1\pm\sqrt{2}$$

$\cos6\theta=-1+\sqrt{2}$ or $\cos6\theta=-1-\sqrt{2}$ (reject as $-1\le\cos6\theta\le1$)

$6\theta=\cos^{-1}(\sqrt{2}-1)$

$\alpha=1.14\text{rad}$ (3s.f.) M1

6θ belongs in the 1st & 4th quadrant.

$6\theta=1.14,\ 2\pi-1.14,\ -1.14,\ -2\pi+1.14,$

$=1.14,\ 5.14,\ -1.14,\ -5.14$

$\theta=0.191,\ 0.857,\ -0.191,\ -0.857$ (3s.f.) // A1

4 **(i)** Show that $\frac{\mathrm{d}}{\mathrm{d}x} x^6 \ln x^3 = ax^5 \ln x^3 + bx^5$, where $x > 0$. and a and b are constants to be found. [1]

3/10

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$$\frac{d}{dx} x^6 \ln x^3 = \frac{d}{dx} 3x^6 \ln x$$

$$= 3 \frac{d}{dx} x^6 \ln x$$

$$= 3\left[x^6\left(\frac{1}{x}\right) + (\ln x)(6x^5)\right]$$

$$= 18x^5 \ln x + 3x^5$$

$$= \underline{6x^5 \ln x^3 + 3x^5}$$

$a = 6$, $b = 3$ A1

(ii) **Only by expressing** $\int ax^5 \ln x^3 + bx^5\,\mathrm{d}x = x^6 \ln x^3 + c_1$, where c_1 is an arbitrary constant, find $\int x^5 \ln x^3\,\mathrm{d}x$. 2 ~~[3]~~

↳ other methods = 0

☆ 9/10

$$\int 6x^5 \ln x^3 + 3x^5\,dx = x^6 \ln x^3 + c_1$$

$$\int x^5 \ln x^3 + \frac{3}{6} x^5\,dx = \frac{1}{6} x^6 \ln x^3 + c_2$$

$$\int x^5 \ln x^3 + \frac{1}{2} x^5\,dx = \frac{1}{6} x^6 \ln x^3 + c_2$$

$$\int x^5 \ln x^3 + \frac{1}{2} x^5 - \frac{1}{2} x^5\,dx = \frac{1}{6} x^6 \ln x^3 - \underline{\int \frac{1}{2} x^5\,dx} + c_2$$

↓ M1

$$= \frac{1}{6} x^6 \ln x^3 - \underline{\frac{x^6}{2(6)} + c_3} + c_2$$

$$= \underline{\frac{1}{6} x^6 \ln x^3 - \frac{1}{12} x^6 + c_4} \quad //$$

A1

(iii) Look at Diagram **4** below. The shaded area can be expressed in two different ways. One way in expressing the shaded area is first making x the subject, then expressing it as a bound integral below:

$\boxed{\int_n^m x\,dy,}$ where m and n are constants.

Find the values of m and n, and the shaded area below, all in 3 sig fig . ~~[3]~~ 2

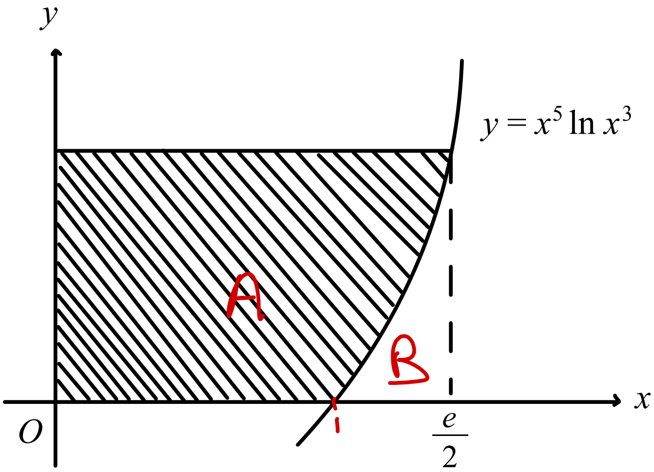


Diagram **4**

Sub $y=0$:

$x^5 \ln x^3 = 0$

$x = 0$ or $x = 1$

$\int_1^{e/2} x^5 \ln x^3 \, dx$

$= \left[\frac{1}{6}x^6 \ln x^3 - \frac{1}{12}x^6\right]_1^{\frac{e}{2}}$

$= \left(\frac{1}{6}\left(\frac{e}{2}\right)^6 \ln\left(\frac{e}{2}\right)^3 - \frac{1}{12}\left(\frac{e}{2}\right)^6\right)$

$-$

$\left(\frac{1}{6}(1)^6 \ln 1^3 - \frac{1}{12}(1)^6\right)$

$= \left(\frac{e^6}{384}(3)(1-\ln 2) - \frac{e^6}{768}\right)$

$-$

$\left(\frac{1}{6} - \frac{1}{12}\right)$

$= \frac{e^6}{128}(1-\ln 2) - \frac{e^6}{768} - \frac{1}{12}$

$= e^6\left(\frac{1-\ln 2}{128} - \frac{1}{768}\right) - \frac{1}{12}$ units^2 //

$\left(\frac{3}{32}e^5(1-\ln 2)\right)\left(\frac{e}{2}\right) - \left[e^6\left(\frac{1-\ln 2}{128} - \frac{1}{768}\right) - \frac{1}{12}\right]$

$= \underline{5.44 \text{ units}^2}$ (3s.f.) A1

Sub $x = \frac{e}{2}$:

$y = \left(\frac{e}{2}\right)^5 \ln\left(\frac{e}{2}\right)^3$

$y = \frac{3e^5}{32}(\ln e - \ln 2)$

$m = \frac{3}{32}e^5(1 - \ln 2) = 4.27$ (3s.f.)

best ans as constants need to be exact

$n = 0$

A1

iv✓

~~(iii)~~ Integrate $\dfrac{2}{(x+3)^2} - \dfrac{5}{2-3x} + \dfrac{x-1}{x+1}$ with respect to x. [2]

7/10

$\int \frac{2}{(x+3)^2} - \frac{5}{2-3x} + \frac{x-1}{x+1}\,dx$

$$x+1 \overline{\big)\, x-1}\quad 1$$
$$-(x+1)$$
$$-2$$

$= 2\int \frac{1}{(x+3)^2}dx - 5\int \frac{1}{2-3x}dx + \int 1 - \frac{2}{x+1}\,dx$

$= 2\int (x+3)^{-2}dx + \frac{5}{3}\int \frac{-3}{2-3x}dx + \int 1\,dx - 2\int \frac{1}{x+1}dx$

$= 2\left[\frac{(x+3)^{-2+1}}{(-2+1)(1)}\right] + \frac{5}{3}\ln(2-3x) + x - 2\ln(x+1) + c$

$= \underline{\overbrace{-\frac{1}{x+3} + \frac{5}{3}\ln(2-3x)}^{M1} + \overbrace{x - 2\ln(x+1)}^{M1} + c}$

5 It is given that $f''(x) = 10\sin^2 3x - 6\tan^2 6x - 11$.

(i) Given that the normal of the line at $x = 0$ is perpendicular to the x-axis, find and simplify an expression for $f'(x)$. [3]

8/10

$f''(x) = 10\sin^2 3x - 6\tan^2 6x - 11$

$= -5(-2\sin^2 3x) - 6(\sec^2 6x - 1) - 11$

$= -5(1 - 2\sin^2 3x) + 5 - 6\sec^2 6x + 6 - 11$

$= -5\cos 6x + 5 - 6\sec^2 6x + 6 - 11$

$= -5\cos 6x - 6\sec^2 6x$ M1 for simplifying $f''(x)$

$f'(x) = \int -5\cos 6x - 6\sec^2 6x \, dx$

$= -5(\frac{1}{6}\sin 6x) - 6(\frac{1}{6}\tan 6x) + c_1$

$= -\frac{5}{6}\sin 6x - \tan 6x + c_1$ M1

At $x = 0$, $f'(x) = 0$:

$-\frac{5}{6}\sin 0 - \tan 0 + c_1 = 0$

$c_1 = 0$

$\underline{f'(x) = -\frac{5}{6}\sin 6x - \tan 6x}$ A1

(ii) Find an expression for $\frac{d}{dx}\ln(2 - 2\sin^2 nx)$, where n is a constant. [1]

5/10

$\frac{d}{dx}\ln(2 - 2\sin^2 nx)$ OR

$= \frac{d}{dx}\ln(2\cos^2 nx)$

$= \frac{d}{dx}\ln 2 + \ln\cos^2 nx$

$= \frac{d}{dx} 2\ln\cos nx$

$= -\underline{\frac{2n\sin nx}{\cos nx}}$ / $\underline{-2n\tan nx}$

A1

$= 2\left(\frac{\cos nx(-n\sin nx)}{\cos^2 nx}\right)$

$= 2\left(\frac{-n\sin nx}{\cos nx}\right)$

$= -2n\tan nx$ //

(iii) Hence, given that $y = \mathrm{f}(x)$ intersects the point $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, find an expression for $\mathrm{f}(x)$. [3]

☆ 9/10

$f'(x) = -\frac{5}{6}\sin 6x - \tan 6x$

$f(x) = \int -\frac{5}{6}\sin 6x - \tan 6x \, dx$

$\int -2n\tan nx \, dx = \ln(2-2\sin^2 nx)$

$\cdot\, 2n \int \tan nx \, dx = \ln(2-2\sin^2 nx)$

$\int \tan nx \, dx = -\frac{1}{2n}\ln(2-2\sin^2 nx)$

$= -\frac{5}{6}\left(-\frac{1}{6}\cos 6x\right) + c_2 + \frac{1}{12}\int -12\tan 6x \, dx$

$= \frac{5}{36}\cos 6x + c_2 + \frac{1}{12}\ln(2-2\sin^2 6x)$

$f(x) = \frac{5}{36}\cos 6x + \frac{1}{12}\ln(2-2\sin^2 6x) + c_3$ M1

Sub $x = \frac{\pi}{3}$, $f(x) = \frac{1}{2}$;

$\frac{1}{2} = \frac{5}{36}\cos 2\pi + \frac{1}{12}\ln(2-2\sin^2 2\pi) + c_3$

$\frac{1}{2} = \frac{5}{36} + \frac{1}{12}\ln(2) + c_3$

$c_3 = \frac{13}{36} - \frac{\ln 2}{12}$ M1

$f(x) = \frac{5}{36}\cos 6x + \frac{1}{12}\ln(2-2\sin^2 6x) + \frac{13}{36} - \frac{\ln 2}{12}$ // A1

6 **(a)** It is given that $f(x) = 2e^{-kx} - 3e^{3x} + e^{\ln x}$.

If $-3f'(x) + f''(x) + 3 = 0$, find the exact value(s) of k. [2]

7/10

$f(x) = 2e^{-kx} - 3e^{3x} + x$

$f'(x) = -2ke^{-kx} - 3e^{3x} + 1$

$f''(x) = (-2k)(-ke^{-kx}) - 9e^{3x}$ M1

$= 2k^2e^{-kx} - 9e^{3x}$

$-3f'(x) + f''(x) + 3 = 0$:

$-3(-2ke^{-kx} - 3e^{3x} + 1) + (2k^2e^{-kx} - 9e^{3x}) + 3 = 0$

$6ke^{-kx} + 9e^{3x} - 3 + 2k^2e^{-kx} - 9e^{3x} + 3 = 0$

$(6k + 2k^2)e^{-kx} = 0$

$k(k+3)e^{-kx} = 0$

$k(k+3) = 0$ or $e^{-kx} = 0$

$\underline{k=0}$ or $k+3=0$ (reject as $e^{-kx} > 0$)

$\underline{k=-3}$

A1

(b) The equation of a curve is given by $y = 3x^2e^{-\frac{1}{2}x}$

A point P lies on the curve such that the normal to the curve at P has a negative gradient.

tangent → positive m

(i) State the range of values of the possible x coordinates of P. [2]

8/10

$y = 3x^2e^{-\frac{1}{2}x}$

$\frac{dy}{dx} = 3\left\{(x^2)(-\frac{1}{2}e^{-\frac{1}{2}x}) + (e^{-\frac{1}{2}x})(2x)\right\}$

$= 3\left\{-\frac{1}{2}x^2e^{-\frac{1}{2}x} + 2xe^{-\frac{1}{2}x}\right\}$

$\underline{= (-\frac{3}{2}x^2 + 6x)e^{-\frac{1}{2}x}}$ M1

$\frac{dy}{dx} > 0$:

$(-\frac{3}{2}x^2 + 6x)e^{-\frac{1}{2}x} > 0$

Since $e^{-\frac{1}{2}x} > 0$ for all real values of x,

$-\frac{3}{2}x^2 + 6x > 0$

$3x^2 - 12x < 0$

$x(x-4) < 0$

$\underline{0 < x < 4}$ A1

(ii) Find the equation of the normal at P, given that the x coordinate of P is 1. [2]

8/10

Sub $x=1$ into $\frac{dy}{dx}$:

$\frac{dy}{dx} = (-\frac{3}{2}(1)^2 + 6(1))e^{-\frac{1}{2}(1)}$

$= \frac{9}{2}e^{-\frac{1}{2}}$

gradient of normal $= -1 \div \frac{9}{2\sqrt{e}}$

$= \underline{-\frac{2\sqrt{e}}{9}}$ M1

Sub $x=1$ into y:

$y = 3(1)^2e^{-\frac{1}{2}(1)}$

$= \frac{3}{\sqrt{e}}$

$\underline{(1, \frac{3}{\sqrt{e}})}$

$y - \frac{3}{\sqrt{e}} = -\frac{2\sqrt{e}}{9}(x-1)$

$y - \frac{3}{\sqrt{e}} = -\frac{2\sqrt{e}}{9}x + \frac{2\sqrt{e}}{9}$

$\underline{y = -\frac{2\sqrt{e}}{9}x + (\frac{2\sqrt{e}}{9} + \frac{3}{\sqrt{e}})}$ //

or

$y = -\frac{2}{9}e^{\frac{1}{2}}x + \frac{2}{9}e^{\frac{1}{2}} + 3e^{-\frac{1}{2}}$

A1

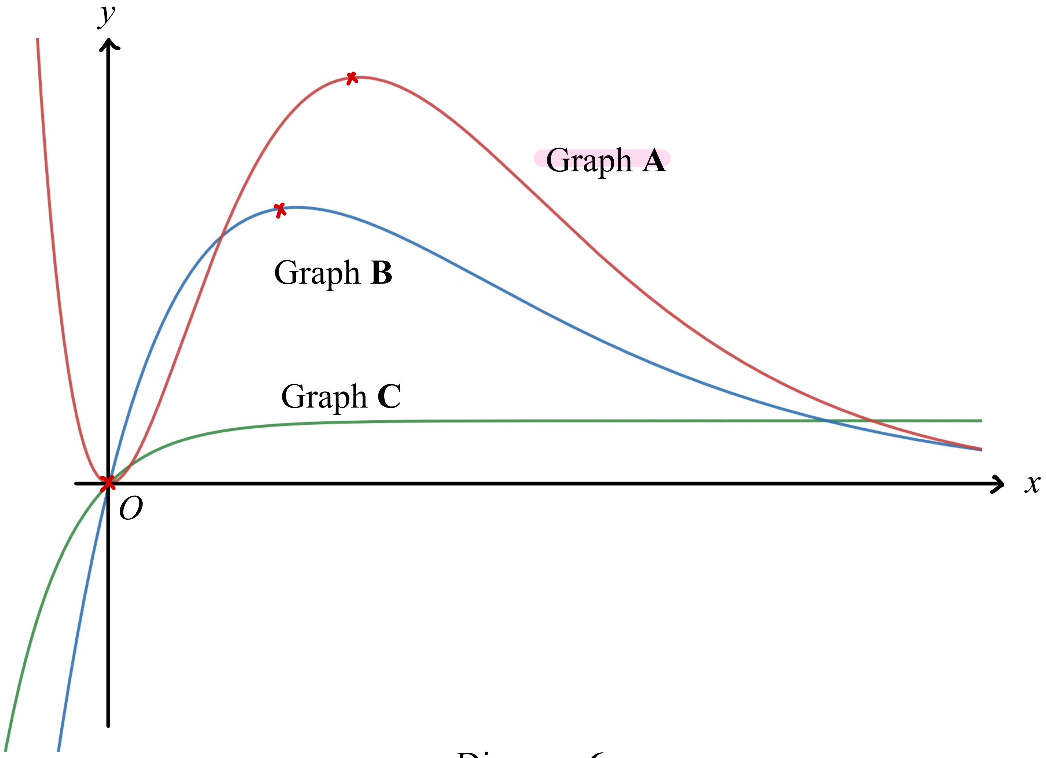
(iii) Look at the diagram below. It shows three different graphs, Graph **A**, **B**, and **C**.

Determine whether which graph is most suitable for the shape of the line $y = 3x^2 e^{-\frac{1}{2}x}$.

Explain your answer with reasonings.

9/10

Graph A. → Award 0 if not graph A



[2]

Diagram **6**

$y=3x^2e^{-\frac{1}{2}x}$

$\frac{dy}{dx}=(-\frac{3}{2}x^2+6x)e^{-\frac{1}{2}x}$

$\frac{dy}{dx}=0$:

$(-\frac{3}{2}x^2+6x)e^{-\frac{1}{2}x}=0$

$-\frac{3}{2}x^2+6x=0$ or $e^{-\frac{1}{2}x}=0$

$x^2-4x=0$ (reject as $e^{-\frac{1}{2}x}>0$)

$x(x-4)=0$

$x=0$ or $x-4=0$

$x=4$

$x=0$

	0^-	0	0^+
$\frac{dy}{dx}$:	-0.647 (negative)	0	0.556 (positive)
outline of graph:	\	—	/

By First Derivative Test, $x=0$ is a minimum.

$x=4$

4^-	4	4^+
0.0832 (positive)	0	-0.0792 (negative)
/	—	\

By First Derivative Test, $x=4$ is a maximum.

Graph **A.**

Firstly, Graph A is the only graph above that shows that it reaches to a minimum at the origin or @ $x=0$ (0 , 0). When finding the stationary points of *y*, it is also clear that the graph reaches to a minimum at (0 , 0) by depends on working First / Second Derivative Test. [1]

(required explanation above the other 2 below)

Secondly, from my answer in (b) (i), the **gradient** of this graph should be **positive/increasing** from $0 < x < 4$, and Graphs A and B both possibly show that the graph is increasing from $x = 0$ to 4, while graph C seems to increase beyond $x > 4$. + *workings* [1]

OR

Graphs A and B also show that there is a **maximum after** the **first** stationary point $x = 0$ by First/Second Derivative Test, and the other stationary point of this graph is at $x = 4$. + *workings* [1]

Award **0** for no explanation but have working and **1** for no working but have explanation

7 It is given that ~~f(x)~~ y $= \dfrac{2\sin x}{\cos x - 1}$, for $0 < x < 2\pi$.

(i) Explain, with reasoning~~s~~, whether ~~f(x)~~ y is increasing or decreasing. [3]

8/10

$$y = \frac{2\sin x}{\cos x - 1}$$

$$\frac{dy}{dx} = \frac{(\cos x - 1)(2\cos x) - (2\sin x)(-\sin x)}{(\cos x - 1)^2}$$

$$= \frac{2\cos^2 x - 2\cos x + 2\sin^2 x}{(\cos x - 1)^2}$$

$$= \frac{2 - 2\cos x}{(\cos x - 1)^2} \quad \text{M1}$$

$$= \frac{-2(\cos x - 1)}{(\cos x - 1)^2}$$

$$= -\frac{2}{\cos x - 1}$$

$$= \frac{2}{1 - \cos x} \quad \text{M1 ONLY THIS FORM}$$

Since $-1 < \cos x < 1$ for $0 < x < 2\pi$

$-1 < -\cos x < 1$

$0 < 1 - \cos x < 2$, $1 - \cos x > 0$

Hence, $\dfrac{2}{1-\cos x} > 0$, $\dfrac{dy}{dx} > 0$.

Thus, y is increasing for $0 < x < 2\pi$.

A1

(ii) Find the range of values of x for which ~~f'(x)~~ $\frac{dy}{dx}$ is increasing. [2]

☆ 9/10

$$\frac{dy}{dx} = \frac{2}{1 - \cos x}$$

$$\frac{d^2y}{dx^2} = \frac{(1-\cos x)(0) - (2)(\sin x)}{(1 - \cos x)^2}$$

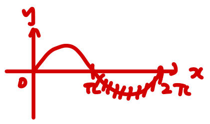
$$= -\frac{2\sin x}{(1-\cos x)^2}. \quad \text{M1}$$

Since $(1-\cos x)^2 > 0$ for $0 < x < 2\pi$,

$-2\sin x > 0$

$2\sin x < 0$

$\sin x < 0$



$\underline{\pi < x < 2\pi}$ A1

OR

Since $\sin x < 0$, x has to belong in the **3rd** or **4th quadrant**.

$\pi < x < 2\pi$

8 In the diagram, a surveillance camera is mounted at a point S that is 7.5 m towards the left of point L_0. A lizard crawls from point L_0 along an upward course at a speed of 1.2 m/s. The surveillance camera tracks the motion of the lizard by panning upwards from the fixed point S.
Find the rate of change of the angle that the surveillance camera makes with the lizard and the wall of the building when the runner is 10 m from L_0. Give your answer in radians per second.

☆ 9.5/10

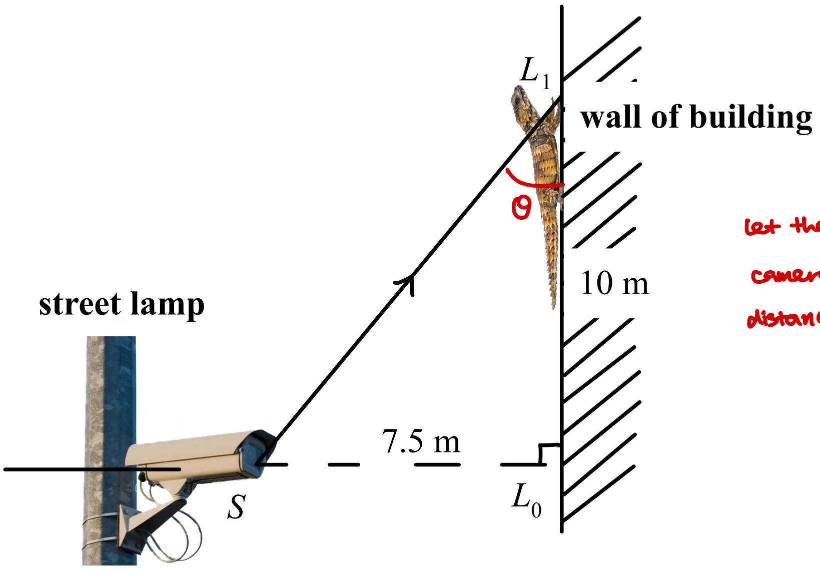


Diagram **8**

[4]

let the angle the surveillance camera makes be θ and the distance from L_0 to L_1 be s.

$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \times \frac{ds}{dt}$$

$\frac{ds}{dt} = 1.2\ m/s$

$\tan\theta = \frac{7.5}{s}$

$s = \frac{7.5}{\tan\theta}$

$= 7.5(\tan\theta)^{-1}$

$\frac{ds}{d\theta} = 7.5\left[(-1)(\tan\theta)^{-2}(\sec^2\theta)\right]$

$= -\frac{7.5\sec^2\theta}{\tan^2\theta}$

$= -7.5\left(\frac{1}{\cos^2\theta} \div \frac{\sin^2\theta}{\cos^2\theta}\right)$

$= -\frac{7.5}{\sin^2\theta}$ M1 for $\frac{ds}{d\theta}$

$\frac{d\theta}{ds} = -\frac{\sin^2\theta}{7.5}$

$\frac{d\theta}{dt} = \frac{d\theta}{ds} \times \frac{ds}{dt}$

$= -\frac{\sin^2\theta}{7.5} \times 1.2$

$= -0.16\sin^2\theta$ M1

sub $\theta = \tan^{-1}\frac{3}{4}$:

$= -0.16\sin^2(\tan^{-1}\frac{3}{4})$

$= -0.0576$ rad/s

$\tan\theta = \frac{7.5}{10}$

$\theta = \tan^{-1}\frac{3}{4}$ M1

The rate is –0.0576 rad/s. OR
The angle decreases at a rate of 0.0576 rad/s. } final statement A1

9 To study the effects of water waves, a wave generator and a rubber duck were placed in a water tank as shown in the diagram above. The height, h metres, from the bottom of water tank was modelled by $h = a \cos kt + b$, where t is the time in hours after midnight and a, b and k are constants. The motion of the rubber duck was observed for 60 hours.
The minimum height of 1.4 m from bottom of water tank was first recorded at midnight on Day 1.
The duck reaches minimum height again at 16 00 on Day 2.
After 60 hours, the duck reaches a height of 2.4 m.

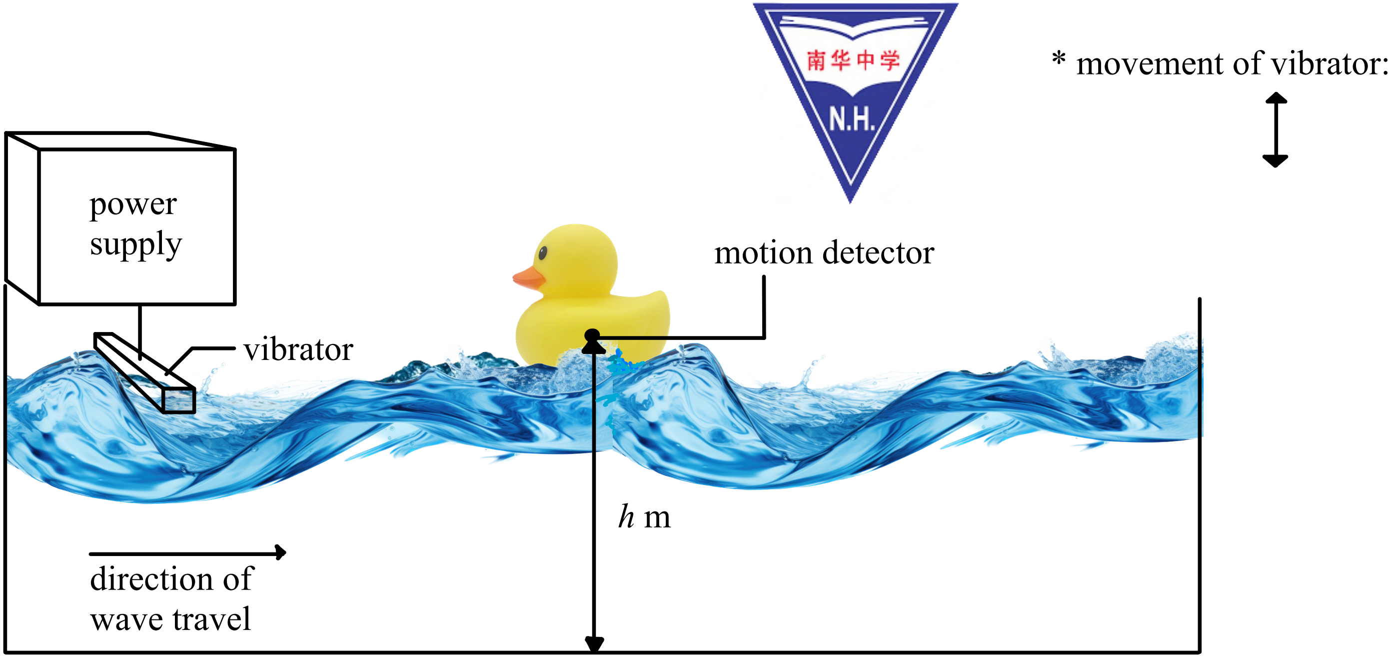


Diagram **9**

(i) Explaining each of your values, find a, b, and k. [3]

☆ 9/10

explanation + answer → 1m

($a = -0.5$)
Explanation: $|a|$ / positive value of a represents **amplitude** which is **maximum displacement** from rest position/**equilibrium**, and amplitude = (maximum h – minimum h) / 2 = (2.4 – 1.4) / 2 = 0.5 m, and since the graph starts at **minimum** point, a is **negative**, hence $a = -0.5$. [1]

($b = 1.9$)
Explanation: b represents the **constant** distance of h **from equilibrium position** which **translates** the graph up, hence $b = (2.4 + 1.4) / 2 = 1.9$ [1]

($k = \frac{\pi}{20}$)

Explanation: Since the duck reaches a minimum height of 1.4m **again** only on Day 2 after 16 00, it took **40 hours** for the duck to reach minimum height again, hence the **time taken** for one **complete wave**, otherwise known as the **period**, is 40 hours.

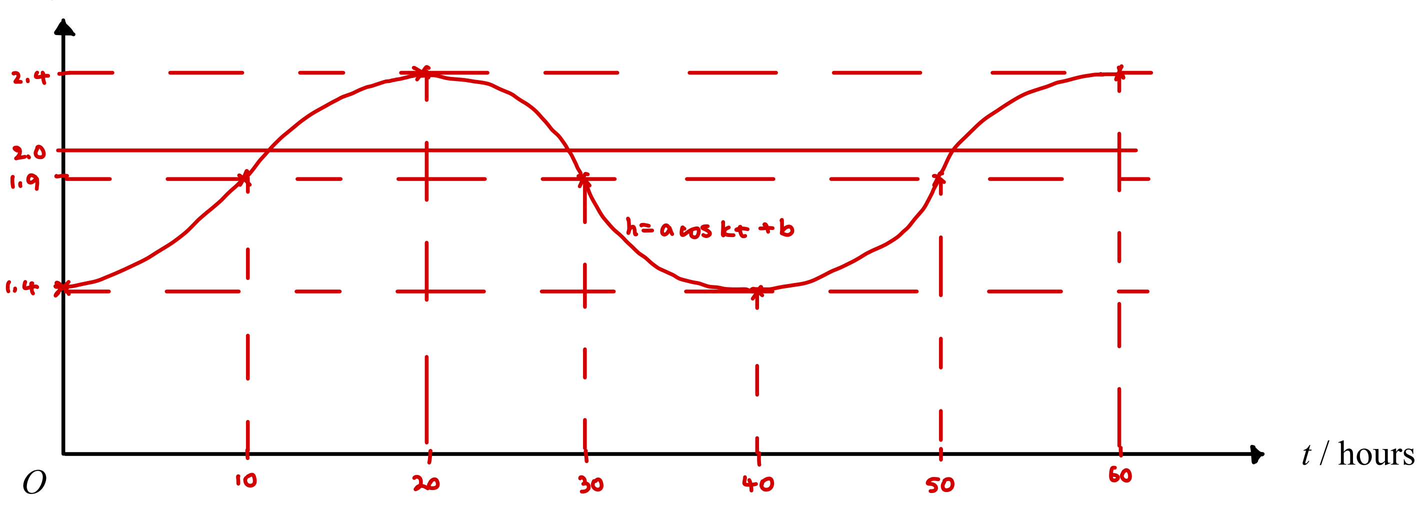
Hence, since period $= \frac{2\pi}{k}$, $\frac{2\pi}{k} = 40$, $k = \frac{2\pi}{40} = \frac{\pi}{20}$
[1]

(ii) Hence, sketch the graph of $h = a \cos kt + b$, for $0 \le t \le 60$. [2]

M1 for smooth graph

M1 for axes + points indication

5/10



(iii) How long does the rubber duck remain above 2.0 m during this period of 60 hours? [2]

☆ 9/10

$h = -0.5\cos\frac{\pi}{20}t + 1.9$

$0 \le t \le 60$

$0 \le \frac{\pi}{20}t \le 3\pi$

$0 \le \frac{\pi}{20}t \le 9.4248$

$-0.5\cos\frac{\pi}{20}t + 1.9 > 2$

$-0.5\cos\frac{\pi}{20}t > 0.1$

$\cos\frac{\pi}{20}t < -\frac{1}{5}$

$\frac{\pi}{20}t < \cos^{-1}(-\frac{1}{5})$ M1

α of $\frac{\pi}{20}t = 1.3694$ (5s.f.)

$\frac{\pi}{20}t = 1.7722, 4.5110, 8.0554$ (5s.f.)

$t = 11.282, 28.718, 51.282$ (5s.f.)

Time taken $= 28.718 - 11.282 + 60 - 51.282$

$= 26.154$ hours $=$ $\underline{26 \text{ hours } 9 \text{ mins}}$ A1 (nearest min)

(iv) It is further given that after 60 hours, the operator adjusted the vibrator such that it oscillates up-and-down at a greater displacement from equilibrium and at a faster rate.

☆ The graph starts from $t = 0$ again at the lowest point.
State and explain how the values of a, b, and k change, if at all. [1]

$T\downarrow$ $\frac{2\pi}{\uparrow k} = T\downarrow$

9/10 a

a **decreases**. Due to **greater displacement** from equilibrium, and since *a* is **negative**, the **magnitude** of a **increases** hence *a* becomes **more negative**, *a* thus decreases.
b remains the same. ← not needed
k **increases**. **Period** decreases, hence *k* **increases** by T = 2π / *k* / period is inversely proportional to *k*.

10 Look at the figure below. In the diagram, two circles with centres *A* and *B* and radii *r* cm and 5 cm, are right next to each other as shown. Lines *CD* and *EP* are tangents to both circles, and *AEB* is a right angle. *AP* = *r* cm, *BC* = 5 cm, and *CD* = 8 cm. Angle *EAP* = θ, and θ is always an acute angle.

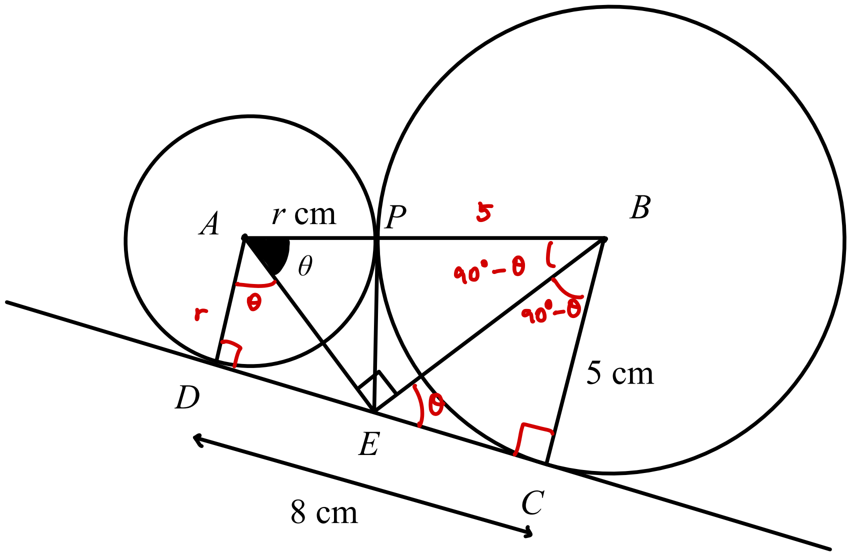


Diagram **10**

★ error

(i) Show that the area of trapezium *ABCD*, $A = 4(\frac{8\tan\theta - 5}{\tan^2\theta} + 5)$ cm^2. [2]

7/10

$AD = AP = r$ cm (tangents from an external point)

$\angle DAE = \angle EAP = \theta$ (tangents from an external point)

$\tan\theta = \frac{DE}{r}$

$DE = r\tan\theta$

$\angle ABE = 180° - 90° - \theta$ (angle sum in a Δ)

$= 90° - \theta$

$\angle CBE = \angle ABE = 90° - \theta$ (tangents from an external point)

$\tan(90° - \theta) = \frac{CE}{5}$ M1

$\tan\theta = \frac{5}{CE}$

$CE = \frac{5}{\tan\theta}$

$r\tan\theta + \frac{5}{\tan\theta} = 8$

$r\tan\theta = \frac{8\tan\theta - 5}{\tan\theta}$

$r = \frac{8\tan\theta - 5}{\tan^2\theta}$ M1

$A = \frac{1}{2}(r+5)(8)$

$= 4(\frac{8\tan\theta - 5}{\tan^2\theta} + 5)$ cm^2

(shown)

(ii) Hence, show that $\frac{dA}{d\theta} = \frac{4(-8\sin\theta + 10\cos\theta)}{\sin^3\theta}$.

☆ Explain the significance of this expression. [3]

9.5/10

$$A = 4\left(\frac{8\tan\theta - 5}{\tan^2\theta} + 5\right)$$

$$= 4\left(\frac{8}{\tan\theta} - \frac{5}{\tan^2\theta} + 5\right)$$

$$= 4\left(8(\tan\theta)^{-1} - 5(\tan\theta)^{-2} + 5\right)$$

$$\frac{dA}{d\theta} = 4\left[8(-1)(\tan\theta)^{-2}(\sec^2\theta) - 5(-2)(\tan\theta)^{-3}(\sec^2\theta)\right]$$

$$= 4\left[-\frac{8\sec^2\theta}{\tan^2\theta} + \frac{10\sec^2\theta}{\tan^3\theta}\right]$$

$$= 4\left(\frac{8\sec^2\theta\tan\theta + 10\sec^2\theta}{\tan^3\theta}\right) \quad \text{M1}$$

$$= 4\left[\frac{-\frac{8\sin\theta}{\cos^3\theta} + \frac{10}{\cos^2\theta}}{\frac{\sin^3\theta}{\cos^3\theta}}\right]$$

$$= 4\left(\frac{\frac{-8\sin\theta + 10\cos\theta}{\cos^3\theta}}{\frac{\sin^3\theta}{\cos^3\theta}}\right) \quad \text{M1}$$

$$= \frac{4(-8\sin\theta + 10\cos\theta)}{\sin^3\theta}$$

[shown)

This suggests that the ‸inst. **rate of change** of the area of the trapezium with respect to the angle θ / angle *EAP* is $\frac{4(-8\sin\theta + 10\cos\theta)}{\sin^3\theta}$. E1 not accepting 'gradient function'

↳ implies there is a line / graph

(iii) It is given that as θ varies, the value of $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{4}{\sin^3\theta}$.

By expressing the **numerator of the expression of** $\frac{\mathrm{d}A}{\mathrm{d}\theta}$ **from** (**ii**) in the form

$R\cos(\theta + \alpha)$, where $R > 0$ and $0^\circ < \alpha < 90^\circ$, solve for the area of the trapezium. 4 ~~[5]~~

8/10

$$\frac{dA}{d\theta} = \frac{4}{\sin^3\theta}$$

$$\frac{4(-8\sin\theta + 10\cos\theta)}{\sin^3\theta} = \frac{4}{\sin^3\theta}$$

Let $8(5\cos\theta - 4\sin\theta) = 8\left[R\cos(\theta+\alpha)\right]$, $R>0$, $0^\circ<\alpha<90^\circ$.

$5\cos\theta - 4\sin\theta = R\cos(\theta+\alpha)$

$5\cos\theta - 4\sin\theta = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$5\cos\theta - 4\sin\theta = (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$

By comparing,

$R\cos\alpha = 5$ —①

$R\sin\alpha = 4$ —②

②÷①:

$\frac{R\sin\alpha}{R\cos\alpha} = \frac{4}{5}$

$\tan\alpha = \frac{4}{5}$

$\alpha = \tan^{-1}\frac{4}{5}$

$= 38.7^\circ$ (1d.p.)

①^2+②^2:

$R^2(\sin^2\alpha + \cos^2\alpha) = 25+16$

$R^2 = 41$

$R = \sqrt{41}$ $(R>0)$

$= 6.40$ (3s.f.)

M2

-1 if one-two parts missing

-2 if three parts missing

common mistakes: loss:

$a = \alpha$ -1

not expanding -1

3 for R-formula

1 for A

$5\cos\theta - 4\sin\theta = \sqrt{41}\cos(\theta + 38.660^\circ)$

$8(5\cos\theta - 4\sin\theta) = 8\sqrt{41}\cos(\theta + 38.660^\circ)$ /

$= 51.2\cos(\theta + 38.7^\circ)$ A1

$8\sqrt{41}\cos(\theta + 38.660^\circ) = 4$

$\cos(\theta + 38.660^\circ) = \frac{1}{2\sqrt{41}}$

$\theta + 38.660^\circ = 85.521^\circ$ (3d.p.)

$\theta = 85.521^\circ - 38.660^\circ$

$= 46.861^\circ \leftarrow$

$A = 4\left(\frac{8\tan 46.861^\circ - 5}{\tan^2 46.861^\circ} + 5\right)$

$= 32.424$ (5s.f.)

$= 32.4\text{cm}^2$ (3s.f.) A1

11 The diagram below shows the values of the gradient function and its second derivative function of a real function $y = \text{f}(x)$, where $\text{f}(x)$ is a function not to be determined.

maximum → $\frac{d^2y}{dx^2} < 0$

minimum → $\frac{d^2y}{dx^2} > 0$

Value	**0^-**	**0**	**0^+**	**1^-**	**1**	**1^+**	**2^-**	**2**	**2^+**	**3^-**	**3**	**3^+**
Value of $\frac{dy}{dx}$	+	+	+	+ /	0 –	– \	–	–	–	– \	0 –	+ /
Value of $\frac{d^2y}{dx^2}$	–	–	– ↓ most suitable	–	– close to 1	–	– ↓ most suitable	–	– –	+ –	+ close to 3	+

Legend:
'+' denotes **positive**
'–' denotes **negative**

Diagram **11**

(i) In Diagram 11, fill in the empty boxes with the correct symbol '+', '–', or 0. [1]

2/10

(ii) Using Diagram 11, identify the stationary point(s) of $y = \text{f}(x)$ from $0 \leq x \leq 3.1$ and determine the nature of the stationary point(s). ~~[1]~~

6/10

Stationary point: $x = 1$
Nature: By First/Second Derivative Test, $x = 1$ is a **maximum**.
Stationary point: $x = 3$
Nature: By First/Second Derivative Test, $x = 3$ is a **minimum**.

Note: allow 'Second' only if **(i)** is correct

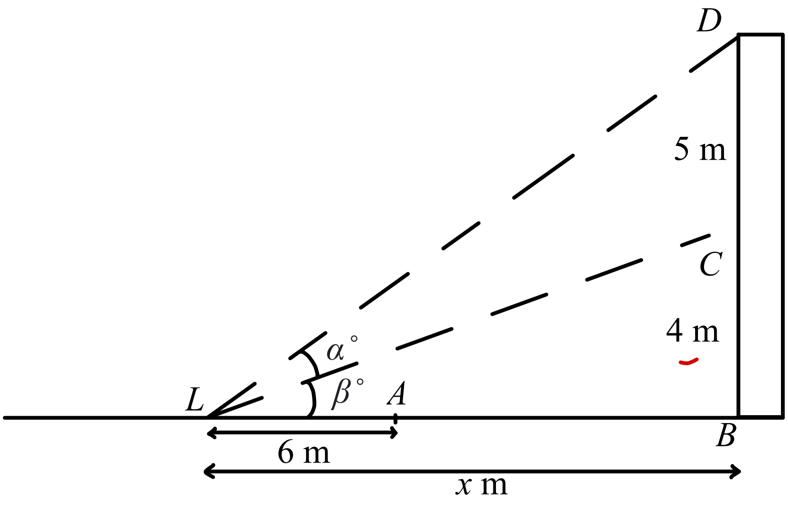
(iii) Determine the number of stationary point(s) of the graph $y = \text{f}'(x)$, if any, from $0 \leq x \leq 3.1$ and determine the nature of the stationary point(s), if any. [1]

8/10

1 stationary point.
By First Derivative Test, this stationary point is a **minimum**.

(Updated)

12* In the diagram, *A* and *B* are two fixed points on a horizontal ground and a projector is positioned on the ground at *L* which is *x* m away from *B*. The projector casts a beam of light on a screen *CD*, of fixed height 5 m. *C* is the bottom of the screen, where *BC* = 4 m. Angle *CLD* is $\alpha°$ and Angle *BLC* is $\beta°$. Assume that the thickness of the screen is negligible.



(i) Express tan $\alpha°$ in terms of *x*. [2]

8/10

$\tan\beta° = \frac{4}{x}$ $\quad$ $\tan(\alpha+\beta)° = \frac{9}{x}$

$$\frac{\tan\alpha° + \tan\beta°}{1 - \tan\alpha° \tan\beta°} = \frac{9}{x}$$

$$\frac{\frac{4}{x} + \tan\alpha°}{1 - \frac{4}{x}\tan\alpha°} = \frac{9}{x}$$ M1

$$x\left(\frac{4}{x} + \tan\alpha°\right) = 9 - \frac{36}{x}\tan\alpha$$

$$4 + x\tan\alpha° = 9 - \frac{36}{x}\tan\alpha$$

$$x\tan\alpha° + \frac{36}{x}\tan\alpha° = 5$$

$$\left(x + \frac{36}{x}\right)\tan\alpha° = 5$$

$$\tan\alpha° = \frac{5}{x + \frac{36}{x}}$$

$$= \frac{5}{\left(\frac{x^2+36}{x}\right)}$$

$$= \frac{5x}{x^2+36} \quad //$$ — A1

(ii) ☆ Due to numerous faults in projector L, projector L is replaced with projector M that has a much wider range of projection.

In order for projector M to cast its full image on a screen, the screen itself has to extend long enough in order to capture the full virtual image, and the image now reaches point B, the lowest point on the screen. The projector is able to cast its image, yet some of the image is still cut off and unavailable to view.

The angle the projector needs to cast such that a full virtual image is shown on the screen is given by acute $(\alpha + 2\beta)^\circ$.

Hence, the operator moves the projector in front by 6 m to position A, and now a full image is shown on the screen.

Find the value of x. [2]

☆ 9/10

$\tan\beta = \frac{4}{x}$ $\tan(\alpha+\beta) = \frac{9}{x}$

$$\tan(\alpha+2\beta) = \frac{\tan(\alpha+\beta)+\tan\beta}{1-\tan(\alpha+\beta)\tan\beta}$$

$$= \frac{(\frac{9}{x}+\frac{4}{x})}{1-(\frac{9}{x})(\frac{4}{x})}$$

$$= \frac{\frac{13}{x}}{1-\frac{36}{x^2}}$$

$$= \frac{13}{x} \div (\frac{x^2-36}{x^2})$$

$$= \frac{13}{x} \times \frac{x^2}{x^2-36}$$

$$= \frac{13x}{x^2-36} \quad \text{M1}$$

$AB = (x-6)\text{m}$

$$\tan(\alpha+2\beta) = \frac{9}{x-6}$$

$$\frac{13x}{x^2-36} = \frac{9}{x-6}$$

$$\frac{13x}{(x+6)(x-6)} = \frac{9}{x-6}$$

$$\frac{13x}{x+6} = 9$$

$13x = 9x + 54$

$4x = 54$

$\underline{x = 13.5}$ A1

(Updated)

13* **(a)** It is given that $x = f(y)$ is drawn on a graph.

(i) Given that $\int_a^{2a} x \, dy = 0$, what will this imply about the geometrical interpretation of the graph of $f(y)$? [1]

☆ 9/10

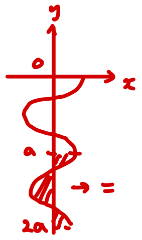
This implies that the curve $x = f(y)$ **intersects the y axis from** $y = a$ **to** $y = 2a$, such that the **area** of the **curve** from $x < 0$ is **equal** to the area of the curve from $x > 0$, from $a < y < 2a$. [1]

(ii) Given further that $3x = -8 \sin^2 y \cos^2 y + 1$, what is the smallest possible value of a? [1]

$3x = -8(\sin y \cos y)^2 + 1$
$= -8(\frac{1}{2}\sin 2y)^2 + 1$
$= -2\sin^2 2y + 1$
$= 1 - 2\sin^2 2y$
$3x = \cos 4y$
$x = \frac{4}{3}\cos 4y$

period $= \frac{2\pi}{4} = \frac{\pi}{2}$

$\underline{a = -\frac{\pi}{2}}$



(b) Look at the diagram below. It shows the graph of $y = 2 \cos x - 3 \cos (3x)$ plotted for $x > 0$.

Two vertical lines, $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{6}$ are shown below.

Calculate the **exact** shaded area bounded by the two lines and the curve below. [4]

8.5/10

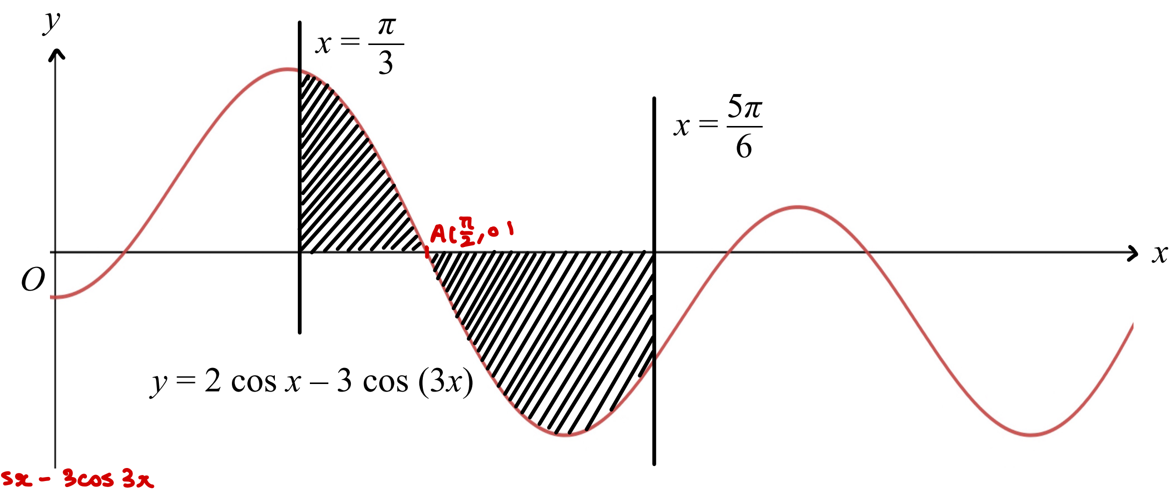


Diagram **13.1**

$y = 2\cos x - 3\cos 3x$

Sub $y = 0$:

$2\cos x - 3\cos 3x = 0$

$2\cos x = 3(\cos(2x + x))$

$2\cos x = 3(\cos 2x \cos x - \sin 2x \sin x)$

$2\cos x = 3[(2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x]$

$2\cos x = 3[2\cos^3 x - \cos x - 2\sin^2 x \cos x]$

$2\cos x = 3[2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x]$

$2\cos x = 3[2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x]$

$2\cos x = 3(4\cos^3 x - 3\cos x)$ M1

$2\cos x = 3(4\cos^3 x - 3\cos x)$

$2\cos x = 12\cos^3 x - 9\cos x$

$0 = 12\cos^3 x - 11\cos x$

$0 = \cos x(12\cos^2 x - 11)$

$\cos x = 0$ or $12\cos^2 x - 11 = 0$

$x = \frac{\pi}{2}$ $\quad \cos^2 x = \frac{11}{12}$

$= 1.5708$ (5s.f.) $\quad \cos x = \pm\sqrt{\frac{11}{12}}$

$\cos x = \sqrt{\frac{11}{12}}$ or $\cos x = -\sqrt{\frac{11}{12}}$

$x = 0.29284$ $\quad x = 2.8487$ (5s.f.)

From the diagram, $A(\frac{\pi}{2}, 0)$. M1

(Continuation of working space for question **13 (b)**)

$\int 2\cos x - 3\cos 3x \, dx$

$= 2\sin x - 3(\frac{1}{3}\sin 3x) + c$

$= 2\sin x - \sin 3x + c$

$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\cos x - 3\cos 3x \, dx + \left| \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} 2\cos x - 3\cos 3x \, dx \right|$

$= \left[2\sin x - \sin 3x\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} + \left|\left[2\sin x - \sin 3x\right]_{\frac{\pi}{2}}^{\frac{5\pi}{6}}\right|$

$= \left\{(2\sin\frac{\pi}{2} - \sin\frac{3\pi}{2}) - (2\sin\frac{\pi}{3} - \sin\pi)\right\} + \left|\left\{(2\sin\frac{5\pi}{6} - \sin\frac{5\pi}{2}) - (2\sin\frac{\pi}{2} - \sin\frac{3\pi}{2})\right\}\right|$

$= \left\{(2(1) - (-1)) - (2(\frac{\sqrt{3}}{2}) - 0)\right\} + \left|\left\{(2(\frac{1}{2}) - (1)) - (2(1) - (-1))\right\}\right|$ M1

$= 3 - \sqrt{3} + |-3|$

$= \underline{(6 - \sqrt{3})}$ units A1

(c) The diagram below shows part of a curve of $y = \frac{1}{x^2}$.

☆ 9/10

M1 for sketch or relevant explanation

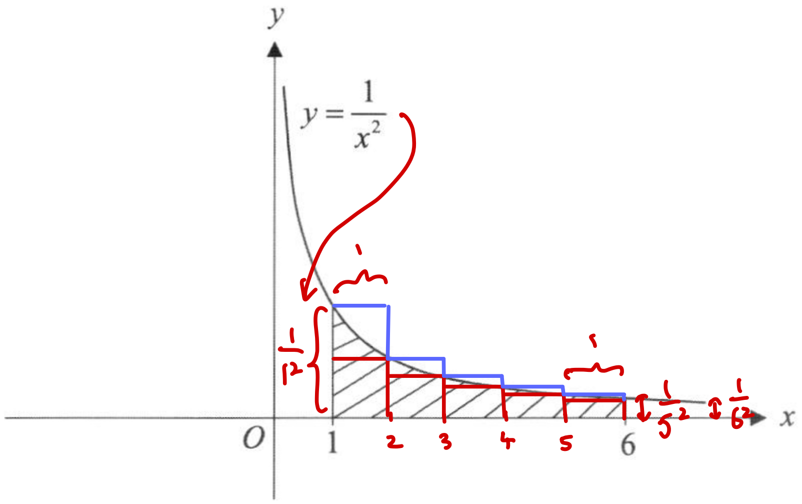


Diagram **13.2**

Show that the area of the shaded region is $\frac{5}{6}$ units^2.

Hence, without any other calculations, explain why

$$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} < \frac{5}{6} < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

[2]

You may annotate or sketch out anything on Diagram **13.2** to explain your answer.

$\int_1^6 \frac{1}{x^2} dx = \int_1^6 x^{-2} dx$

$= \left[-\frac{1}{x}\right]_1^6$

$= \left\{(-\frac{1}{6}) - (-\frac{1}{1})\right\} = \frac{5}{6}$ $\underline{\text{units}^2}$ A1

The left hand side of the inequality represents the **summation** of **rectangles** in Diagram 13.2, as each rectangle would have a **breadth** of 1 unit, and length $\frac{1}{x^2}$ units, where $2 \le x \le 6$. The right hand side of the inequality represents a similar summation of rectangles, except the length is now $\frac{1}{(x-1)^2}$ units^2, where $2 \le x \le 6$. E1

Since

summation of rect. length $\frac{1}{x^2}$ units $< \int_1^6 \frac{1}{x^2} dx <$ summation of rect. length $\frac{1}{(x-1)^2}$ units

$\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} < \frac{5}{6} < \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$

14 The diagram shows a right circular cone in a sphere with centre O and radius 40 cm.
The vertex of the cone, N, and the circumference of its base lies on the sphere and the centre of the sphere is on the axis of the cone.

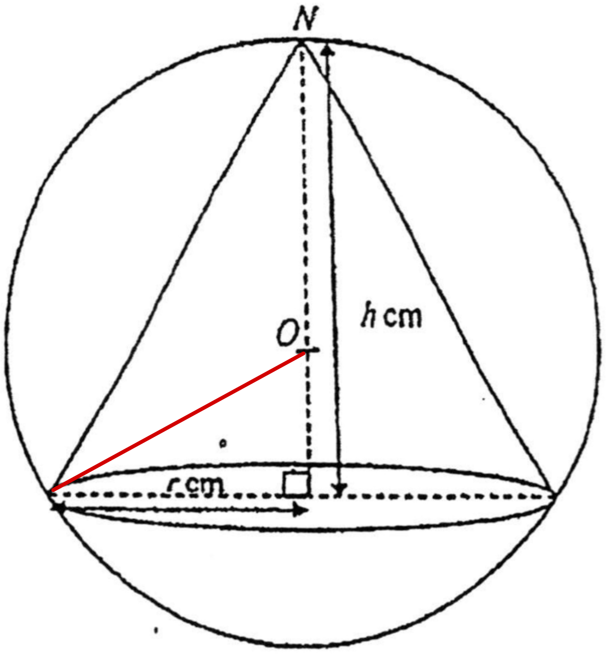


Diagram **14**

Given that the radius, height and volume of the right circular cone are r cm, h cm and V cm^3 respectively, by expressing V in terms of h, find the stationary value of V, and determine the nature of this value **without any stationary tests**. [5]

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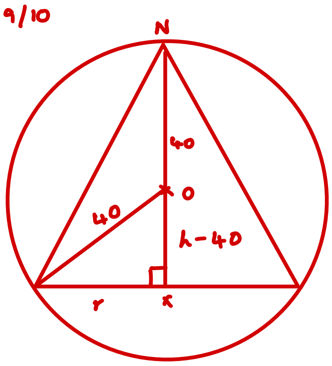


let x be the centre of the base area of the cone.

NX passes through centre of circle. (perpendicular bisector of chord)

NX = h cm

OX = h − 40 cm

Using Pythagoras' Theorem,

$(h-40)^2+r^2=40^2$

$h^2-80h+1600+r^2=1600$

$r^2=80h-h^2$

$r=\sqrt{80h-h^2}$ $(r>0)$

$V=\frac{1}{3}\pi r^2 h$

$=\frac{1}{3}\pi(80h-h^2)h$

$=\frac{80h^2-h^3}{3}\pi$

$=\frac{\pi}{3}(80h^2-h^3)$ M1

$\frac{dV}{dh}=\frac{\pi}{3}\left[160h-3h^2\right]$

$=\frac{\pi}{3}(160h-3h^2)$ M1

$\frac{dV}{dh}=0$:

$\frac{\pi}{3}(160h-3h^2)=0$

$160h-3h^2=0$

$h(160-3h)=0$

$h=0$ or $160-3h=0$

(reject as $h>0$)

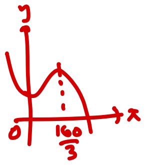
$h=\frac{160}{3}$ cm A1

$V=\frac{\pi}{3}\left[80\left(\frac{160}{3}\right)^2-\left(\frac{160}{3}\right)^3\right]$

$=79\,431.9$ (6s.f.)

$=79\,400$ cm^3 (3s.f.) A1

The original equation of V when plotted against h would give a **negative cubic graph**, hence the graph would first reach to a minimum, then a maximum. Since $h = 0$ is the first stationary point albeit not possible to be V, the second point $h = 160/3$ has to be a **maximum.** **[1]**



Continuation of working of question **14**

15 In predator-prey relationships, the number of animals in each category tends to vary periodically.

In a certain habitat, the populations of foxes (F) in thousands and rabbits (R) in thousands are modelled by the equations: $F = 300 - 125\sin\frac{\pi t}{4}$ and $R = 1200 - 650\cos\frac{\pi t}{4}$.

(i) Using the above models, determine which month(s) in the first year will the population of rabbits be four times as much as the population of foxes. [2]

$0 \leq t \leq 12$

$0 \leq \frac{\pi t}{4} \leq 3\pi$

$R = 4F$

$1200 - 650\cos\frac{\pi t}{4} = 4(300 - 125\sin\frac{\pi t}{4}) \leftarrow$

$1200 - 650\cos\frac{\pi t}{4} = 1200 - 500\sin\frac{\pi t}{4}$

$-650\cos\frac{\pi t}{4} = -500\sin\frac{\pi t}{4}$

$\frac{650}{500} = \tan\frac{\pi t}{4}$

$\frac{\pi t}{4} = \tan^{-1}\frac{650}{500}$

$\alpha = 0.915$ (3s.f.) M1

$\frac{\pi t}{4} = 0.915, 4.057, 7.198$ (3d.p.)

$t = 1.17, 5.17, 9.16$

2nd month, 6th month, 10th month A1

(ii) It is given that t is in months.

Without using the R Formula, predict the number of months it would take for the total population of foxes and rabbits to be the highest within a period of 8 months, and show your workings clearly. [1]

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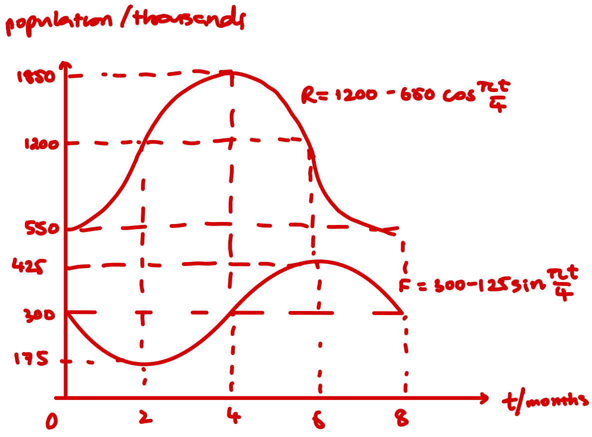
Method 1

graphs



Mth 2 = 1200 + 175
= 1375

Mth 6 = 1200 + 425
= 1625

Mth 4 = 1850 + 300
= 2150

Mth 8 = 550 + 300
= 850

Method 2

	amplitude:	period:
$R = 1200 - 650\cos\frac{\pi t}{4}$	650	$\frac{2\pi}{\pi/4} = 8$ weeks
$F = 300 - 125\sin\frac{\pi t}{4}$	125	

Given that the **amplitude** of R (600) will be higher than the **amplitude** of F (125) , the **increase** in **population** of R will notably be higher than the **increase** in **population** of F, hence the **highest** population of R will have a **greater magnitude** than the **highest** population of F.

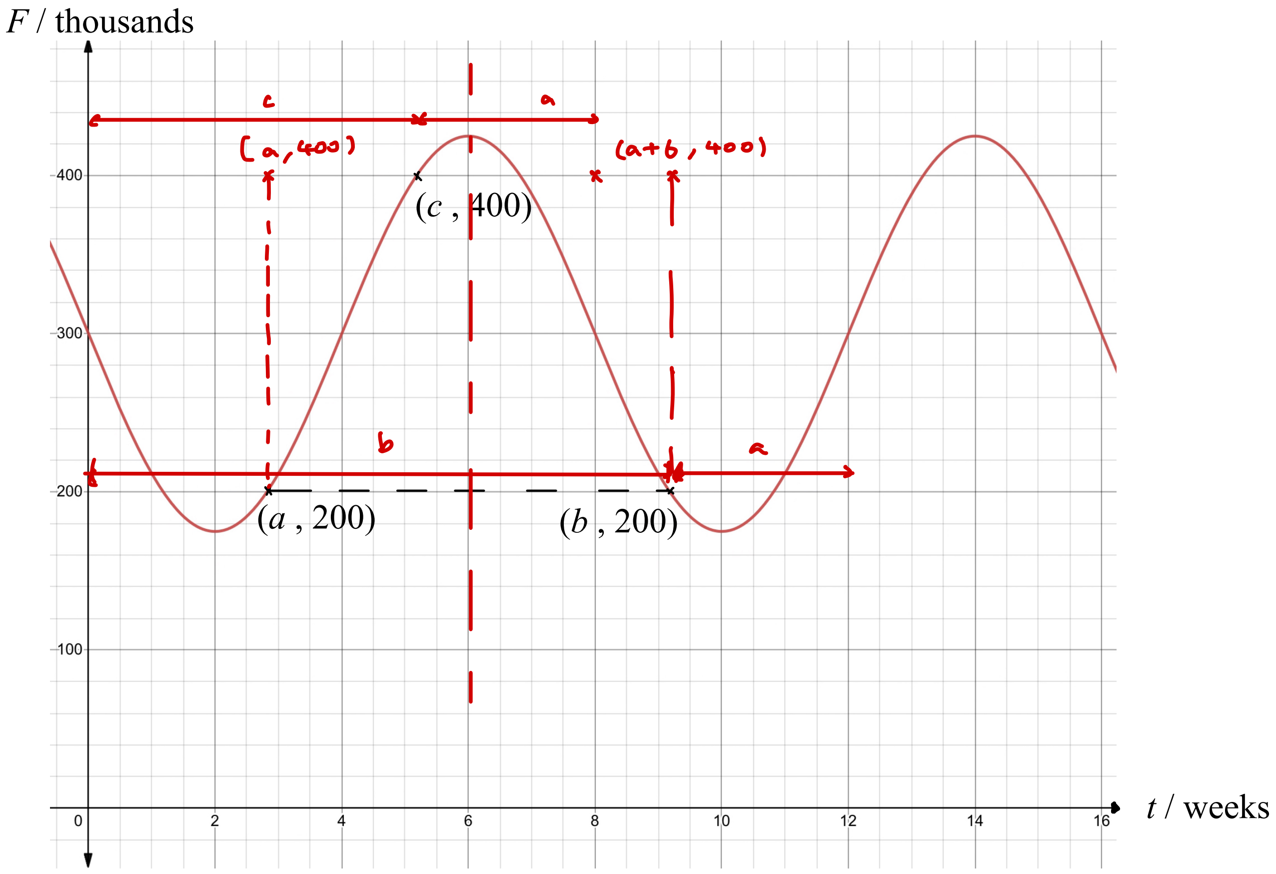
The highest population of R occurs at half of the period which is 8 / 2 = month 4.

Hence, month 4 will have the highest population of foxes and rabbits.

$\therefore$ 4 months

(iii) The graph below shows the population of foxes F in thousands against time t, from $0 \leq t \leq 16$.

8/10



The curve passes through the points $(a , 200)$, $(b , 200)$, and $(c , 400)$.

Form an equation connecting b and c. Show all workings. [1]

$a+b=12$ —①

$a+c=8 \mid a=8-c$ } required

$a=8-c$ —②

Sub ② into ①:

$8-c+b=12$

$b-c=4$ A1

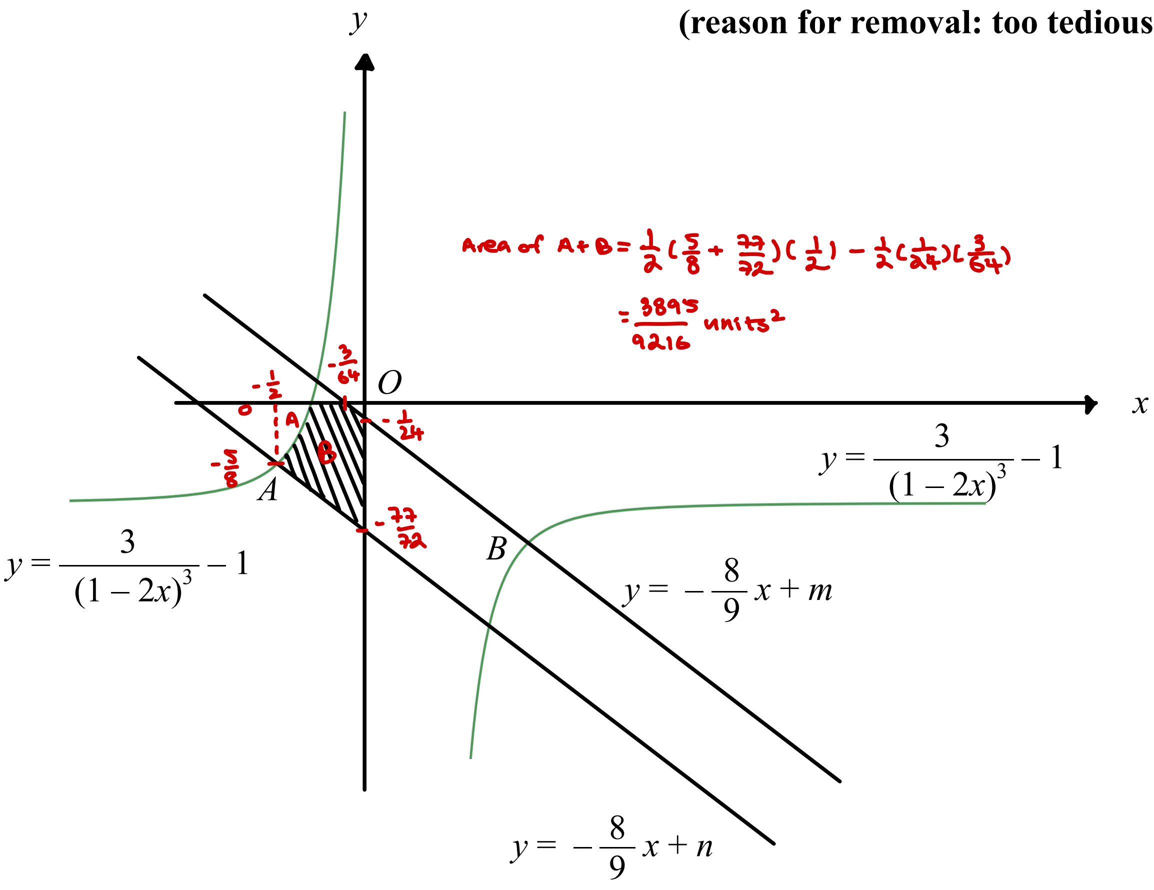
or

$b=c+4$

End of paper :)

13 Look at Diagram 13 below. It shows the graph $y = \dfrac{3}{(1-2x)^3} - 1$ plotted along with two different normals that are parallel to each other. The equation $y = -\dfrac{8}{9}x + m$ is a normal to the curve at point B while the equation $y = -\dfrac{8}{9}x + n$ is a normal to the curve at point A below.

(reason for removal: too tedious)



Area of A+B $= \frac{1}{2}(\frac{5}{8} + \frac{77}{72})(\frac{1}{2}) - \frac{1}{2}(\frac{1}{24})(\frac{3}{64})$

$= \frac{3895}{9216}$ units^2

Diagram **13**

(i) Find the coordinates of A and B. [3]

$y = \frac{3}{(1-2x)^3} - 1$ —①

$= 3(1-2x)^{-3} - 1$

$\frac{dy}{dx} = 3[(-3)(1-2x)^{-4}(-2)]$

$= 3[6(1-2x)^{-4}]$

$= \frac{18}{(1-2x)^4}$ M1

$\frac{dy}{dx} = -1 \div -\frac{8}{9}$

$\frac{18}{(1-2x)^4} = \frac{9}{8}$

$\frac{2}{(1-2x)^4} = \frac{1}{8}$

$(1-2x)^4 = 16$

$1-2x = \pm\sqrt[4]{16}$

$1-2x = 2$ or $1-2x = -2$

$-2x = 1$ $\quad$ $-2x = -3$

$x = -\frac{1}{2}$ $\quad$ $x = \frac{3}{2}$ $\quad$ M1 for x coordinates

Sub $x = -\frac{1}{2}$ into ①: $\quad$ Sub $x = \frac{3}{2}$ into ①:

$y = \frac{3}{[1-2(-\frac{1}{2})]^3} - 1$ $\quad$ $y = \frac{3}{[1-2(\frac{3}{2})]^3} - 1$

$= -\frac{5}{8}$ $\quad$ $= -\frac{11}{8}$

$\underline{A(-\frac{1}{2}, -\frac{5}{8})}$ $\quad$ $\underline{B(\frac{3}{2}, -\frac{11}{8})}$

A1

(ii) Find the shaded area bounded by the curve and the two different normals. [4]

$y = \frac{3}{(1-2x)^3} - 1$

Sub $y=0$:

$\frac{3}{(1-2x)^3} = 1$

$(1-2x)^3 = 3$

$1-2x = \sqrt[3]{3}$

$-2x = \sqrt[3]{3} - 1$

$x = \frac{1-\sqrt[3]{3}}{2}$

$$\text{Area } A = \int_{-\frac{1}{2}}^{\frac{1-\sqrt[3]{3}}{2}} \left(\frac{3}{(1-2x)^3} - 1\right) dx = 3\int_{-\frac{1}{2}}^{\frac{1-3\sqrt{3}}{2}} (1-2x)^{-3} - [x]_{-\frac{1}{2}}^{\frac{1-\sqrt[3]{3}}{2}}$$

$$= 3\left[\frac{(1-2x)^{-3+1}}{(-3+1)(-2)}\right]_{-\frac{1}{2}}^{\frac{1-\sqrt[3]{3}}{2}} - [x]_{-\frac{1}{2}}^{\frac{1-3\sqrt{3}}{2}}$$

$$= 3\left[\frac{(1-2x)^{-2}}{4}\right]_{-\frac{1}{2}}^{\frac{1-\sqrt[3]{3}}{2}} - [x]_{-\frac{1}{2}}^{\frac{1-\sqrt[3]{3}}{2}}$$

$$= 3\left\{\left[\frac{\left(1-2\left(\frac{1-\sqrt[3]{3}}{2}\right)\right)^{-2}}{4}\right] - \left[\frac{\left(1-2\left(-\frac{1}{2}\right)\right)^{-2}}{4}\right]\right\} - \left\{\frac{1-\sqrt[3]{3}}{2} - \left(-\frac{1}{2}\right)\right\}$$

$= |-0.10581| \text{ units}^2$ (5s.f.)

$= 0.10581 \text{ units}^2$

equation of normal:

$y - \left(-\frac{11}{8}\right) = -\frac{8}{9}\left(x - \frac{3}{2}\right)$

$y + \frac{11}{8} = -\frac{8}{9}x + \frac{4}{3}$

$y = -\frac{8}{9}x - \frac{1}{24}$

$y - \left(-\frac{5}{8}\right) = -\frac{8}{9}\left(x + \frac{1}{2}\right)$

$y + \frac{5}{8} = -\frac{8}{9}x - \frac{4}{9}$

$y = -\frac{8}{9}x - \frac{77}{72}$

Sub $y=0$:

$\frac{8}{9}x = -\frac{1}{24}$

$x = -\frac{1}{24} \div \frac{8}{9}$

$= -\frac{3}{64}$

$\text{Area of } A+B = \frac{1}{2}\left(\frac{5}{8} + \frac{77}{72}\right)\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{24}\right)\left(\frac{3}{64}\right)$

$= \frac{3895}{9216} \text{ units}^2$

$\text{Area } B = \frac{3895}{9216} - 0.10581$

$= 0.31683$ (5s.f)

$= 0.317 \text{ units}^2$ (3s.f.)