Chapter 8 (Pure Mathematics): Applications of Integration

Objectives:

At the end of the chapter, you should be able to:

(a)	evaluate definite integrals
(b)	find the area of a region bounded by a curve and lines parallel to the coordinate axes, between a curve and a line, or between two curves
(c)	find the numerical value of a definite integral using a graphing calculator

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- 8.2.1 Evaluating Definite Integral Algebraically
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8.4 Area Under a Curve
- 8.4.1 Area under the curve with respect to *x*-axis
- 8.4.2 Area between the curve and *y*-axis
- 8.4.3 Area between two curves

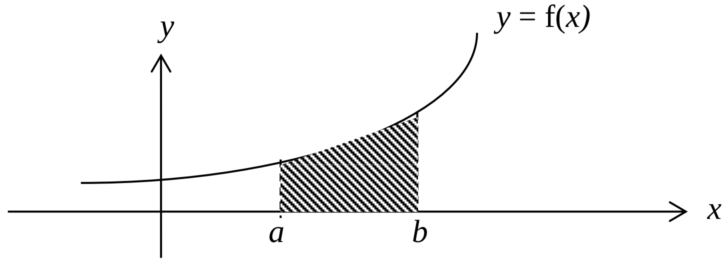
References

1. New Additional Mathematics, Ho Soo Thong (Msc, Dip Ed), Khor Nyak Hiong (Bsc, Dip Ed)

2. New Syllabus Additional Mathematics (7^{th} Edition), Shinglee Publishers Ptd Ltd

8.1 What is a Definite Integral

In the previous chapter you learnt that integration is the reverse of differentiation but what does the integral function do? In this chapter, you will learn that you can use the integral to find the area of a function over an interval [*a*, *b*] where ***a* is called the lower limit and *b* is called the upper limit**.



Area under the curve over the interval [*a*, *b*] = $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$.

We will go into more details in Section 8.4.

8.2 Evaluate Definite Integrals

Steps to find a definite integral

1. Integrate the function.
2. Substitute the upper limit, *b*, into *x* to obtain $\mathrm{F}(b)$.
3. Substitute the lower limit, *a*, into *x* to obtain $\mathrm{F}(a)$.
4. $\mathrm{F}(b) - \mathrm{F}(a)$. The answer is the area under the curve.

8.2.1 Evaluating Definite Integral Algebraically

Example 1

Evaluate (a) $\int_1^2 x^3 dx$ (b) $\int_0^2 (2x+5)^{-3} dx$.

Solution:

(a)

$\int_1^2 x^3 \mathrm{d}x = \left[\frac{x^4}{4}\right]_1^2$ Step 1: Integrate the function x^3 to get $\frac{x^4}{4}$.

$= \frac{1}{4}\left[x^4\right]_1^2$ Step 2: Substitute the upper limit, 2, into *x* to obtain $\frac{2^4}{4}$.

$= \frac{1}{4}\left[2^4 - 1^4\right]$ Step 3: Substitute the lower limit, 1, into *x* to obtain. $\frac{1^4}{4}$

$= \frac{15}{4} = 3\frac{3}{4}$ Step 4: $\frac{2^4}{4} - \frac{1^4}{4}$. The answer is the area under the curve.

(b)

$$\int_0^2 \left(2x+5\right)^{-3} \mathrm{d}x$$

$$= \left[\frac{\left(2x+5\right)^{-2}}{-2(2)} \right]_0^2$$

$$= \frac{1}{-4}\left[\frac{1}{\left(2x+5\right)^2} \right]_0^2$$

$$= \frac{1}{-4}\left[\frac{1}{(4+5)^2} - \frac{1}{(5)^2} \right]$$

$$= \frac{14}{2025}$$

Step 1: Integrate the function $(2x+5)^{-3}$ to get $\frac{(2x+5)^{-2}}{(-2)(2)}$

Step 2: Substitute the upper limit, 2, into x to obtain $-\frac{1}{4}\left(\frac{1}{(4+5)^2}\right)$.

Step 3: Substitute the lower limit, 0, into x to obtain $-\frac{1}{4}\left(\frac{1}{(0+5)^2}\right)$

Step 4: $-\frac{1}{4}\left[\frac{1}{(4+5)^2} - \frac{1}{(5)^2}\right]$ The answer is the area under the curve.

Example 2

Integrate the following with respect to x, leaving your answer in exact form.

Solution:

(i)

$$\int_0^8 \sqrt{x+1}\ \mathrm{d}x$$

$$= \int_0^8 \left(x+1\right)^{\frac{1}{2}}\ \mathrm{d}x$$

$$= \left[\frac{\left(x+1\right)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^8$$

$$= \frac{2}{3}\left[\left(\sqrt{x+1}\right)^3 \right]_0^8$$

$$= \frac{2}{3}\left[\left(\sqrt{8+1}\right)^3 - \left(\sqrt{0+1}\right)^3 \right]$$

$$= \frac{2}{3}\left[27-1\right]$$

$$= \frac{52}{3}$$

$$= 17\frac{1}{3}$$

(iii)

$$\int_0^1 \frac{2}{\sqrt{3x+1}}\ \mathrm{d}x$$

$$= 2\int_0^1 \left(3x+1\right)^{-\frac{1}{2}}\ \mathrm{d}x$$

$$= 2 \times \frac{1}{3} \times \frac{1}{\frac{1}{2}} \left[\left(3x+1\right)^{\frac{1}{2}} \right]_0^1$$

$$= \frac{4}{3}\left[\sqrt{3x+1} \right]_0^1$$

$$= \frac{4}{3}\left[\left(\sqrt{3(1)+1}\right) - \left(\sqrt{3(0)+1}\right) \right]$$

$$= \frac{4}{3}\left[2-1\right]$$

$$= \frac{4}{3}$$

(ii)

$$\int_{-3}^{0} \frac{5}{\sqrt{1-x}}\,dx$$
$$= 5\int_{-3}^{0} (1-x)^{-\frac{1}{2}}\,dx$$
$$= 5 \times \frac{1}{-1} \times \frac{1}{\frac{1}{2}}\left[(1-x)^{\frac{1}{2}}\right]_{-3}^{0}$$
$$= -10\left[\sqrt{1-x}\right]_{-3}^{0}$$
$$= -10\left[\sqrt{1-0} - \sqrt{1-(-3)}\right]$$
$$= 10$$

(iv)

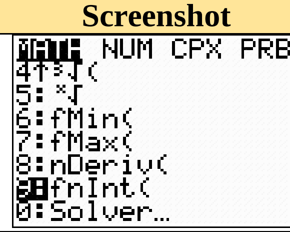
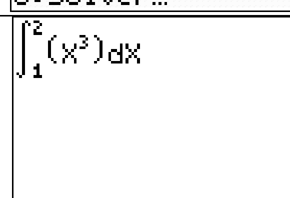
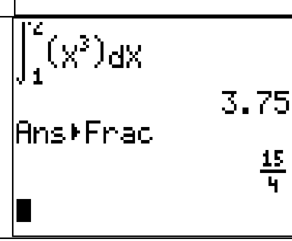
$$\int_{-2}^{\frac{1}{3}} \frac{3}{1-2x}\,dx$$
$$= -\frac{3}{2}\Big[\ln|1-2x|\Big]_{-2}^{\frac{1}{3}}$$
$$= -\frac{3}{2}\left[\ln\left|1-2\left(\frac{1}{3}\right)\right| - \ln|1-2(-2)|\right]$$
$$= -\frac{3}{2}\left[\ln\left|\frac{1}{3}\right| - \ln|5|\right]$$
$$= -\frac{3}{2}(\ln 1 - \ln 3 - \ln 5)$$
$$= -\frac{3}{2}(-\ln 3 - \ln 5)$$
$$= \frac{3}{2}(\ln 3 + \ln 5)$$
$$= \frac{3}{2}\ln 15$$

8.2.2 Evaluating Definite Integral Using GC

Example 3

Evaluate $\int_{1}^{2} x^3\,dx$.

Solution:

Steps	Screenshot
Press [MATH] You should see this screen. Press [9]	
You should see this screen. Complete the function as shown on the right. Then press [ENTER]	
Note: Press [MATH] followed by [1] and [ENTER] to change the final answer into a fraction.	

Using GC,

$\int_1^2 x^3 \mathrm{d}x = \frac{15}{4}$

8.3 Basic Properties of Definite Integral

Let f and g be two functions. Then,

1. $\int_a^a \mathrm{f}(x)\ \mathrm{d}x = 0$
2. $\int_a^b \mathrm{f}(x)\ \mathrm{d}x = -\int_b^a \mathrm{f}(x)\ \mathrm{d}x$
3. $\int_a^b k\ \mathrm{f}(x)\ \mathrm{d}x = k\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ where k is a constant.
4. $\int_a^b [\mathrm{f}(x) \pm \mathrm{g}(x)]\ \mathrm{d}x = \int_a^b \mathrm{f}(x)\ \mathrm{d}x \pm \int_a^b \mathrm{g}(x)\ \mathrm{d}x$
5. $\int_a^b \mathrm{f}(x)\ \mathrm{d}x + \int_b^c \mathrm{f}(x)\ \mathrm{d}x = \int_a^c \mathrm{f}(x)\ \mathrm{d}x$ where $a < b < c$

Example 4

Given that $\int_1^4 \mathrm{f}(x)\mathrm{d}x = 10$ and $\int_3^4 \mathrm{f}(x)\mathrm{d}x = 3$, find

(a) $\int_4^1 \mathrm{f}(x)\mathrm{d}x$ (b) $\int_1^4 6\mathrm{f}(x)\mathrm{d}x$ (c) $\int_1^3 \mathrm{f}(x)\mathrm{d}x$ (d) $\int_1^4 (\mathrm{f}(x) - 2)\mathrm{d}x$

Solution:

(a) $\int_4^1 \mathrm{f}(x)\mathrm{d}x = -\int_1^4 \mathrm{f}(x)\mathrm{d}x$
$= -10$

(b) $\int_1^4 6\mathrm{f}(x)\mathrm{d}x = 6\int_1^4 \mathrm{f}(x)\mathrm{d}x$
$= 6(10)$
$= 60$

(c) $\int_1^3 \mathrm{f}(x)\mathrm{d}x = \int_1^4 \mathrm{f}(x)\mathrm{d}x - \int_3^4 \mathrm{f}(x)\mathrm{d}x$
$= 10 - 3$
$= 7$

(d) $\int_1^4 (\mathrm{f}(x) - 2)\mathrm{d}x = \int_1^4 \mathrm{f}(x)\mathrm{d}x - \int_1^4 2\ \mathrm{d}x$
$= 10 - 2[x]_1^4$
$= 10 - 2(4 - 1)$
$= 4$

Exercise 1

1. Evaluate the following **without the use of a GC**:

(i) $\int_2^3 (2x-3)(x+2)\,\mathrm{d}x$ (ii) $\int_0^{\frac{1}{2}} (2x+1)^4\,\mathrm{d}x$

(iii) $\int_0^1 20e^{-4x}\,\mathrm{d}x$ (iv) $\int_1^3 \frac{2x^2+x^3}{4x^3}\,\mathrm{d}x$

Solution:

1(i) $\int_2^3 (2x-3)(x+2)\,\mathrm{d}x$ $=\int_2^3 \left(2x^2+x-6\right)\,\mathrm{d}x$ $=\left[\frac{2x^3}{3}+\frac{x^2}{2}-6x\right]_2^3$ $=\left(\frac{2(3)^3}{3}+\frac{3^2}{2}-18\right)-\left(\frac{2(2)^3}{3}+\frac{2^2}{2}-12\right)$ $=\frac{9}{2}-\left(-4\frac{2}{3}\right)=9\frac{1}{6}$	1(ii) $\int_0^{\frac{1}{2}} (2x+1)^4\,\mathrm{d}x=\left[\frac{(2x+1)^5}{5(2)}\right]_0^{\frac{1}{2}}$ $=\frac{2^5}{10}-\frac{1}{10}$ $=3\frac{1}{10}$
1(iii) $\int_0^1 20e^{-4x}\,\mathrm{d}x=\left[\frac{20e^{-4x}}{-4}\right]_0^1$ $=-5\left(e^{-4}-1\right)$ $=-5\left(\frac{1}{e^4}-1\right)$	1(iv) $\int_1^3 \frac{2x^2+x^3}{4x^3}\,\mathrm{d}x=\int_1^3\left(\frac{1}{2x}+\frac{1}{4}\right)\mathrm{d}x$ $=\left[\frac{1}{2}\ln\|x\|+\frac{x}{4}\right]_1^3$ $=\left(\frac{1}{2}\ln 3+\frac{3}{4}\right)-\left(\frac{1}{2}\ln 1+\frac{1}{4}\right)$ $=\frac{1}{2}\ln 3+\frac{1}{2}$ $=\ln\sqrt{3}+\frac{1}{2}$

2. Given that $\int_0^2 g(x)\,dx = 12$ and $\int_2^5 g(x)\,dx = 8$, evaluate

(i) $\int_0^5 g(x)\,dx$ (ii) $\int_0^5 (3g(x)-1)\,dx$

(iii) $\int_5^2 g(x)\,dx$ (iv) $\int_0^2 4x - g(x)\,dx$

Solution:

2(i) $\int_0^5 g(x)\,dx = \int_0^2 g(x)\,dx + \int_2^5 g(x)\,dx$ $= 12 + 8$ $= 20$	2(ii) $\int_0^5 (3g(x)-1)\,dx = 3\int_0^5 g(x)\,dx - \int_0^5 1\,dx$ $= 3(20) - [x]_0^5$ $= 60 - (5-0)$ $= 55$
2(iii) $\int_5^2 g(x)\,dx = -\int_2^5 g(x)\,dx$ $= -8$	2(iv) $\int_0^2 (4x - g(x))\,dx = 4\int_0^2 x\,dx - \int_0^2 g(x)\,dx$ $= 4\left[\frac{x^2}{2}\right]_0^2 - 12$ $= 2(4-0) - 12$ $= -4$

3. Given that $y = \ln\left(\frac{x-2}{x+2}\right)$, find $\frac{dy}{dx}$. Hence evaluate $\int_4^8 \frac{1}{x^2-4}\,dx$ in exact form.

Solution:

$y = \ln\frac{x-2}{x+2}$ $= \ln(x-2) - \ln(x+2)$ $\frac{dy}{dx} = \frac{1}{x-2} - \frac{1}{x+2}$ $= \frac{x+2-(x-2)}{x^2-4}$ $= \frac{4}{x^2-4}$	$\int \frac{4}{x^2-4}\,dx = \ln\left(\frac{x-2}{x+2}\right)$ $4\int_4^8 \frac{1}{x^2-4}\,dx = \left[\ln\left(\frac{x-2}{x+2}\right)\right]_4^8$ $\int_4^8 \frac{1}{x^2-4}\,dx = \frac{1}{4}\left[\ln\left(\frac{x-2}{x+2}\right)\right]_4^8$ $= \frac{1}{4}\left[\ln\left(\frac{6}{10}\right) - \ln\left(\frac{2}{6}\right)\right]$ $= \frac{1}{4}\ln\left(\frac{6}{10}\times\frac{6}{2}\right)$ $= \frac{1}{4}\ln\left(\frac{9}{5}\right)$

Answer:

1(i) $9\frac{1}{6}$ 1(ii) $3\frac{1}{10}$ 1(iii) $-5\left(\frac{1}{e^4}-1\right)$ 1(iv) $\ln\sqrt{3}+\frac{1}{2}$

2(i) 20 2(ii) 55 2(iii) -8 2(iv) -4

3. $\frac{4}{x^2-4}$; $\frac{1}{4}\ln\frac{9}{5}$

8.4 Area under curve

8.4.1 Area under the curve with respect to *x*-axis

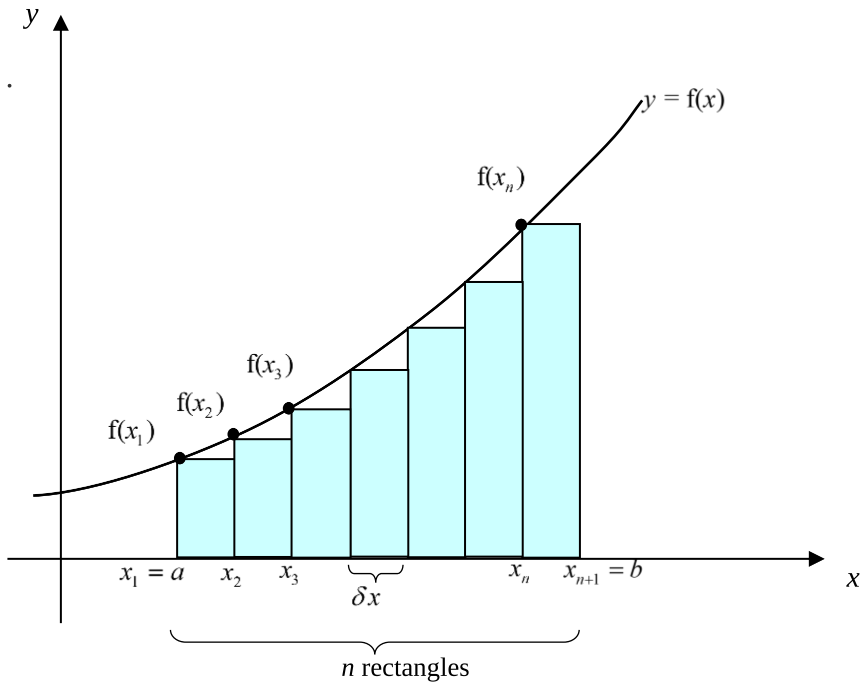


Fig (1)

To find, *S*, the area of region bounded by the curve $y = \mathrm{f}(x)$ which lies above the *x*-axis, the *x*-axis and the lines $x = a$ and $x = b$, we can subdivide the area into *n* rectangles. The total area of these rectangles can be used to approximate the value of *S*.

This is how we can do it :
We divide the interval [*a*, *b*] into *n* equal subintervals by the points $x_1, x_2, ..., x_n, x_{n+1}$ where $x_1 = a$ and $x_{n+1} = b$ and the width of each subinterval is $\delta x = \frac{b-a}{n}$. Then we draw *n* rectangles, each of width δx and height $\mathrm{f}(x_r)$. The height of each rectangle depends on *x*.

At x_1, the height of the rectangle is $\mathrm{f}(x_1)$.

At x_2, the height of the rectangle is $\mathrm{f}(x_2)$.

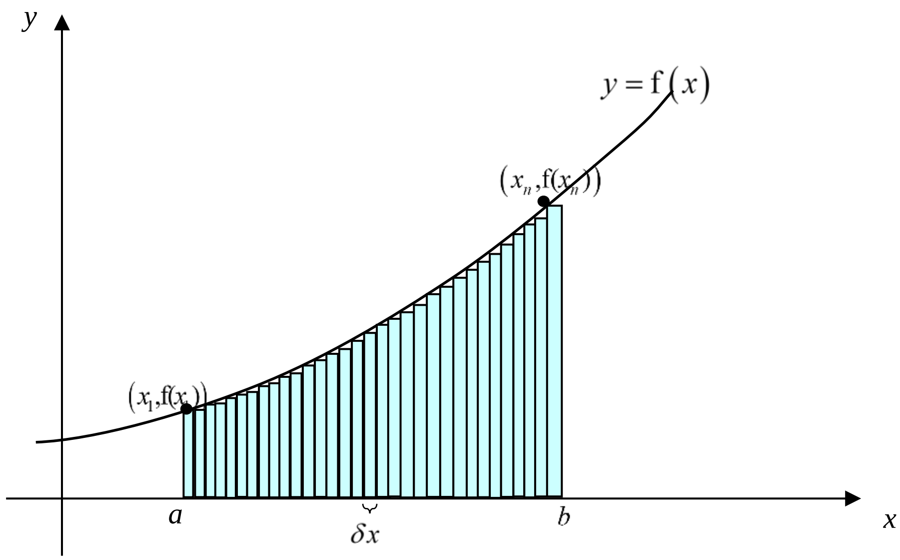
At x_3, the height of the rectangle is $\mathrm{f}(x_3)$.

…
At x_n, the height of the rectangle is $\mathrm{f}(x_n)$.
Therefore, the area of the first rectangle is $\mathrm{f}(x_1)\delta x$ square units.

By summing up the areas of the *n* rectangles, we can get an approximate value of *S*.

$$S \approx \delta x \times \mathrm{f}(x_1) + \delta x \times \mathrm{f}(x_2) + \ ...\ + \delta x \times \mathrm{f}(x_n)$$
$$= \delta x\left[\mathrm{f}(x_1) + \mathrm{f}(x_2) + ... + \mathrm{f}(x_n)\right]$$

From **Fig (1)**, the approximated area of *n* rectangles is an underestimate of *S*. If the number of rectangles increases, i.e. $n \to \infty$, the width of each rectangle gets thinner, i.e. $\delta x \to 0$, then the approximation improves and the sum of the area of the *n* rectangles approaches *S*.

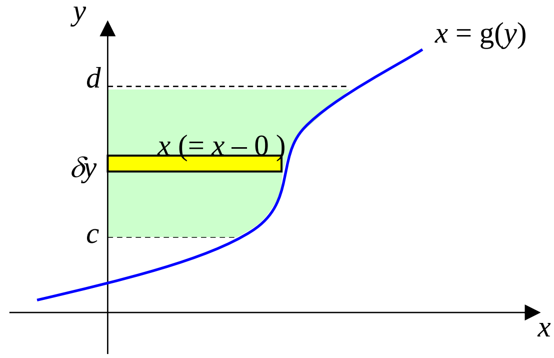


Hence, we can conclude that the limit of the sum of *n* rectangles is defined as the definite integral of $\mathrm{f}(x)$ from $x = a$ to $x = b$, i.e. $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$

We call '*b*' the upper limit and '*a*' the lower limit of the integral.
An integral with limits is termed as a **definite integral.**

In general, $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ denotes the area bounded by the curve $y = \mathrm{f}(x)$, the *x*-axis and the lines $x = a$ and $x = b$.

8.4.2 Area between the curve and *y*-axis



Area of a rectangular strip = $x\,\delta y$

Taking the limiting sum of infinite number of rectangular strips,

we have $\int_c^d x \,\mathrm{d}y$.

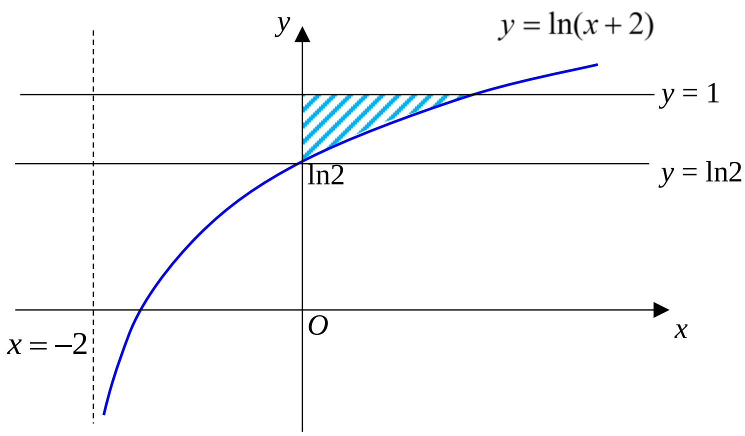
Area bounded by the curve $x = \mathrm{g}(y)$, the y-axis and the lines $y = c$ and $y = d$,

$A = \int_c^d x\,\mathrm{d}y = \int_c^d \mathrm{g}(y)\,\mathrm{d}y$.

Example 5

Find the exact area of the region bounded by the curve $y = \ln(x+2)$, the y-axis and the line $y = 1$.

Solution:



$y = \ln(x+2)$

$x + 2 = \mathrm{e}^y$

$x = \mathrm{e}^y - 2$

When $x = 0, y = \ln 2$

$\therefore (0, \ln 2)$ is the y-intercept.

Required area

$= \int_{\ln 2}^{1} x \, dy$

$= \int_{\ln 2}^{1} \left(e^{y} - 2\right) dy$

$= \left[e^{y} - 2y\right]_{\ln 2}^{1}$

$= (e - 2) - \left(e^{\ln 2} - 2\ln 2\right)$

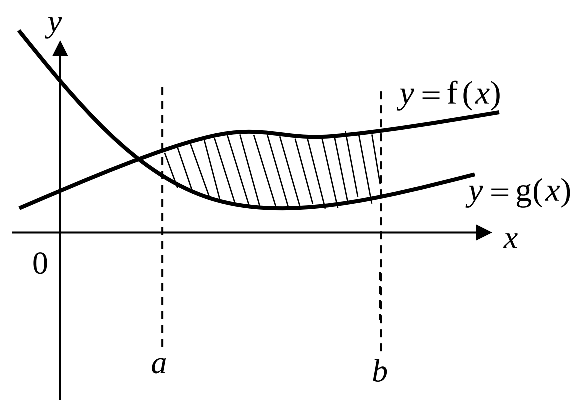
$= e - 2 - 2 + 2\ln 2$

$= (e - 4 + 2\ln 2)$ units^2

8.4.3 Area between 2 curves

Area of $y = f(x)$ over the interval [a,b]

$= \int_{a}^{b} f(x) \, dx$



Area of $y = g(x)$ over the interval [a,b]

$= \int_{a}^{b} g(x) \, dx$

Then, area between $f(x)$ and $g(x)$ over the interval [a,b]

$= \int_{a}^{b} \left(f(x) - g(x)\right) dx$

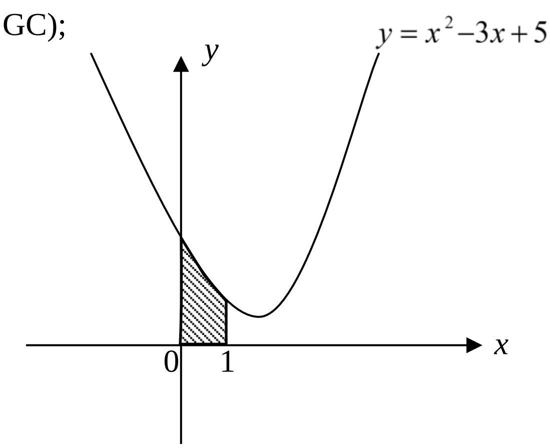
- Rule: Area of top curve – area of bottom curve

Example 6

Find the area bounded by the curve $y = x^2 - 3x + 5$

(a) and x-axis from $x = 0$ to $x = 1$ (without GC);

(b) and $y = -x + 5$ (using GC).



Solution:

a) Required area

$= \int_0^1 \left(x^2 - 3x + 5\right)\, dx$

$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 5x\right]_0^1$

$= \left(\frac{1}{3} - \frac{3}{2} + 5\right) - (0)$

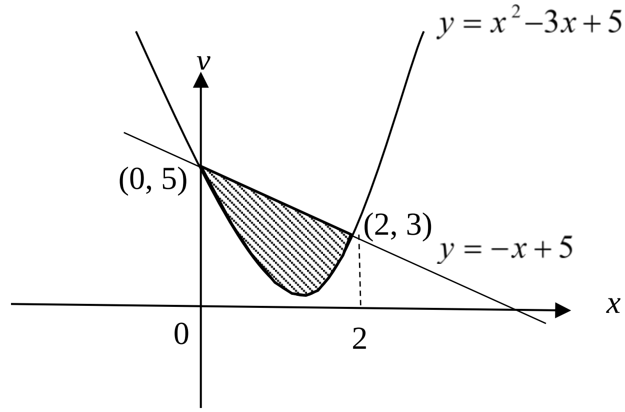
$= \frac{23}{6} = 3\frac{5}{6}$ units^2

(b) Find x-intercepts:
From GC, $x = 0$ or $x = 2$

Required area

$= \int_0^2 \left[\left(-x+5\right) - \left(x^2 - 3x + 5\right)\right] dx$

$= \frac{4}{3}$ units^2

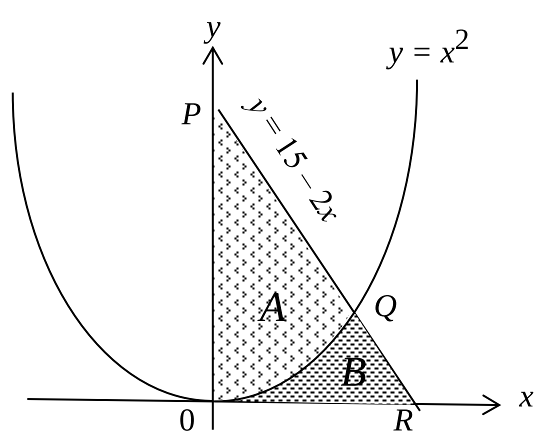


Example 7

The diagram shows part of the curve $y = x^2$ and part of the line $y = 15 - 2x$. Find, without using GC,

(i) the coordinates of P, Q and R,

(ii) the areas of the shaded region A and B.



Solution:

(i)

$y = 15 - 2x \cdots (1)$

When $x = 0$, $y = 15$

Coordinates of P is $(0, 15)$

When $y = 0$, $x = 7.5$

Coordinates of R is $(7.5, 0)$

$y = x^2 \cdots (2)$

(ii)

Area of Region B = $\int_0^3 x^2\, dx + \frac{1}{2}(4.5)(9)$

$= \left[\frac{x^3}{3}\right]_0^3 + 20.25$

$= \frac{3^3}{3} + 20.25$

$= 29.25$ units^2

(1) = (2), $x^2 = 15 - 2x$

$x^2 + 2x - 15 = 0$

$(x+5)(x-3) = 0$

$x = -5$ (rej $\because x > 0$) or $x = 3$

When $x = 3$, $y = 3^2$

$= 9$

Coordinates of Q is $(3,9)$

Area of ΔOPR$= \frac{1}{2}(15)(7.5)$

$= 56.25 \text{ units}^2$

Area of Region A= $56.25 - 29.25$

$= 27 \text{ units}^2$

Example 8 (NJC/2017 Prelim/Q5 parts)

A ship builder manufactures yachts. The rate, C thousand dollars per year, at which the total manufacturing costs change is to be monitored regularly over a period of 5 years. The Chief Financial Officer proposed that the relationship between C and the time, t years, can be modelled by the equation

$C = 25 - 12t + e^{0.8t}$, for $0 \le t \le 5$.

(i) Use differentiation and this model to find the exact minimum value of C. Justify that the value obtained is a minimum.

(ii) Sketch the graph of C against t, giving the coordinates of any intersections with the axes.

(iii) Find the exact area of the region bounded by C, the t-axis and the lines $t = 0$ and $t = 3$. Give an interpretation of the area that you found, in the context of the question.

Solution:

(i) $C = 25 - 12t + e^{0.8t} \quad \Rightarrow \quad \frac{dC}{dt} = -12 + 0.8e^{0.8t}$

For turning points, $\frac{dC}{dt} = 0 \quad \Rightarrow \quad 0.8e^{0.8t} = 12$

$\Rightarrow \quad e^{0.8t} = 15$

$\Rightarrow \quad 0.8t = \ln(15)$

$\Rightarrow \quad t = \frac{5}{4}\ln(15)$ years

The minimum $C = 25 - 15\ln15 + 15 = 40 - 15\ln15$

Using First derivative test,

t	3.3	$\frac{5}{4}\ln 15$	3.4
$\frac{dC}{dt}$	−0.789	0	0.144
Sketch of tangent	\	—	/

Therefore, C is minimum when $t = \frac{5}{4}\ln 15$

OR

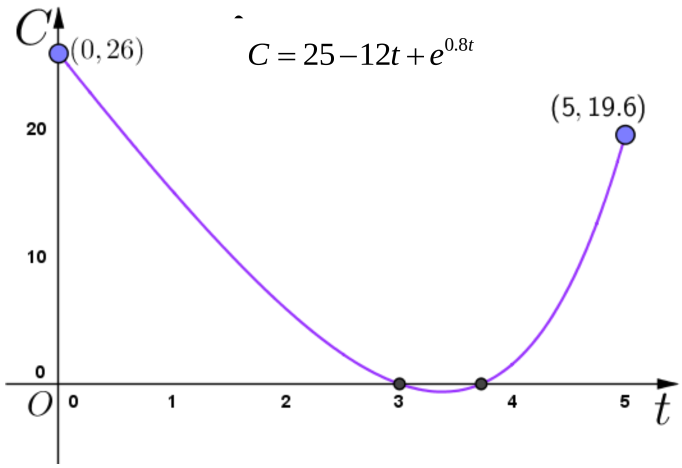
Using Second derivative test,

$$\frac{d^2C}{dt^2} = (0.8)^2 e^{0.8t}$$

When $t = \frac{5}{4}\ln(15)$, $\frac{d^2C}{dt^2} = (0.8)^2 e^{0.8\left(\frac{5}{4}\ln 15\right)} > 0$.

Therefore, C is minimum when $t = \frac{5}{4}\ln 15$

(ii)



(3.01, 0) (3.73, 0)

(iii) $\int_0^3 \left(25 - 12t + \mathrm{e}^{0.8t}\right)\,\mathrm{d}t$

$= \left[25t - 6t^2 + \frac{5}{4}\mathrm{e}^{0.8t}\right]_0^3$

$= \left[25(3) - 6(3)^2 + \frac{5}{4}\mathrm{e}^{2.4}\right] - \left[\frac{5}{4}\right]$

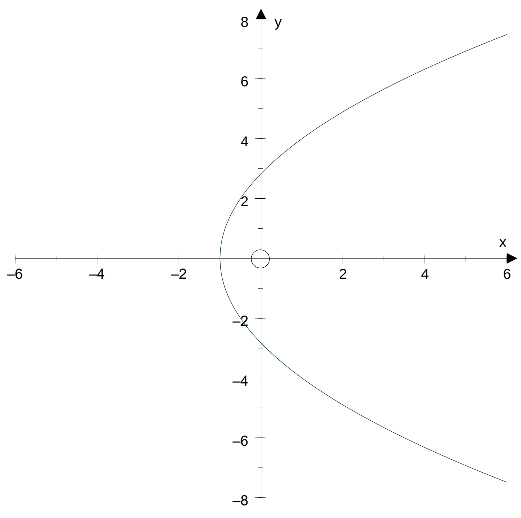
$= 19\frac{3}{4} + \frac{5}{4}e^{2.4}$ units^2

The value of the area found represents the total manufacturing cost (in thousand dollars) during the first three years of monitoring.

Exercise 2

1. Sketch the curve $y^2 = 8x + 8$. Calculate the exact area enclosed by the curve and the line $x = 1$.

Solution:



Required Area

$= 2\int_{-1}^{1} \sqrt{8x+8}\ \mathrm{d}x$

$= 2\int_{-1}^{1} (8x+8)^{\frac{1}{2}}\ \mathrm{d}x$

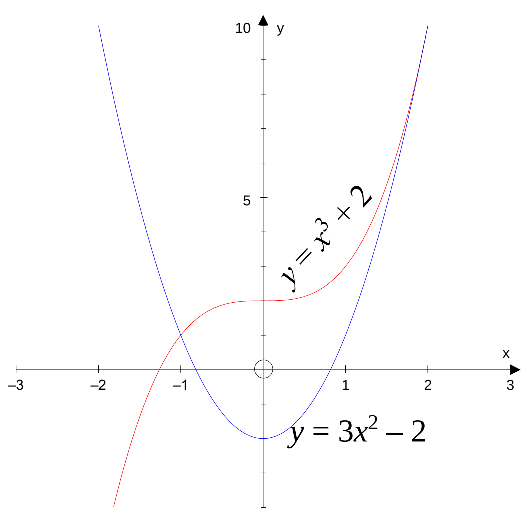
$= 2\left[\dfrac{(8x+8)^{\frac{3}{2}}}{\frac{3}{2}(8)}\right]_{-1}^{1}$

$= \dfrac{1}{6}\left(16^{\frac{3}{2}} - 0^{\frac{3}{2}}\right)$

$= 10\dfrac{2}{3}$ units^2

2. Find the exact area enclosed between the curves $y = x^3 + 2$ and $y = 3x^2 - 2$.

Solution:



To find intersection points, we equate the two curves

$y = x^3 + 2$ and $y = 3x^2 - 2$

i.e. $x^3 + 2 = 3x^2 - 2$

$x^3 - 3x^2 + 4 = 0$

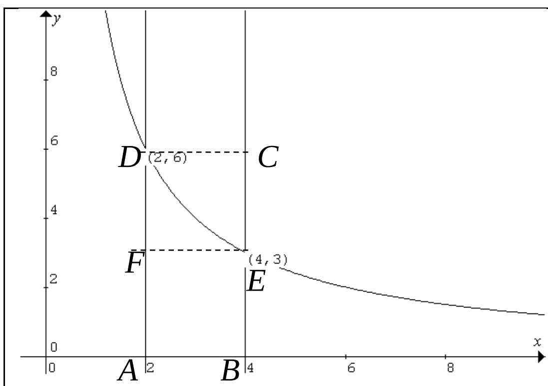
$(x+1)(x^2 - 4x + 4) = 0$

$x = -1$ or $x = 2$

	Required Area $= \int_{-1}^{2} \left(x^3+2\right)-\left(3x^2-2\right) dx$ $= \int_{-1}^{2} x^3-3x^2+4dx$ $= \left[\frac{1}{4}x^4 - x^3 + 4x\right]_{-1}^{2}$ $= \left[\frac{1}{4}(2)^4-(2)^3+4(2)\right]-\left[\frac{1}{4}(-1)^4-(-1)^3+4(-1)\right]$ $= 6.75 \text{ units}^2$

3. Sketch the graph of $y=\frac{12}{x}$ for the domain $2 \le x \le 4$. Hence explain briefly why $6 < \int_2^4 \frac{12}{x}\, dx < 12$.

Solution:

	From the graph, we can see that for 2 < x < 4, $3 < \frac{12}{x} < 6$ Hence, $\int_2^4 3 \text{ d}x < \int_2^4 \frac{12}{x} \text{ d}x < \int_2^4 6 \text{ d}x$ $[3x]_2^4 < \int_2^4 \frac{12}{x} \text{ d}x < [6x]_2^4$ $3(4)-3(2) < \int_2^4 \frac{12}{x} \text{ d}x < 6(4)-6(2)$ $\therefore 6 < \int_2^4 \frac{12}{x} \text{ d}x < 12$

<u>Alternative method</u>

Area of rectangle *ABEF* < Area under graph < Area of rectangle *ABCD*

$$2 \times 3 < \int_2^4 \frac{12}{x} \text{ d}x < 2 \times 6$$

$$\therefore \ 6 < \int_2^4 \frac{12}{x} \text{ d}x < 12$$

4. **RI/2018/Prelim/Q5**

A company manufactures a flat beverage coaster which is made in the shape of the region enclosed by the curves C_1 and C_2 with equations $y = 4x^2$ and $y = -2x^2 + 4$ respectively. The curves intersect at the points P and Q, where the x-coordinate of P is smaller than the x-coordinate of Q.

(i) Find the exact coordinates of P and Q. [3]

(ii) Find the exact area of the beverage coaster. [4]

(iii) As part of its design, a triangle with vertices P, Q and the origin O, is printed on the surface of the beverage coaster. Find the perimeter of the triangle. [2]

It is known that the manufacturing cost $\$C$, for producing n beverage coasters is given by the equation

$$C = 0.02n + 3\sqrt{n}$$

(iv) Find the least selling price for an order of 500 if the company wants to make a profit of at least 20% for the order. [3]

[Profit = selling price – manufacturing cost]

Solution:

(i) $4x^2 = -2x^2 + 4 \Rightarrow x^2 = \frac{2}{3} \Rightarrow x = \pm\frac{\sqrt{2}}{\sqrt{3}} = \pm\frac{\sqrt{6}}{3}$

$y = 4x^2 = \frac{8}{3}$

Hence the coordinates are $P\left(-\frac{\sqrt{6}}{3}, \frac{8}{3}\right)$ and $Q\left(\frac{\sqrt{6}}{3}, \frac{8}{3}\right)$

(ii) Area of the coaster $= \int_{-\frac{\sqrt{6}}{3}}^{\frac{\sqrt{6}}{3}} (-2x^2 + 4) - (4x^2)\, dx$

$$= \int_{-\frac{\sqrt{6}}{3}}^{\frac{\sqrt{6}}{3}} (4 - 6x^2)\, dx = \left[4x - 2x^3\right]_{-\frac{\sqrt{6}}{3}}^{\frac{\sqrt{6}}{3}}$$

$$= \left(4\left(\frac{\sqrt{6}}{3}\right) - 2\left(\frac{\sqrt{6}}{3}\right)^3\right) - \left(4\left(-\frac{\sqrt{6}}{3}\right) - 2\left(-\frac{\sqrt{6}}{3}\right)^3\right)$$

$$= 8\left(\frac{\sqrt{6}}{3}\right) - 4\left(\frac{\sqrt{6}}{3}\right)^3$$

$$= 8\left(\frac{\sqrt{6}}{3}\right) - 4\left(\frac{6}{9}\right)\left(\frac{\sqrt{6}}{3}\right)$$

$= \frac{16}{9}\sqrt{6}$ units^2

(iii) $OP = OQ = \sqrt{\frac{2}{3}+\frac{64}{9}} = \frac{\sqrt{70}}{3}$, $PQ = 2 \times \frac{\sqrt{6}}{3}$

$\therefore$ Perimeter = $OP + OQ + PQ$ = 7.21 units (to 3 s.f.)

(iv) Cost, $C = 0.02\,(500) + 3\sqrt{500} = 10 + 30\sqrt{5}$

For profit to be at least 20%, selling price must be at least $1.2 \times C$
$= 1.2 \times (10 + 30\sqrt{5})$ = \$92.50 (to 2 d.p.)

Answer:

1) $10\frac{2}{3}$ units^2	2) $6\frac{3}{4}$ units^2	
4) (i) $P\left(-\frac{\sqrt{6}}{3}, \frac{8}{3}\right)$, $Q\left(\frac{\sqrt{6}}{3}, \frac{8}{3}\right)$	4)(ii) $\frac{16}{9}\sqrt{6}$ units	4)(iii) 7.21 units 4(iv) \$92.50
Answers (ii) $t = 5.84$ the year is 2015 (v) $\frac{41}{2} - \frac{1}{2}e^{-8}$, The area found is the total cost incurred over the 4 years, which is \$20500.		

Summary

The definite integral of a function $\mathrm{f}(x)$ with lower limits $x = a$ and upper limit $x = b$ is defined by $\boxed{\int_a^b \mathrm{f}(x)\,\mathrm{d}x = [\mathrm{F}(x)]_a^b = \mathrm{F}(b) - \mathrm{F}(a)}$ where $\frac{d}{dx}\mathrm{F}(x) = \mathrm{f}(x)$.

1. Let f and g be two functions. Then,

1. $\int_a^a \mathrm{f}(x)\,\mathrm{d}x = 0$
2. $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$
3. $\int_a^b k\,\mathrm{f}(x)\,\mathrm{d}x = k\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ where k is a constant.
4. $\int_a^b [\mathrm{f}(x) \pm \mathrm{g}(x)]\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x \pm \int_a^b \mathrm{g}(x)\,\mathrm{d}x$
5. $\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x$ where $a < b < c$

2. $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ denotes the area bounded by the curve $y = \mathrm{f}(x)$, the x-axis and the lines $x = a$ and $x = b$.