National Junior College
2016 – 2017 H2 Further Mathematics
Numerical Methods **Tutorial**

Basic Mastery Questions

Interval Bisection, Linear Interpolation and Newton-Raphson's Method

1 Estimate the positive root of the equation $x = \ln(x+2)$, giving your answer to four significant figures. J73/P1/14

Solution:

Let $f(x) = x - \ln(x+2)$

$f'(x) = 1 - \dfrac{1}{x+2}$.

$\therefore f'(x) = 0 \Rightarrow x = -1$

$f''(x) = \dfrac{1}{(x+2)^2} > 0 \ \forall x \in \mathbb{R}$

$\therefore\ x = -1$ gives a minimum point $\Rightarrow$ there exists a root in $(-1, \infty)$.

$f(0) = -\ln 2 < 0$

$f(1) = 1 - \ln 3 < 0$

$f(2) = 2 - \ln 4 > 0$

There exists a positive root in $1 < x < 2$.

Estimate of root	$f(x)$	Interval containing root
1	< 0	
2	> 0	$(1, 2)$
1.5	> 0	$(1, 1.5)$
1.25	> 0	$(1, 1.25)$
1.125	< 0	$(1.125, 1.25)$
1.1875	> 0	$(1.125, 1.1875)$
1.15625	> 0	$(1.125, 1.15625)$
1.140625	< 0	$(1.140625, 1.15625)$
1.1484375	> 0	$(1.140625, 1.1484375)$
1.14453125	< 0	$(1.14453125, 1.1484375)$
1.146484375	> 0	$(1.14453125, 1.146484375)$
1.145507813	< 0	$(1.145507813, 1.146484375)$
1.145996094	< 0	$(1.145507813, 1.145996094)$

Since there exists a positive root in $(1.145507813, 1.145996094)$, the positive root is 1.146 (to 4 significant figures).

2 Use linear interpolation to find an approximation to the root of equation $x - \ln(4\sin x) = 0$ which lies between 1 and 2. Give your answer correct to three decimal places.

Solution:

Let $\text{f}(x) = x - \ln(4\sin x)$ and α be the root.

Estimate of root	f (x)	Interval containing α
1	< 0	
2	> 0	$(1,2)$
1.5	> 0	$(1,1.5)$
$\dfrac{1.5\lvert\text{f}(1)\rvert + 1\lvert\text{f}(1.5)\rvert}{\lvert\text{f}(1)\rvert + \lvert\text{f}(1.5)\rvert} = 1.32387$ (5 .d.p)	< 0	$(1.32387, 1.5)$
$\dfrac{1.5\lvert\text{f}(1.32387)\rvert + 1.32387\lvert\text{f}(1.5)\rvert}{\lvert\text{f}(1.32387)\rvert + \lvert\text{f}(1.5)\rvert} = 1.36155$ (5 .d.p)	< 0	$(1.36155, 1.5)$
$\dfrac{1.5\lvert\text{f}(1.36155)\rvert + 1.36155\lvert\text{f}(1.5)\rvert}{\lvert\text{f}(1.36155)\rvert + \lvert\text{f}(1.5)\rvert} = 1.36468$ (5 .d.p)	< 0	$(1.36468, 1.5)$
$\dfrac{1.5\lvert\text{f}(1.36468)\rvert + 1.36468\lvert\text{f}(1.5)\rvert}{\lvert\text{f}(1.36468)\rvert + \lvert\text{f}(1.5)\rvert} = 1.36494$ (5 .d.p)	< 0	$(1.36494, 1.5)$

Check if $\alpha = 1.365$,

$$\left.\begin{array}{l}\text{f}(1.3645) < 0 \\ \text{f}(1.3654) > 0\end{array}\right\} \therefore \alpha \in (1.3645, 1.3654)$$

$\therefore \alpha = 1.365$(to 3.d.p)

3 Show that the equation $e^x + x - 2 = 0$ has one and only one real root. Use the Newton-Raphson process to find this root correct to three decimal places. J75/P1/14

Solution:

Let $\text{f}(x) = e^x + x - 2$

$\text{f}'(x) = e^x + 1$

For all real values of x, $e^x > 0$. $\therefore \text{f}'(x) = e^x + 1 \Rightarrow \text{f}'(x) > 0$

$\therefore \text{f}(x)$ is strictly increasing function and thus cuts the x-axis only once.

$\therefore e^x + x - 2 = 0$ has one and only one real root.

Let $x_1 = 0$ and α be the root of $\text{f}(x) = 0$.

By Newton – Raphson's method:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0.5$$

$$x_3 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.44385 \text{ (5.d.p)}$$

$$x_4 = 0.44385 - \frac{f(0.44385)}{f'(0.44385)} = 0.44285 \text{ (5.d.p)}$$

$$x_5 = 0.44285 - \frac{f(0.44285)}{f'(0.44285)} = 0.44285 \text{ (5.d.p)}$$

Check if $\alpha = 0.443$,

$$\left.\begin{array}{l} f(0.4425) < 0 \\ f(0.4434) > 0 \end{array}\right\} \therefore \alpha \in (0.4425, 0.4434)$$

$\therefore \alpha = 0.443$ (3.d.p)

4 Sketch the following graphs on a single diagram, stating the x-coordinates of all intersections with the x-axis and the equations of any asymptotes.

(i) $y = x(x^2-4) = x(x-2)(x+2)$

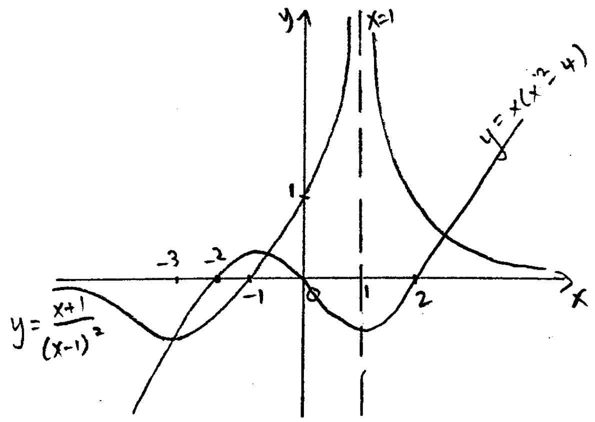
(ii) $y = \dfrac{x+1}{(x-1)^2}$

Use linear interpolation once on the interval $[-1, 0]$ to obtain an approximation to a root of the equation $x(x^2-4) = \dfrac{x+1}{(x-1)^2}$.

The Newton-Raphson method is to be used to find an approximation to another root of the equation. Use the method, with $x = 2$ as a first approximation, to obtain a second approximation to this root, giving your answer correct to 2 places of decimal.

N2000/P2/12

Solution:



$x = 1$ is a vertical asymptote.

Let $\text{f}\left(x\right)=x\left(x^2-4\right)-\dfrac{x+1}{\left(x-1\right)^2}$.

By linear interpolation, approximation to root = $\dfrac{\left(-1\right)\left|\text{f}\left(0\right)\right|}{\left|\text{f}\left(0\right)\right|+\left|\text{f}\left(-1\right)\right|}=\dfrac{-1}{1+3}=-\dfrac{1}{4}$.

$\text{f}'\left(x\right)=3x^2-4+\dfrac{\left(x-1\right)\left(x+3\right)}{\left(x-1\right)^4}$

By Newton-Raphson's method,

2nd approximation to root $=2-\dfrac{\text{f}\left(2\right)}{\text{f}'\left(2\right)}=2-\dfrac{-3}{13}=2.23$ (2.d.p)

<u>Iterations involving recurrence relations of the form</u> $x_{n+1}=\text{F}\left(x_n\right)$

5 The equation $x^3-12x+1=0$ has two positive roots, α and β , $\left(\alpha<\beta\right)$ and one negative root.

(i) Prove that $0<\alpha<1$ and $3<\beta<4$.

(ii) Use the iterative formula $x_{n+1}=\left(12x_n-1\right)^{\frac{1}{3}}$, $n\geq 1$, with 3.5 as a starting value to approximate β correct to two decimal places.

(iii) With the aid of a graph, show that the iterative formula in (ii) will converge to β for some starting values more than α .

Solution:

(i) $\text{f}\left(x\right)=x^3-12x+1$

$\text{f}\left(0\right)=1>0$

$\text{f}\left(1\right)=-10<0$

$\therefore$ There exists a root in $\left(0,1\right)$.

$\text{f}\left(x\right)=x^3-12x+1$

$\text{f}\left(3\right)=-8<0$

$\text{f}\left(4\right)=-17>0$

$\therefore$ There exists a root in $\left(3,4\right)$.

Given that $\alpha<\beta$, $0<\alpha<1$ and $3<\beta<4$.

(ii)

$x_1=3.5$

$x_2=\left(12x_1-1\right)^{\frac{1}{3}}=\left(41\right)^{\frac{1}{3}}\approx 3.4482\left(4.\text{d.p}\right)\right)$

$x_3=\left(12x_2-1\right)^{\frac{1}{3}}=\left(40.3784\right)^{\frac{1}{3}}\approx 3.4307\left(4.\text{d.p}\right)\right)$

$x_4=\left(12x_3-1\right)^{\frac{1}{3}}=\left(40.1684\right)^{\frac{1}{3}}\approx 3.4247\left(4.\text{d.p}\right)\right)$

$x_5=\left(12x_4-1\right)^{\frac{1}{3}}=\left(40.0964\right)^{\frac{1}{3}}\approx 3.4227\left(4.\text{d.p}\right)\right)$

$x_6=\left(12x_5-1\right)^{\frac{1}{3}}=\left(40.0724\right)^{\frac{1}{3}}\approx 3.4220\left(4.\text{d.p}\right)\right)$

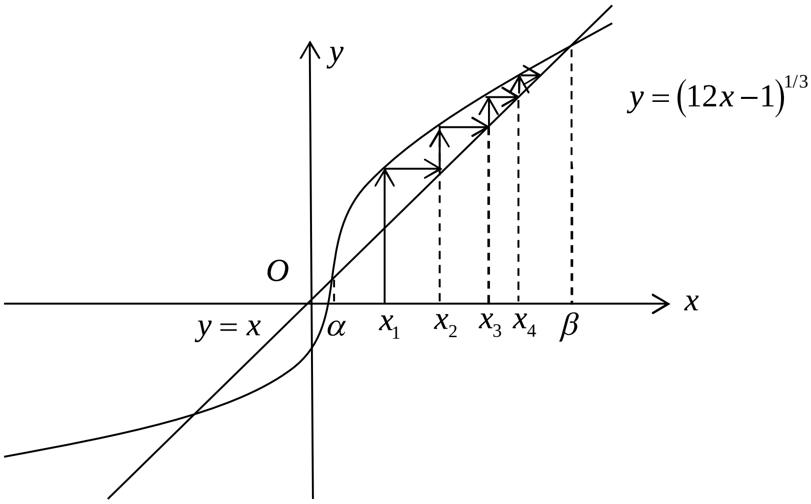
Check if $\beta \approx 3.42$

$\text{f}(3.415) = -0.1535 < 0$

$\text{f}(3.424) = 0.0542 > 0$

$\therefore \beta \approx 3.42 (\text{to 2.d.p})$

(iii)



Trapezium Rule and Simpson's Rule

6 Estimate the values of the following definite integrals, taking the number of ordinates in each case, using (a) the trapezium rule, (b) Simpson's Rule.

(i) $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\, \text{d}x$, 3 ordinates (ii) $\int_0^{0.4} \sqrt{1-x^2}\, \text{d}x$, 5 ordinates

Solution:

(i)	$h = \frac{\frac{\pi}{2}-0}{3-1} = \frac{\pi}{4}$ Let $\text{f}(x) = \frac{1}{1+\cos x}$ $\text{f}(0) = 1/2$, $\text{f}\left(\frac{\pi}{4}\right) = \frac{1}{1+1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}+1}$, $\text{f}\left(\frac{\pi}{2}\right) = 1$ Using Trapezium Rule,	$h = \frac{\frac{\pi}{2}-0}{3-1} = \frac{\pi}{4}$ Let $\text{f}(x) = \frac{1}{1+\cos x}$ $\text{f}(0) = 1/2$, $\text{f}\left(\frac{\pi}{4}\right) = \frac{1}{1+1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}+1}$, $\text{f}\left(\frac{\pi}{2}\right) = 1$ Using Simpson's Rule,

	$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\ \mathrm{d}x$ $\approx \frac{1}{2}\left(\frac{\pi}{4}\right)\left[1/2+1+2\left(\frac{\sqrt{2}}{1+\sqrt{2}}\right)\right]$ ≈ 1.04912 ≈ 1.05 (3 significant figures)	$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\ \mathrm{d}x$ $\approx \frac{1}{3}\left(\frac{\pi}{4}\right)\left(1/2+4\left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)+1\right)$ ≈ 1.00613 ≈ 1.01 (3 significant figures)
(ii)	$h=\frac{0.4-0}{5-1}=0.1$ Let $\mathrm{f}(x)=\sqrt{1-x^2}$ $\mathrm{f}(0)=1$, $\mathrm{f}(0.1)=\sqrt{0.99}$, $\mathrm{f}(0.2)=\sqrt{0.96}$, $\mathrm{f}(0.3)=\sqrt{0.91}$, $\mathrm{f}(0.4)=\sqrt{0.84}$, Using Trapezium Rule, $\int_0^{0.4}\sqrt{1-x^2}\ \mathrm{d}x$ $\approx \frac{0.1}{2}\left[1+\sqrt{0.84}+2\left(\sqrt{0.99}+\sqrt{0.96}+\sqrt{0.91}\right)\right]$ ≈ 0.38870 ≈ 0.389 (3 significant figures)	$h=\frac{0.4-0}{5-1}=0.1$ Let $\mathrm{f}(x)=\sqrt{1-x^2}$ $\mathrm{f}(0)=1$, $\mathrm{f}(0.1)=\sqrt{0.99}$, $\mathrm{f}(0.2)=\sqrt{0.96}$, $\mathrm{f}(0.3)=\sqrt{0.91}$, $\mathrm{f}(0.4)=\sqrt{0.84}$, Using Simpson's Rule, $\int_0^{0.4}\sqrt{1-x^2}\ \mathrm{d}x$ $\approx \frac{0.1}{3}\left(1+4\left(\sqrt{0.99}+\sqrt{0.91}\right)+2\left(\sqrt{0.96}\right)+\sqrt{0.84}\right)$ ≈ 0.38906 ≈ 0.389 (3 significant figures)

Euler's Method and Improved Euler's Method

7 Apply Euler's method with step size of 0.1 to obtain an approximation to the given initial value problems at $y(0.5)$.

(a) $\frac{\mathrm{d}y}{\mathrm{d}t}=1-y^2, \quad y(0)=0.5$ (b) $\frac{\mathrm{d}y}{\mathrm{d}t}=t+\sqrt{ty}, \quad y(0)=1$

Solution:

(a) $\mathrm{f}(t, y) = 1 - y^2$, thus by Euler's method, $y_{n+1}=y_n+\Delta t\cdot\left(1-y^2\right)$.

n	t_n	$y_{n,\ approx}$
0	0.0	0.5000
1	0.1	0.5750
2	0.2	0.6419

Beginning with $t_0 = 0$, $y_0 = 0.5$, the approximate values are

$y_1 = 0.5 + (0.1)\cdot(1 - 0.5^2) = 0.575$

$y_2 = 0.575 + (0.1)\cdot(1 - 0.575^2) = 0.6419375$

$y_3 = 0.6419375 + (0.1)\cdot(1 - 0.6419375^2) = 0.7007291124$

$y_4 = 0.7007291124 + (0.1)\cdot(1 - 0.7007291124^2) = 0.751626994$

$y_5 = 0.751626994 + (0.1)\cdot(1 - 0.751626994^2) = 0.79513268$

3	0.3	0.7007
4	0.4	0.7516
5	0.5	0.7951

Thus, the approximate value of $y(0.5) = 0.795$ when $\Delta t = 0.1$.

(b) f(t, y) = $t + \sqrt{ty}$, thus by Euler's method, $y_{n+1} = y_n + \Delta t \cdot \left(t_n + \sqrt{t_n y_n}\right)$.

Beginning with $t_0 = 0$, $y_0 = 1.0$, the approximate values are

$y_1 = 1 + (0.1)\cdot\left[0 + \sqrt{(0)(1)}\right] = 1$

$y_2 = 1 + (0.1)\cdot\left[0.1 + \sqrt{(0.1)(1)}\right] = 1.04162$

$y_3 = 1.04162 + (0.1)\cdot\left[0.2 + \sqrt{(0.2)(1.04162)}\right] = 1.10726$

$y_4 = 1.10726 + (0.1)\cdot\left[0.3 + \sqrt{(0.3)(1.10726)}\right] = 1.19489$

$y_5 = 1.19489 + (0.1)\cdot\left[0.4 + \sqrt{(0.4)(1.19489)}\right] = 1.30402$

n	t_n	$y_{n,\ approx}$
0	0.0	1.0000
1	0.1	1.0000
2	0.2	1.0416
3	0.3	1.1073
4	0.4	1.1949
5	0.5	1.3040

Thus, the approximate value of $y(0.5) = 1.30$ when $\Delta t = 0.1$.

8 Apply improved Euler's method with step size of 0.1 to the initial value problems given in **Question 7** to obtain an approximation to the value at $y(0.5)$.

Solution:

(a) f(t, y) = $1 - y^2$, thus by improved Euler's method,

$$y^*_{n+1} = y_n + \Delta t \cdot \left(1 - {y_n}^2\right)$$

$$y_{n+1} = y_n + \Delta t \cdot \left[\frac{(1 - y_n^2) + (1 - {y^*_{n+1}}^2)}{2}\right]$$

n	t_n	y^*_n	$y_{n,\ approx}$
0	0.0	---	0.5000
1	0.1	0.5750	0.5710
2	0.2	0.6384	0.6343
3	0.3	0.6941	0.6901
4	0.4	0.7425	0.7387
5	0.5	0.7841	0.7807

Beginning with $t_0 = 0$, $y_0 = 0.5$, the first two approximate values are

$y^*_1 = 0.5 + 0.1\cdot\left(1 - 0.5^2\right) = 0.575$

$y_1 = 0.5 + 0.1\cdot\left[\frac{(1 - 0.5^2) + (1 - 0.575^2)}{2}\right] = 0.57096875$

$y^*_2 = 0.57096875 + 0.1\cdot\left(1 - 0.57096875^2\right) = 0.638368$

$$y_2 = 0.57096875 + 0.1 \cdot \left[\frac{(1-0.57096875^2)+(1-0.638368^2)}{2} \right] = 0.63429$$

The rest of the approximations are represented in the table.
Thus, the approximate value of $y(0.5) = 0.781$ when $\Delta t = 0.1$.

(b) f(t, y) = $t + \sqrt{ty}$, thus by improved Euler's method,

$$y^*_{n+1} = y_n + \Delta t \cdot \left(t_n + \sqrt{t_n y_n} \right)$$

$$y_{n+1} = y_n + \Delta t \cdot \left[\frac{(t_n + \sqrt{t_n y_n}) + (t_{n+1} + \sqrt{t_{n+1} y^*_{n+1}})}{2} \right]$$

n	t_n	y^*_n	$y_{n,\text{ approx}}$
0	0.0	---	1.0000
1	0.1	1.0000	1.0208
2	0.2	1.0628	1.0748
3	0.3	1.1412	1.1523
4	0.4	1.2411	1.2519
5	0.5	1.3627	1.373556

Beginning with $t_0 = 0$, $y_0 = 1.0$, the first two approximate values are

$$y^*_1 = 1 + 0.1 \cdot \left(0 + \sqrt{(0)(1)} \right) = 1$$

$$y_1 = 1 + 0.1 \cdot \left[\frac{(0+\sqrt{(0)(1)})+(0.1+\sqrt{(0.1)(1)})}{2} \right] = 1.02081$$

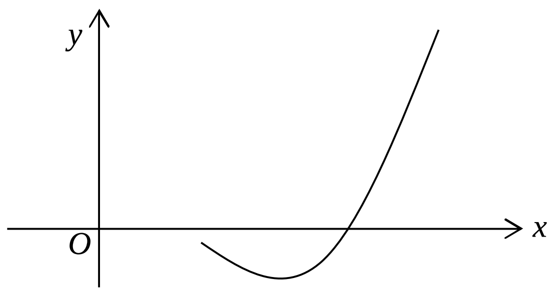
$$y^*_2 = 1.02081 + 0.1 \cdot \left(0.1 + \sqrt{(0.1)(1.02081)} \right) = 1.06276$$

$$y_2 = 1.06276 + 0.1 \cdot \left[\frac{(0.1+\sqrt{(0.1)(1.02081)})+(0.2+\sqrt{(0.2)(1.06276)})}{2} \right] = 1.0748$$

The rest of the approximations are represented in the table.
Thus, the approximate value of $y(0.5) = 1.37$ when $\Delta t = 0.1$.

Practice Questions

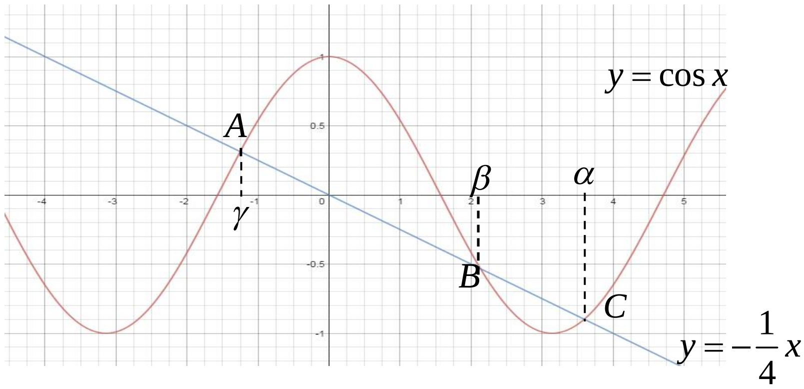
1 By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.
Use linear interpolation, once only, on the interval to find an approximation to the negative root of the equation $x + 4\cos x = 0$, giving 2 decimal places in your answer.



The diagram shows part of the graph $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.
Use the Newton-Raphson method to find α correct to 2 significant figures.

N94/P2/Q13

Solution:



From the sketch, the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$ intersect at *A*, *B* and *C*.

$\cos x = -\frac{1}{4}x \Rightarrow 4\cos x = -x \Rightarrow 4\cos x + x = 0$

Therefore, the equation has 1 negative root, γ, and 2 positive roots, α and β.

Let $\mathrm{f}(x) = x + 4\cos x$,

By linear interpolation, an approximation to γ

$= \frac{(-1)|\mathrm{f}(-1.5)| + (-1.5)|\mathrm{f}(-1)|}{|\mathrm{f}(-1.5)| + |\mathrm{f}(-1)|}$

$\approx -1.24413(\text{to 5 decimal places})$

$\approx -1.24(\text{to 2 decimal places})$

From the given diagram, there exists a minimum point on the left of α. Taking initial approximation smaller than α, has a gradient which is small. The tangent at this point will cut the x-axis results in an approximation that is further away from α, which is not a good estimate of α.

$\mathrm{f}(x) = x + 4\cos x \Rightarrow \mathrm{f}'(x) = 1 - 4\sin x$

$\mathrm{f}(3) = -0.95997 < 0$

$\mathrm{f}(4) = 1.38543 > 0$

$\Rightarrow \alpha \in (3, 4)$

$\mathrm{f}(3.5) = -0.24583 < 0$

$\Rightarrow \alpha \in (3.5, 4)$

Taking the initial value as 4, (inferred from earlier part that we should take initial value bigger than α.

By Newton-Raphson method, $x_1 = 4$

$$x_2 = 4 - \frac{f(4)}{f'(4)} = 3.656 \text{ (4 significant figures)}$$

$$x_3 = 3.656 - \frac{f(3.656)}{f'(3.656)} = 3.597 \text{ (4 significant figures)}$$

$$x_4 = 3.597 - \frac{f(3.597)}{f'(3.597)} = 3.595 \text{ (4 significant figures)}$$

Check if $\alpha = 3.6(\text{to 2 significant figures})$,

$f(3.55) = -0.12102 < 0$

$f(3.64) = 0.12662 > 0$

$\Rightarrow \alpha \in (3.55, 3.64)$

$\therefore \alpha \approx 3.6(\text{to 2 significant figures})$

2 (a) A function f is such that $f(4) = 1.158$ and $f(5) = -3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $f(x) = 0$, use linear interpolation to estimate this value.

For the case where $f(x) = \tan x$, and x is measured in radians, the values of $f(4)$ and $f(5)$ are as given above. Explain with the aid of a sketch why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x = 0$.

(b) Show, by means of a graphical argument that the equation $\ln(x-1) = -2x$ has exactly one real root, and show that this root lies between 1 and 2.

The equation may be written in the form $\ln(x-1) + 2x = 0$. Show that neither $x = 1$ nor $x = 2$ is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form $x - 1 - e^{-2x} = 0$. For this form, use two applications of the Newton-Raphson method, starting with $x = 1$, to obtain an approximation to the root, giving three decimal places in your answer.

N95/P2/13

Solution:

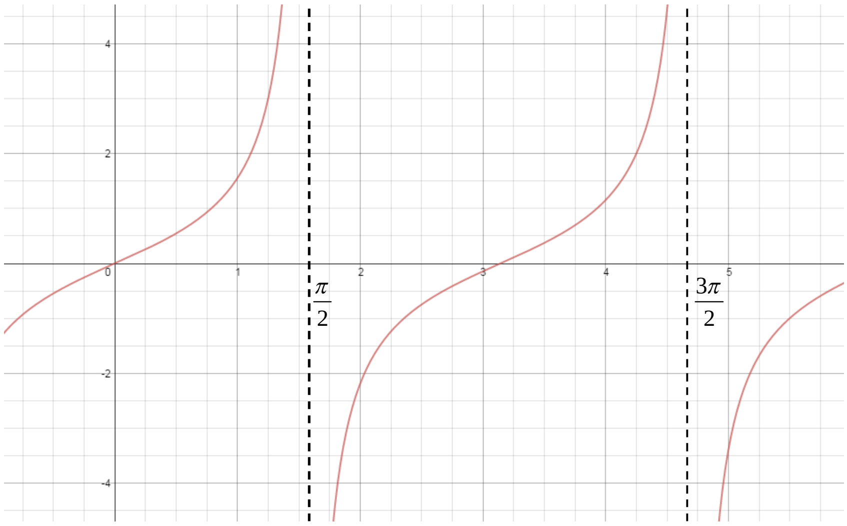
(a) By linear interpolation, an approximation to the root

$$= \frac{(4)|f(5)| + (5)|f(4)|}{|f(4)| + |f(5)|}$$

$\approx 4.25512(\text{to 5 decimal places})$

$\approx 4.255(\text{to 3 decimal places})$

Graph of $y = \tan x$

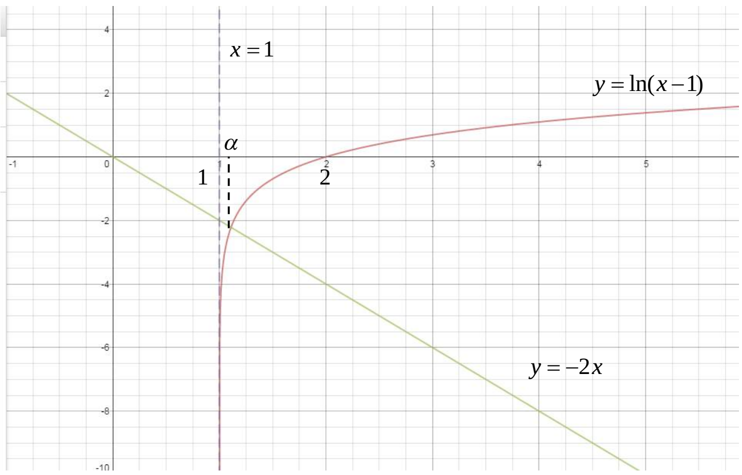


For function $f(x)$ to have a root in the interval (a, b), $f(a)f(b) < 0$ and the curve of $y = f(x)$ must be continuous in the interval (a, b). From the graph of $y = \tan x$, $f(x) = \tan x$ is not continuous in (4,5). Therefore, linear interpolation does not produce an approximation to a solution of $\tan x = 0$.

Or

The graph of $y = \tan x$ does not cut the *x*-axis in the interval $4 < x < 5$, so there are no real roots. Therefore, linear interpolation cannot give an approximate root.

(b)



From the graphs of $y = \ln(x-1)$ and $y = -2x$, there is exactly one point of inetsection, α, in the interval (1,2). Theer fore the root of the equation $\ln(x-1) = -2x$ lies between 1 and 2.

Let

$\text{f}(x)=\ln(x-1)+2x$

$\Rightarrow \text{f}'(x)=\dfrac{1}{x-1}+2$

If initial value $x=1$, $\text{f}'(1)$ is undefined. Therefore, $x=1$ is not a suitable initial value.

If initial value $x_1=2$, $x_2=2-\dfrac{\text{f}(2)}{\text{f}'(2)}=\dfrac{2}{3}$. The iteration stops as $\text{f}\left(\dfrac{2}{3}\right)$is undefined.

$\ln\left(\dfrac{2}{3}-1\right)$ cannot be evaluated. Therefore, $x=2$ is not a suitable initial value.

Let $\text{g}(x)=x-1-\text{e}^{-2x}$

$\text{g}'(x)=1+2\text{e}^{-2x}$

By Newton-Raphson method, taking $x_1=1$,

$x_2=1-\dfrac{\text{g}(1)}{\text{g}'(1)}=1.1065$

$x_3=1.1065-\dfrac{\text{g}(1.1065)}{\text{g}'(1.1065)}=1.1088$

An approximation to $\alpha\approx 1.109$ $(\text{to 3 decimal places})$

3 The equation $x^3-5x^2+2=0$ has one negative root and two positive roots.

(i) Find the integer n such that the smaller positive root α , denoted by, lies in $(n,n+1)$.

(ii) Show that the graph of $y=x^3-5x^2+2$ is strictly increasing for $x<0$.

(iii) If β_0 is a first approximation to the negative root, β , obtained by linear interpolation method, explain why $\beta_0>\beta$.

(iv) An iteration formula for finding β is given by $x_{n+1}=-\sqrt{\dfrac{2}{5-x_n}}$.

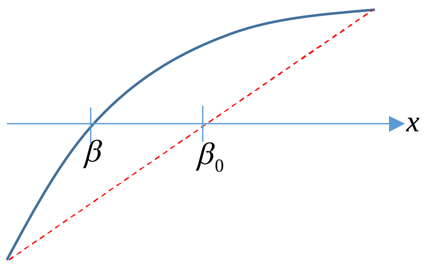
Taking $\beta_1=-0.6$, apply this formula to find β , correct to 3 significant figures.

Solution:

(i) Let $\text{f}(x)=x^3-5x^2+2$. Since $\text{f}(0)=2>0$ and $\text{f}(1)=-2<0$, α lies in $(0,1)$. $\therefore n=0.$

(ii) $\text{f}'(x)=3x^2-10x=x(3x-10)$. For $x<0$, $3x-10<0$. $\therefore \text{f}'(x)>0$ for $x<0$. The graph of of $y=x^3-5x^2+2$ is strictly increasing for $x<0$.

(iii) $\text{f}''(x)=6x-10<0$ for $x<0$. The graph of is concave downwards for $x<0$. From part (ii), the graph of $y=x^3-5x^2+2$ is strictly increasing for $x<0$. As shown from the diagram below, $\beta_0>\beta$ if linear interpolation is applied.



(iv) Taking $x_1 = -0.6$,

$$x_2 = -\sqrt{\frac{2}{5-x_1}} = -0.59761 \text{ (to 5 s.f.)}$$

$$x_3 = -\sqrt{\frac{2}{5-x_2}} = -0.59774 \text{ (to 5 s.f.)}$$

$$\therefore \beta = -0.598 \text{ (to 5 s.f.)}$$

4 The area of the finite region bounded by the curve $y = \frac{\ln x}{x^2}$, the x-axis and the line $x = 2$ is given by A.

(i) Evaluate A, leaving your answer in terms of natural logarithms.

(ii) Use the trapezium rule with two strips to obtain an approximation A_1 to the value of A, giving your answer to three significant figures.

With the aid of a diagram, explain whether A_1 is an underestimation or overestimation.

Solution:

(i)

$$A = \int_1^2 \frac{\ln x}{x^2}\, dx$$

$$= \left[-\frac{1}{x}\ln x\right]_1^2 - \int_1^2 -\frac{1}{x^2}\, dx$$

$$= -\frac{1}{2}\ln 2 - \left[\frac{1}{x}\right]_1^2$$

$$= -\frac{1}{2}\ln 2 - \left(\frac{1}{2} - 1\right)$$

$$= \frac{1}{2} - \frac{1}{2}\ln 2$$

(ii) Let $f(x) = \frac{\ln x}{x^2}$ and $h = \frac{2-1}{2} = 0.5$,

$$f(1) = 0,$$

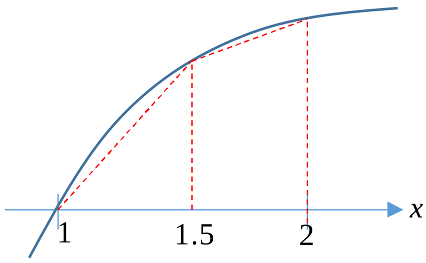
$$f(1.5) = \frac{\ln 1.5}{2.25},$$

$$f(2) = \frac{\ln 2}{4}.$$

Using Trapezium Rule,

$$\int_1^2 \frac{\ln x}{x^2}\,\mathrm{d}x$$
$$\approx \frac{0.5}{2}\left[0+\frac{\ln 2}{4}+2\left(\frac{\ln 1.5}{2.25}\right)\right]$$
$$\approx 0.133\left(3 \text{ significant figures}\right)$$

Since the curve $y=\frac{\ln x}{x^2}$ from $x=1$ to $x=2$ concave downwards, applying the trapezium rule will lead to an underestimation of the actual area as the sum of the areas of the trapezia is less than the actual area. $\therefore A_1$ is an underestimation.



5 Using the Simpson's Rule with 5 ordinates, find an approximation to

$$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x .$$

Hence use your result to obtain an estimated value of π. Give all your answers correct to 2 decimal places.

Without evaluating the approximation to $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$ using Trapezium Rule, state with reason which approximation, using Simpson's Rule or Trapezium Rule, is a better estimation.

Solution:

Let $\mathrm{f}\left(x\right)=\frac{1}{\sqrt{1-x^2}}$ and $h=\frac{0.5-0}{4}=0.125$,

$\mathrm{f}\left(0\right)=1$,

$\mathrm{f}\left(0.125\right)=\frac{1}{\sqrt{0.984375}}$,

$\mathrm{f}\left(0.25\right)=\frac{1}{\sqrt{0.9375}}$,

$\mathrm{f}\left(0.375\right)=\frac{1}{\sqrt{0.859375}}$,

$\mathrm{f}\left(0.5\right)=\frac{1}{\sqrt{0.75}}$.

Using Simpson's Rule,

$$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x$$
$$\approx \frac{0.125}{3}\left[1+4\left(\frac{1}{\sqrt{0.984375}}+\frac{1}{\sqrt{0.859375}}\right)+2\left(\frac{1}{\sqrt{0.9375}}\right)+\frac{1}{\sqrt{0.75}}\right]$$
$$\approx 0.52362$$
$$\approx 0.52\left(2 \text{ decimal places}\right)$$

$$\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\ \mathrm{d}x = \left[\sin^{-1} x\right]_0^{0.5}$$
$$= \sin^{-1} 0.5 - \sin^{-1} 0$$
$$= \frac{\pi}{6}$$

$$\frac{\pi}{6} \approx 0.52362$$
$$\Rightarrow \pi \approx 3.14172 \approx 3.14 \ \left(\text{to 2 decimal places}\right)$$

$$\mathrm{f}\left(x\right)=\frac{1}{\sqrt{1-x^2}} \Rightarrow \mathrm{f}\,'\left(x\right)=\frac{x}{\left(1-x^2\right)^{3/2}} \ \text{ and } \ \mathrm{f}\,''\left(x\right)=\frac{1}{\left(1-x^2\right)^{3/2}}+\frac{3x^2}{\left(1-x^2\right)^{5/2}}$$

For $0 < x < 0.5,$ since $\mathrm{f}\,'\left(x\right) > 0$ and $\mathrm{f}\,''\left(x\right) > 0$, graph of $y = \dfrac{1}{\sqrt{1-x^2}}$ strictly increases and concave upwards in $0 < x < 0.5$. The sum of the areas of the trapezia is greater than the area bounded by the parabola, x-axis and the lines x = 1 and x = 0.5. Thus the approximation using Simpson's Rule is a better estimation.

6 (a) Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + y, \qquad y(0) = 1 \qquad \text{------} \quad (1)$$

Apply Euler's method with step size Δt = 0.1, to find the approximate value of y when t = 1, correct to 4 decimal places. Compare the approximated results with the exact solution.

(b) Use improved Euler's method with step size Δt = 0.1 to find the approximate value of y when t = 1, correct to 4 decimal places. Compare the approximated results with the exact solution and that obtained from the Euler's method.

Solution:

(a) f(t , y) = t + y, thus by Euler's method, $y_{n+1} = y_n + \Delta t \cdot (t_n + y_n)$

Beginning with t_0 = 0, y_0 = 1, the first three approximate values are

$$y_1 = 1.0000 + (0.1)\cdot(0.0 + 1.0000) = 1.1000$$
$$y_2 = 1.1000 + (0.1)\cdot(0.1 + 1.1000) = 1.2200$$
$$y_3 = 1.2200 + (0.1)\cdot(0.2 + 1.2200) = 1.3620$$

The rest are shown in the table below.

The "integrating factor" method can be used to solve (1).

$$\frac{\mathrm{d}y}{\mathrm{d}t} = t + y \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} - y = t$$

Integrating Factor = $\mathrm{e}^{\int -1 dt}$ = e^{-t}

Therefore, $\left(\mathrm{e}^{-t}\right)\frac{\mathrm{d}y}{\mathrm{d}t} - y\left(\mathrm{e}^{-t}\right) = t\left(\mathrm{e}^{-t}\right) \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(y\mathrm{e}^{-t}) = t\mathrm{e}^{-t}$

$\Rightarrow \quad y\mathrm{e}^{-t} = \int t\mathrm{e}^{-t} dt = -t\mathrm{e}^{-t} - \mathrm{e}^{-t} + C$ (using integration by parts)

$\Rightarrow \quad y = -t - 1 + C\mathrm{e}^{t}$

At $t = 0$, $y = 1$, we get $C = 2$. Thus, $y = -t - 1 + 2\mathrm{e}^{t}$.

The table below shows the approximate values of $y(t_n)$ obtained in all ten steps, the actual value $y(t_n)$, the absolute error and the percentage error in each approximate value.

From the above table, we observe that the error in y_n increases as n increases. The percentage errors increase from 0.93% at $t_1 = 0.1$ to 7.25% at $t_{10} = 1.0$.

Euler's Method

n	t_n	$y_{n,\ approx}$	$y_{n,\ actual}$	$\lvert y_{n,\ \text{actual}} - y_{n,\ \text{approx}}\rvert$	**Percentage Error**
0	0.0	1.0000	1.0000	0.0000	0.00%
1	0.1	1.1000	1.1103	0.0103	0.93%
2	0.2	1.2200	1.2428	0.0228	1.84%
3	0.3	1.3620	1.3997	0.0377	2.69%
4	0.4	1.5282	1.5836	0.0554	3.50%
5	0.5	1.7210	1.7974	0.0764	4.25%
6	0.6	1.9431	2.0442	0.1011	4.95%
7	0.7	2.1974	2.3275	0.1301	5.59%
8	0.8	2.4872	2.6511	0.1639	6.18%
9	0.9	2.8159	3.0192	0.2033	6.73%
10	1.0	3.1875	3.4366	0.2491	7.25%

(b) f(t , y) = $t + y$, thus by improved Euler's method,

$$y_{n+1}^{*} = y_n + \Delta t \cdot (t_n + y_n)$$

$$y_{n+1} = y_n + \Delta t \cdot \left[\frac{(t_n + y_n) + (t_{n+1} + y_{n+1}^{*})}{2}\right]$$

Beginning with $t_0 = 0$, $y_0 = 1$, the first two approximate values are

$$y_1^{*} = 1.0000 + (0.1)\cdot(0.0 + 1.0000) = 1.1000$$

$$y_1 = 1.0000 + (0.1)\cdot\left[\frac{(0.0 + 1.0000) + (0.1 + 1.1000)}{2}\right] = 1.1100$$

$$y_2^{*} = 1.1100 + (0.1)\cdot(0.1 + 1.1100) = 1.2310$$

$$y_2 = 1.1100 + (0.1)\cdot\left[\frac{(0.1 + 1.1100) + (0.2 + 1.2310)}{2}\right] = 1.2421$$

The rest are shown in the table below.

The table below shows the approximate values of $y(t_n)$ obtained in all ten steps, the actual value $y(t_n)$, the absolute error and the percentage error in each approximate value.

Improved Euler's Method

n	t_n	y^*_n	$y_{n,\ approx}$	$y_{n,\ actual}$	$\lvert y_{n,\ actual} - y_{n,\ approx}\rvert$	**Percentage Error**
0	0.0	---	1.0000	1.0000	0.0000	0.00%
1	0.1	1.1000	1.1100	1.1103	0.0003	0.03%
2	0.2	1.2310	1.2421	1.2428	0.0008	0.06%
3	0.3	1.3863	1.3985	1.3997	0.0013	0.09%
4	0.4	1.5683	1.5818	1.5836	0.0018	0.12%
5	0.5	1.7800	1.7949	1.7974	0.0025	0.14%
6	0.6	2.0244	2.0409	2.0442	0.0034	0.17%
7	0.7	2.3049	2.3231	2.3275	0.0044	0.19%
8	0.8	2.6255	2.6456	2.6511	0.0055	0.21%
9	0.9	2.9901	3.0124	3.0192	0.0068	0.23%
10	1.0	3.4036	3.4282	3.4366	0.0084	0.24%

We notice that the improved Euler's method gives a better approximation to the values of $y(t_n)$ as compared to the Euler's method. This corresponds to the decrease in the error between the actual value and the approximate value.

7 Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} - y = -\frac{1}{2}\mathrm{e}^{\frac{t}{2}}\sin(5t) + 5\mathrm{e}^{\frac{t}{2}}\cos(5t), \quad y(0) = 0$$

Use Euler's method and the step sizes of Δt = 0.1, Δt = 0.05, Δt = 0.01, Δt = 0.005, and Δt = 0.001 to find the approximations to the solution at t = 1, t = 2, t = 3, t = 4, and t = 5. How does changing the values of t affect the accuracy of the approximations? Justify your answer.

Solution:

f(t , y) = $-\frac{1}{2}\mathrm{e}^{\frac{t}{2}}\sin(5t) + 5\mathrm{e}^{\frac{t}{2}}\cos(5t) + y$, thus by Euler's method,

$$y_{n+1} = y_n + \Delta t \cdot \left[-\frac{1}{2}\mathrm{e}^{\frac{t}{2}}\sin(5t) + 5\mathrm{e}^{\frac{t}{2}}\cos(5t) + y \right]$$

The "integrating factor" method with integrating factor e^{-t} can be used to solve the DE to obtain $y = \mathrm{e}^{\frac{t}{2}}\sin(5t)$.

Below are two tables that give the approximations and percentage error for each approximation.

		$y_{n,\text{ approx.}}$				
t	y_{actual}	**Δt = 0.1**	**Δt = 0.05**	**Δt = 0.01**	**Δt = 0.005**	**Δt = 0.001**
1	-1.58100	-0.97167	-1.26512	-1.51580	-1.54826	-1.57443
2	-1.47880	0.65270	-0.34327	-2.18657	-1.35810	-1.45453
3	2.91439	7.30209	5.34682	3.44488	3.18259	2.96851
4	6.74580	15.56128	11.84839	7.89808	7.33093	6.86429
5	-1.61237	21.95465	12.24018	1.56056	0.0018864	-1.28498

Percentage Errors (%)					
t	**Δt = 0.1**	**Δt = 0.05**	**Δt = 0.01**	**Δt = 0.005**	**Δt = 0.001**
1	38.54	19.98	4.12	2.07	0.42
2	144.14	76.79	16.21	8.16	1.64
3	150.55	83.46	18.20	9.20	1.86
4	130.68	75.64	17.08	8.67	1.76
5	1461.63	859.14	196.79	100.12	20.30

Increasing *t* results in an increasing error. Each successive approximation is found using a previous approximation. Therefore, at each step we introduce error and so approximations should, in general, get worse as *t* increases.

8 Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t}+2y=2-\mathrm{e}^{-4t}, \qquad y(0)=1$$

Use Euler's method and the step sizes of Δt = 0.1, Δt = 0.05, Δt = 0.01, Δt = 0.005, and Δt = 0.001 to obtain approximations at t = 1, t = 2, t = 3, t = 4, and t = 5. How does changing the values of Δt affect the accuracy of the approximations?

Solution:

f(t , y) = $2-\mathrm{e}^{-4t}-2y$, thus by Euler's method, $y_{n+1}=y_n+\Delta t\cdot(2-\mathrm{e}^{-4t_n}-2y_n)$.

The "integrating factor" method with integrating factor e^{2t} can be used to solve the DE to obtain $y = 1+\frac{1}{2}\mathrm{e}^{-4t}-\frac{1}{2}\mathrm{e}^{-2t}$.

Below are two tables, one gives approximations to the solution and the other gives the errors for each approximation.

		$y_{n,\text{ approx.}}$				
t	y_{actual}	**Δt = 0.1**	**Δt = 0.05**	**Δt = 0.01**	**Δt = 0.005**	**Δt = 0.001**
1	0.9414902	0.9313244	0.9364698	0.9404994	0.9409957	0.9413914
2	0.9910099	0.9913681	0.9911126	0.9910193	0.9910139	0.9910106

3	0.9987637	0.9990501	0.9988982	0.9987890	0.9987763	0.9987662
4	0.9998323	0.9998976	0.9998657	0.9998390	0.9998357	0.9998330
5	0.9999773	0.9999890	0.9999837	0.9999786	0.9999780	0.9999774

	Percentage Errors (%)				
t	**Δ*t* = 0.1**	**Δ*t* = 0.05**	**Δ*t* = 0.01**	**Δ*t* = 0.005**	**Δ*t* = 0.001**
1	1.08	0.53	0.105	0.053	0.0105
2	0.036	0.010	0.00094	0.00041	0.0000703
3	0.029	0.013	0.0025	0.0013	0.00025
4	0.0065	0.0033	0.00067	0.00034	0.000067
5	0.0012	0.00064	0.00013	0.000068	0.000014

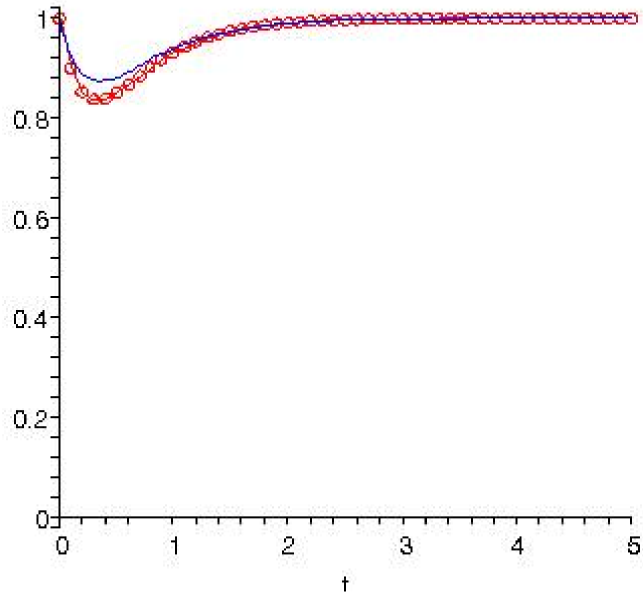
We observe that decreasing Δ*t* improves the accuracy of the approximation.

Additional remarks for Question 8:

There are a couple of other interesting things to note from the data. First, notice that in general, decreasing the step size Δ*t*, by a factor of 10 also decreased the error by about a factor of 10 as well.

In this particular question, notice that as *t* increases, the approximation actually tends to get better (as can be observed by the reduction in the percentage errors). This is not the case completely as we can see that in all but the first case, the error at *t* = 3 error is worse than the error at *t* = 2, but after that point, it only gets better. This should not be expected in general. In this case, this is due to the shape of the function of the solution.

The solution curve as well as the approximations for Δ*t* = 0.1 is shown in the graph below.



Notice that the approximation is worse where the function is changing rapidly. This should not be too surprising. Recall that we are using tangent lines to get the approximations and so the value of the tangent line at a given *t* will often be significantly different than the function due to the rapidly changing function at that point.

Also, in this case, because the function ends up fairly flat as *t* increases, the tangents start looking like the function itself and so the approximations are very accurate. This will not always be the case in general.

9 Consider the initial value problem

$$\frac{dy}{dt} = y^2 - 2y + 1, \qquad y(0) = 2$$

Use improved Euler's method with Δt = 0.5 to find the approximate solutions for $0 \le t \le 2$. Your answer should include a table of approximate values of the dependable variable and compare your values obtained from Euler's method.

Solution:

$f(t, y) = y^2 - 2y + 1$.

By Euler's method, $y_{n+1} = y_n + \Delta t \cdot (y_n^2 - 2y_n + 1)$.

Beginning with $t_0 = 0$, $y_0 = 2$, the first two approximate values are

$y_1 = 2.0000 + (0.5)\cdot(2^2 - 2(2) + 1) = 2.5$, $y_2 = 2.5 + (0.5)\cdot(2.5^2 - 2(2.5) + 1) = 3.625$

By improved Euler's method,

$$y_{n+1}^* = y_n + \Delta t \cdot (y_n^2 - 2y_n + 1) \quad , \quad y_{n+1} = y_n + \Delta t \cdot \left[\frac{(y_n^2 - 2y_n + 1) + \left(\left(y_{n+1}^* \right)^2 - 2y_{n+1}^* + 1 \right)}{2} \right]$$

Beginning with $t_0 = 0$, $y_0 = 2$, the first two approximate values are

$$y_1^* = 2 + (0.5)\cdot(2^2 - 2(2) + 1) = 2.5$$

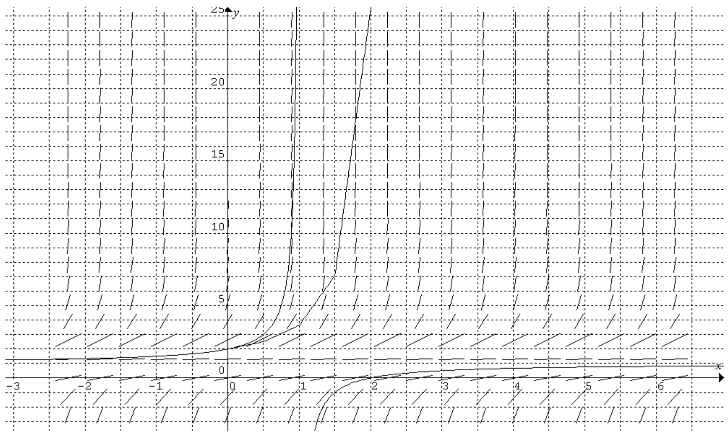
$$y_1 = 2 + (0.5)\cdot\left[\frac{(2^2 - 2(2) + 1) + (2.5^2 - 2(2.5) + 1)}{2}\right] = 2.8125$$

$$y_2^* = 2.8125 + (0.5)\cdot(2.8125^2 - 2(2.8125) + 1) = 4.4551$$

$$y_2 = 2.8125 + (0.5)\cdot\left[\frac{(2.8125^2 - 2(2.8125) + 1) + (4.4551^2 - 2(4.4551) + 1)}{2}\right] = 6.6182$$

The DE can be solved exactly to give $y = 1 - \frac{1}{t-1}$.

Putting the values obtained from the Euler's method, the improved Euler's method and the actual values in a table, we get:



n	***t_n***	**Euler, y_n**	**iEuler, y_{in}**	**$y_{n,\ actual}$**
0	0.0	2	2	2.0000
1	0.5	2.5	2.8125	3.0000
2	1.0	3.625	6.6182	undefined
3	1.5	7.0703	129.0008	−1.0000
4	2.0	25.4947	17310268.86	0.0000

You will notice that neither the Euler's method nor improved Euler's method are able to approximate the actual value well after $t = 0.5$. This is largely due to the presence of a vertical asymptote at $t = 1$.

<u>Numerical Answers to Numerical Methods Tutorial</u>

Basic Mastery Questions

1. 1.146
2. 1.365
3. 0.443
4. 2.23
5. 3.42
6. (i) 1.05 ; 1.01 (ii) 0.389 ; 0.389
7. (a) 0.795 (b) 1.30
8. (a) 0.781 (b) 1.37

Practice Questions

1. 3.6
2. (a) 4.255 (b) 1.109
3. (i) $n = 0$ (iv) -0.598
4. (i) $\frac{1}{2}-\frac{1}{2}\ln 2$ (ii) 0.133
5. 0.52; 3.14