SAINT ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2 **9740/01**
Paper 1

Thursday **30 August 2007** **3 hours**

Additional materials : Answer paper
List of Formulae(MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages including this page.

1 Sketch, on a single diagram, the graphs of $y = 2x + 9$ and $y = 4(|x| + 2)$, showing clearly the point(s) of intersection by giving the *x*-coordinates in **exact** values.
Solve the inequality $\dfrac{2x+9}{|x|+2} \le 4$. [3]

2 The functions f, and g are defined by

$$\mathrm{f}: x \mapsto 2 - x^2, \qquad x \in \mathbb{R},\ x \le a$$
$$\mathrm{g}: x \mapsto \ln(3 - x), \qquad x \in \mathbb{R},\ x < 3$$

(i) Find the largest value of a given that the inverse f^{-1} exists, and define f^{-1}. [3]

(ii) Show that gf exists. [1]

3 (i) Express $\mathrm{f}(x) = \dfrac{2}{(1+x)(1+x^2)}$ in partial fractions. [2]

(ii) Given that x is sufficiently small for x^4 and higher powers of x to be neglected, show that $\mathrm{f}(x) \approx 2 - 2x$. [3]

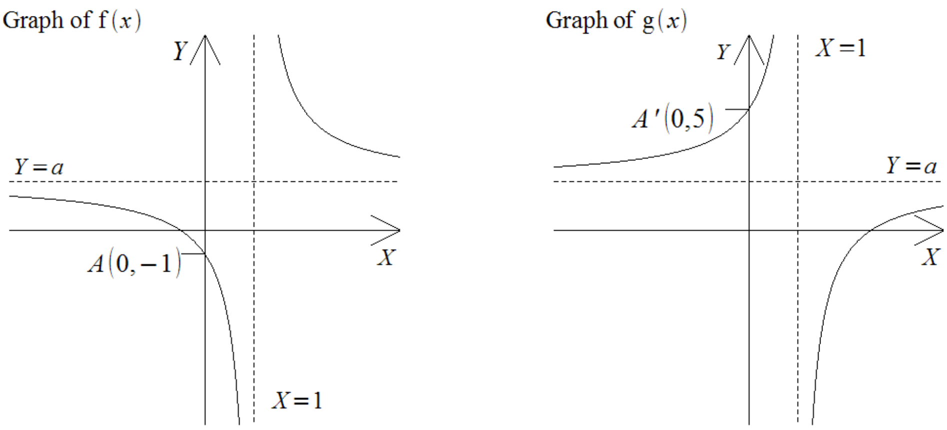
4 (i) Prove by induction that for every integer $n \ge 2$, $\displaystyle\sum_{r=2}^{n} \ln\left(1 - \frac{1}{r^2}\right) = \ln\left(\frac{n+1}{2n}\right)$. [4]

(ii) Hence find $\displaystyle\sum_{r=n+1}^{2n} \ln\left(1 - \frac{1}{r^2}\right)$ giving your answer in the form of $\ln \mathrm{f}(n)$. [2]

5 Mr Tan invested \$25,000, part in a structured deposit account, part in bonds, and part in a mutual fund. He invested \$6,600 more in the bonds than the mutual fund. After one year, he received a total of \$1,580 in simple interest from the three investments. The structured deposit account paid an interest of 6% annually, the bonds paid 7% annually, and the mutual fund paid 8% annually. Find the amount Mr Tan invested in each category, giving your answer to the nearest dollar.
[5]

[Turn over

6 The diagram below shows the curve $f(x)=\dfrac{ax+b}{x+c}$ and $g(x)$, the reflection of $f(x)$ about the line $y=a$.



(i) Find the values of a, b and c. [4]

(ii) Find the equation of $g(x)$. [2]

7 The curve C has equation $y = \dfrac{ax^2 + 2}{x - 1}$ where $x \neq 1$ and a is a non-zero constant.

(i) Show that if C has no stationary points, then $-2 < a < 0$. [3]

(ii) It is given that the line $y = -x - 1$ is an asymptote of C. Find the value of a. [2]

(iii) Sketch C, showing clearly the asymptotes and coordinates of any intersections with the coordinate axes. [3]

8 Find A and B such that $\dfrac{2k+1}{k^2(k+1)^2} = \dfrac{A}{k^2} + \dfrac{B}{(k+1)^2}$. [2]

Show that the sum of the following series

$$\frac{3}{1^2(2)^2} + \frac{5}{2^2(3)^2} + \frac{7}{3^2(4)^2} + \dots + \frac{2n+1}{n^2(n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$ [3]

Deduce that sum to infinity of this series $\dfrac{3}{2^4} + \dfrac{5}{3^4} + \dfrac{7}{4^4} + \dots$ is less than 1. [3]

[Turn over

9 (a) On a string of 31 pearls, the middle pearl is the largest and most expensive of all. Starting from one end, each pearl is worth $100 more than the one before, up to the middle pearl. From the other end, each pearl is $150 more than the one before, up to the middle pearl. The string of pearls is worth $63,000. What is the value of the middle pearl? [4]

(b) An ant of negligible size crawls a distance of 1 metre to the right from the origin (0, 0) along the *x*-axis. It then turns back, and crawls $\frac{1}{2}$ metre to the left from its current point to the point $\left(\frac{1}{2},0\right)$. If the ant continues going back and forth, each time going half the distance it previously went, and repeating the pattern, where does the ant eventually end up? Give your answer as the coordinates of the final position of the ant. [3]

10 Consider the sequence defined by $u_{n+1}=\dfrac{2}{u_n-2}+3$, for all positive integers of *n*, and $u_0=3$.

(i) Find the values of u_1, u_2 and u_3. [2]

(ii) Suppose u_n converges to *l* as $n\to\infty$. By forming and solving a suitable quadratic equation, find the value of *l*. [4]

(iii) Under what condition will u_n converge to the other limit you rejected in (ii)? You should use calculations obtained from your G.C. to substantiate your answer. [1]

11 (a) Given that $u=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $w=4\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$, find the modulus and argument of $\dfrac{u^*}{w^3}$. [4]

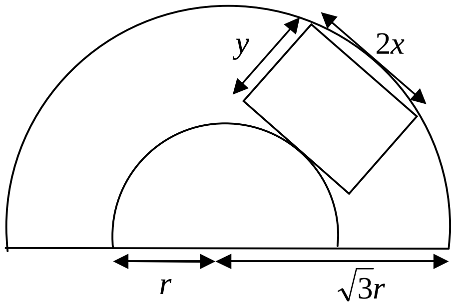
(b) Let *z* be the complex number $-1+i\sqrt{3}$. Find the value of the real number *a* such that $\arg\left(z^2+az\right)=-\dfrac{\pi}{2}$. [5]

[Turn over

12 (i) Show that the series expansion of $\ln(\cos 2x)$ in ascending powers of x up to and including the term in x^4 is $-2x^2 - \frac{4}{3}x^4$. Hence show that $\int_0^{\frac{\pi}{8}} \ln(\cos 2x)\mathrm{d}x \approx -0.043$. [6]

(ii) The region R is bounded by the axes, the curve $y = 1 + x\tan 2x$ and the line $x = \frac{\pi}{8}$. Using integration by parts and the result in (i), find the **approximate** area of R. [4]

13



It is required to take a rectangular frame in a horizontal position along a corridor bounded by vertical walls of which a horizontal cross-section is two concentric semicircles of radii r and $\sqrt{3}$ r; the frame is of length $2x$ and breadth y (see above figure). One side of length $2x$ is tangential to the inner wall, and the two ends of the opposite side are in contact with the outer wall, as shown in the diagram above.

Prove that $x^2 = 2r^2 - 2ry - y^2$. [2]

Prove that, if x and y may vary, the greatest possible area enclosed by the frame is $\frac{r^2\sqrt{3}}{2}$. [8]

[Please turn over for Question 14]

[Turn over

14 (a) By using the substitution $ax = 3\sin\theta$ or otherwise, find the value of a if

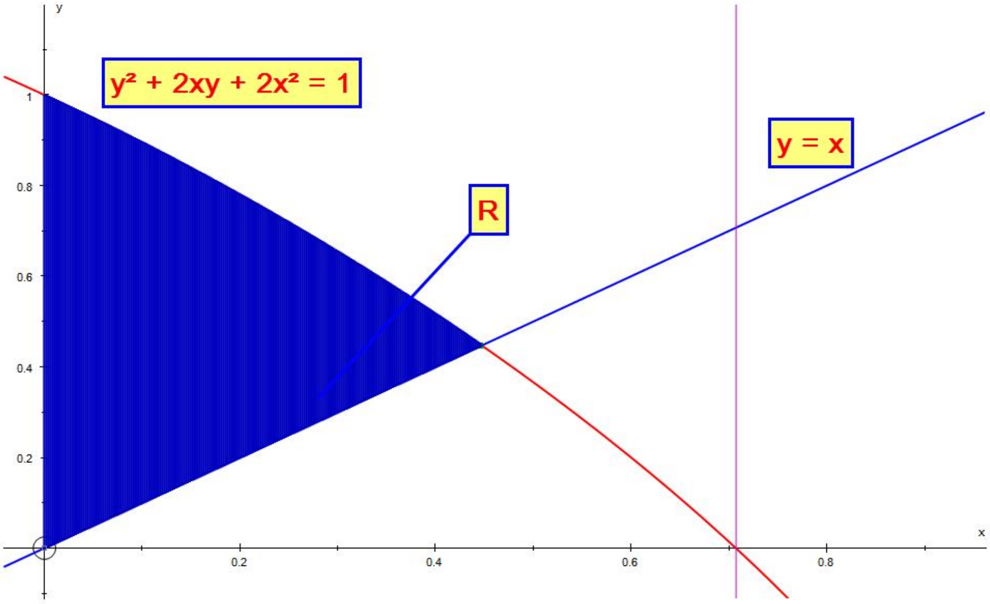
$$\int_0^{\frac{3}{a}} \sqrt{9-(ax)^2}\,\mathrm{d}x = \frac{9\pi}{8}.$$ [5]

(b) (i) Given that x and y are positive numbers such that $y^2 + 2xy + 2x^2 = 1$, show that

$$y = \sqrt{1-x^2} - x.$$

Show also that the gradient of the curve with equation $y^2 + 2xy + 2x^2 = 1$ is negative for all positive values of x and y. [4]

(ii) The diagram shows the curve with the equation $y^2 + 2xy + 2x^2 = 1$ for $x \geq 0$, $y \geq 0$. R is the shaded region bounded by the curve $y^2 + 2xy + 2x^2 = 1$, the line $y = x$ and the y-axis.



Find the volume of the solid formed when R is rotated through 2π radians about the x-axis. Give your answer correct to 3 decimal places. [3]

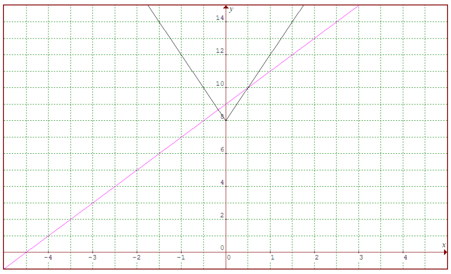
End of Paper 1

[Turn over

2007 H2 Maths Prelim Exam(Paper 1) Solutions

1 x-coordinates of point of intersections = $\frac{1}{2}$, $-\frac{1}{6}$

Graphs of $y = 2x + 9$ and $y = 4\left(|x| + 2\right)$:



For $\frac{2x+9}{|x|+2} \leq 4$,

$\therefore$ the solution set is $\{x : x \leq -\frac{1}{6} \text{ or } x \geq \frac{1}{2}\}$.

2(i) $a = 0$

$\mathrm{f} : x \mapsto 2 - x^2, \qquad x \in \mathbb{R},\ x \leq 0$

$y = 2 - x^2$

$x = -\sqrt{2-y}$ since $x \leq 0$

$\mathrm{f}^{-1}(x) = -\sqrt{2-x},\ x \leq 2$

(ii) Since $R_{\mathrm{f}} = (-\infty, 2] \subseteq (-\infty, 3) = D_{\mathrm{g}}$, gf exists.

3(i) $\frac{2}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{1+x^2}$

$2 = A(1+x^2) + (Bx+C)(1+x)$

Subts. x = -1, A = 1

Comparing coefficients of

$x^2 : B = -1$

$x^0 : C = 1$

$A = 1, B = -1, C = 1$

(ii) $\mathrm{f}(x) = \frac{2}{(1+x)(1+x^2)}$

$= \frac{1}{1+x} + \frac{1-x}{1+x^2}$

$= (1+x)^{-1} + (1-x)(1+x^2)^{-1}$

$= (1 - x + x^2 - x^3 + \cdots) + (1-x)(1 - x^2 + \cdots)$

$$= \begin{array}{cccccc} 1 & -x & +x^2 & -x^3 & +\cdots+ \\ 1 & -x & -x^2 & +x^3 & +\cdots \end{array}$$

$= 2 - 2x + \cdots$ (Shown)

4 (i) Let P_n be the statement $\sum_{r=2}^{n} \ln\left(1-\frac{1}{r^2}\right) = \ln\left(\frac{n+1}{2n}\right)$ where $n \geq 2$.

When $n = 2$,

$LHS = \ln\left(1-\frac{1}{2^2}\right) = \ln\frac{3}{4}$,

$RHS = \ln\left(\frac{2+1}{2(2)}\right) = \ln\left(\frac{3}{4}\right)$.

$\therefore P_2$ is true.

Assume P_k is true for some positive integer $k \geq 2$, i.e.

$$\sum_{r=2}^{k} \ln\left(1-\frac{1}{r^2}\right) = \ln\left(\frac{k+1}{2k}\right)$$

Want to prove that P_{k+1} is true based on P_k is true,

i.e. $\sum_{r=2}^{k+1} \ln\left(1-\frac{1}{r^2}\right) = \ln\left(\frac{k+2}{2(k+1)}\right)$

$$LHS = \sum_{r=2}^{k+1} \ln\left(1-\frac{1}{r^2}\right)$$

$$= \ln\left(\frac{k+1}{2k}\right) + \ln\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \ln\left(\frac{k+1}{2k}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$$= \ln\left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k+1-1}{(k+1)^2}\right)$$

$$= \ln\left(\frac{k+1}{2k}\right)\left(\frac{k^2+2k}{(k+1)^2}\right)$$

$$= \ln\left(\frac{k+2}{2(k+1)}\right) = RHS$$

So P_{k+1} is true.

Since P_1 is true and P_{k+1} is true based on P_k is true, P_n is true for all $n \geq 2$,
by Mathematical Induction.

(ii)

$$\sum_{r=n+1}^{2n} \ln\left(1-\frac{1}{r^2}\right)$$

$$= \sum_{r=2}^{2n} \ln\left(1-\frac{1}{r^2}\right) - \sum_{r=2}^{n} \ln\left(1-\frac{1}{r^2}\right)$$

$$= \ln\left(\frac{2n+1}{2(2n)}\right) - \ln\left(\frac{n+1}{2n}\right)$$

$$= \ln\left(\frac{2n+1}{4n}\right)\left(\frac{2n}{n+1}\right)$$

$$= \ln\left(\frac{2n+1}{2(n+1)}\right)$$

$$= \ln\left(\frac{2n+1}{2n+2}\right)$$

5 Let x be the amount of money invested in the structured deposit account.

y be the amount of money invested in bonds.

z be the amount of money invested in the mutual fund.

$$x + y + z = 25000$$

$$0.06x + 0.07y + 0.08z = 1580$$

$$y - z = 6600$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 0.06 & 0.07 & 0.08 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 25000 \\ 1580 \\ 6600 \end{pmatrix}$$

$x = \$17{,}467$, $y = \$7{,}067$, $z = \$467$

6(i) Let $y = \mathrm{f}(x)$.

$x = -c$ is the vertical asymptote. So $c = -1$

By long division, $y = a + \dfrac{b+a}{x-1}$

$y = a$ is the horizontal asymptote.

Since A' is the reflection of A about the line $y = a$. The point $(0, a)$ is the midpoint of A' and A.

$$a = \frac{5+(-1)}{2} = 2$$

$$y = \frac{2x+b}{x-1}$$

When $x = 0, y = -1$.

$$-1 = \frac{b}{-1}$$

$$b = 1$$

(ii) To get g(x) from f(x)

1) Shift y down 2 units along y-axis
2) Reflect about x-axis
3) Shift up 2 units along y-axis

$$g(x) = -\left(f(x) - 2\right) + 2 = \frac{2x-5}{x-1}$$

<u>Alternative solution 1:</u>

(ii)

$$y = f(x) = \frac{2x+1}{x-1}$$

$$g(x) = -f(x) + 4$$

$$= \frac{2x-5}{x-1}$$

<u>Alternative solution 2:</u>

(ii) Since y=g(x) is a rectangular hyperbola, $y = 2$ and $x = 1$ are its asymptotes

Let $g(x) = \dfrac{2x+d}{x-1}$

When $x = 0, y = 5$

$$5 = \frac{d}{-1}$$

$$d = -5$$

7(i)

$$y = \frac{ax^2+2}{x-1}$$

$$\frac{dy}{dx} = \frac{ax^2-2ax-2}{(x-1)^2}$$

Since C has no stationary points, there are no real roots for $ax^2 - 2ax - 2 = 0$

$$(-2a)^2 - 4a(-2) < 0$$

$$4a(a+2) < 0$$

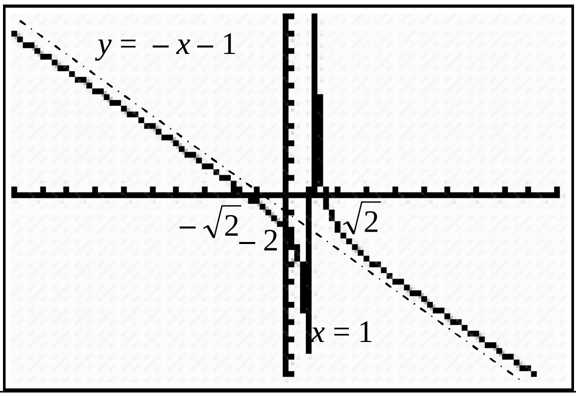
$$-2 < a < 0$$

(ii)

$$y = \frac{ax^2+2}{x-1} = ax + a + \frac{2+a}{x-1}$$

Asymptote is $y = ax + a$.
$a = -1$

(iii) Sketch of $y = \dfrac{-x^2+2}{x-1}$:



8

$$\frac{2k+1}{k^2(k+1)^2} = \frac{A}{k^2} + \frac{B}{(k+1)^2}$$

$$\frac{2k+1}{k^2(k+1)^2} = \frac{A(k+1)^2 + Bk^2}{k^2(k+1)^2}$$

$$\frac{2k+1}{k^2(k+1)^2} = \frac{A(k^2+2k+1) + Bk^2}{k^2(k+1)^2}$$

$A + B = 0$

$2 = 2A$

$A = 1, B = -1$

$$\frac{2k+1}{k^2(k+1)^2} = \frac{1}{k^2} - \frac{1}{(k+1)^2}$$

$$\frac{3}{1^2(2)^2} + \frac{5}{2^2(3)^2} + \frac{7}{3^2(4)^2} + \dots + \frac{2n+1}{n^2(n+1)^2}$$

$$= \sum_{r=1}^{n} \left(\frac{1}{r^2} - \frac{1}{(r+1)^2} \right)$$

$$= \frac{1}{1} - \cancel{\frac{1}{2^2}}$$

$$+ \cancel{\frac{1}{2^2}} - \cancel{\frac{1}{3^2}}$$

$$+ \cancel{\frac{1}{3^2}} - \cancel{\frac{1}{4^2}}$$

$$+ \dots$$

$$+ \cancel{\frac{1}{n^2}} - \frac{1}{(n+1)^2}$$

$$= 1 - \frac{1}{(n+1)^2} \quad \text{(Shown)}$$

8 Note: $(n+1)^2(n+1)^2 > n^2(n+1)^2$ for all positive integers n. Thus,

$$\frac{3}{2^2(2)^2}+\frac{5}{3^2(3)^2}+\frac{7}{4^2(4)^2}+...+\frac{2n+1}{(n+1)^2(n+1)^2}$$

$$<\frac{3}{1^2(2)^2}+\frac{5}{2^2(3)^2}+\frac{7}{3^2(4)^2}+...+\frac{2n+1}{n^2(n+1)^2}$$

Since $\frac{3}{1^2(2)^2}+\frac{5}{2^2(3)^2}+\frac{7}{3^2(4)^2}+...+\frac{2n+1}{n^2(n+1)^2}=1-\frac{1}{(n+1)^2}<1$ as $n\to\infty$.

$$\frac{3}{2^4}+\frac{5}{3^4}+\frac{7}{4^4}+...<\frac{3}{1^2(2)^2}+\frac{5}{2^2(3)^2}+\frac{7}{3^2(4)^2}+...<1 \text{ as } n\to\infty.$$

9(a) Let the cost of the middle pearl be x.

Sum of 16 pearls (which differ by \$100) =

$$\frac{16}{2}\left(2x+15(-100)\right)=16x-12000$$

Sum of 16 pearls (which differ by \$150) =

$$\frac{16}{2}\left(2x+15(-150)\right)=16x-18000$$

Cost of necklace = $(16x-12000)+(16x-18000)-x=31x-30000=63000$

Solving, $x=\$3000$.

(b) Let $a=1$, $r=-0.5$

$$S_\infty=\frac{a}{1-r}=\frac{2}{3}.$$

Coordinates of the final position of the ant = $\left(\frac{2}{3},0\right)$.

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10 (i) $u_1=\frac{2}{3-2}+3=5$;

$u_2=\frac{2}{5-2}+3=3\frac{2}{3}$; $u_3=\frac{2}{3\frac{2}{3}-5}+3=4\frac{1}{5}$

(ii) Suppose u_n converges as $n\to\infty$, we can assume that $u_n\approx u_{n+1}$.

Write $u_n=\frac{2}{u_n-2}+3$.

Expand and solve the equation $u_n^2-5u_n+4=0$.

$(u_n-4)(u_n-1)=0$; $u_n=4$ or 1

By G.C. or observation from part (i), limit is 4. (reject $u_n=1$). $l=4$

(iii) When $u_n=1^+$ or $u_n=1^-$, the sequence diverges (from G.C.). u_n will only converge to 1 when $u_0=1$.

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11 (i)

Given $u=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ and $w=4\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$,

$|u|=2, \arg u=\frac{\pi}{6}$ and $|w|=4, \arg w=-\frac{\pi}{3}$

$\left|\frac{u^*}{w^3}\right|=\frac{|u^*|}{|w^3|}=\frac{|u|}{|w|^3}=\frac{2}{4^3}=\frac{1}{32}$.

$\arg\left(\frac{z^*}{w^3}\right)=\arg z^*-\arg w^3$

$=-\arg z-3\arg w$

$=-\left(\frac{\pi}{6}\right)-3\left(-\frac{\pi}{3}\right)$

$=\frac{5\pi}{6}$

(ii)

$\arg(z)=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

$\arg\left(z^2+az\right)=-\frac{\pi}{2}$

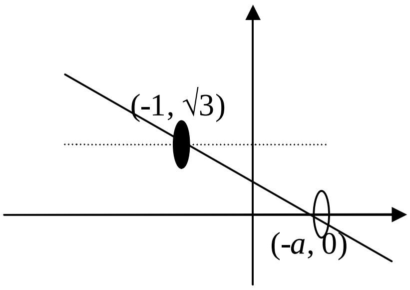
$\arg\left(z(z+a)\right)=-\frac{\pi}{2}$

$\arg(z)+\arg(z+a)=-\frac{\pi}{2}$

$\arg(z+a)=-\frac{\pi}{2}-\frac{2\pi}{3}$

$\arg(z+a)=-\frac{7\pi}{6}$

$\arg(z-(-a))=\frac{5\pi}{6}$



Equation of this line:

$y-0=-\tan^{-1}\left(\frac{\pi}{6}\right)\left(x-(-a)\right)$

$y=-\frac{1}{\sqrt{3}}(x+a)$

Since point $\left(-1,\sqrt{3}\right)$ intersect this line,

$\sqrt{3}=-\frac{1}{\sqrt{3}}(-1+a)$

$-3=-1+a$

$a=-2$

Alternative solution :

$z^2+az=(-1+\sqrt{3}i)^2+a(-1+\sqrt{3}i)$

$=-(2+a)+\sqrt{3}(a-2)i$

$\arg(z^2+az)=-\dfrac{\pi}{2}$

$\tan^{-1}\left[\dfrac{\sqrt{3}(a-2)}{-(2+a)}\right]=-\dfrac{\pi}{2}$

$-(2+a)=0$

$a=-2$

12

$$\ln(\cos 2x)\approx \ln\left[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}\right]$$

$$=\ln\left[1+\left(-2x^2+\frac{2}{3}x^4\right)\right]$$

$$\approx\left(-2x^2+\frac{2}{3}x^4\right)-\frac{\left(-2x^2+\frac{2}{3}x^4\right)^2}{2}$$

$$=-2x^2+\frac{2}{3}x^4-2x^4$$

Alternative solution :

Let $y=\ln(\cos 2x)$

$\dfrac{dy}{dx}=\dfrac{-2\sin 2x}{\cos 2x}=-2\tan 2x$

$\dfrac{d^2y}{dx^2}=-4\sec^2 2x$

$\dfrac{d^3y}{dx^3}=-16\sec 2x\sec 2x\tan 2x=-16\sec^2 2x\tan 2x$

$\dfrac{d^4y}{dx^4}=-16[\sec^2 2x(2\sec^2 2x)+4\sec 2x\sec 2x\tan 2x\tan 2x]$

$=-32\sec^4 2x-64\sec^2 2x\tan^2 2x$

When $x=0, y=0,$

$\dfrac{dy}{dx}=0, \dfrac{d^2y}{dx^2}=-4, \dfrac{d^3y}{dx^3}=0, \dfrac{d^4y}{dx^4}=-32$

$\therefore$ The series expansion is $y\approx 0+\dfrac{-4}{2!}x^2+\dfrac{-32}{4!}x^4=-2x^2-\dfrac{4}{3}x^4$. (Shown)

$$\int_0^{\frac{\pi}{8}} \ln(\cos 2x) \approx \left[-\frac{2}{3}x^3 - \frac{4}{15}x^5 \right]_0^{\frac{\pi}{8}}$$

$$\approx -0.043 \quad \text{(Shown)}$$

Area $R = \int_0^{\frac{\pi}{8}} (1 + x\tan 2x)dx$

$$= [x]_0^{\frac{\pi}{8}} - \left[\frac{1}{2}x\ln(\cos 2x) \right]_0^{\frac{\pi}{8}} + \frac{1}{2}\int_0^{\frac{\pi}{8}} \ln(\cos 2x)dx$$

$$\approx \frac{\pi}{8} - \frac{1}{2}\frac{\pi}{8}\ln\frac{\sqrt{2}}{2} + \frac{1}{2}(-0.043)$$

$$= 0.439$$

By Parts :

$u = x \quad \frac{dv}{dx} = \tan 2x$

$\frac{du}{dx} = 1 \quad v = -\frac{1}{2}\ln(\cos 2x)$

13

$$(r + y)^2 + x^2 = 3r^2$$

$$r^2 + 2ry + y^2 + x^2 = 3r^2$$

$$x^2 = 2r^2 - 2ry - y^2$$

Area of the frame, $A = 2xy$

$$= 2\sqrt{2r^2 - 2ry - y^2}\,y$$

$$= 2\sqrt{2r^2y^2 - 2ry^3 - y^4}$$

$$\frac{dA}{dy} = 2\frac{1}{2\sqrt{2r^2y^2 - 2ry^3 - y^4}}\left(4yr^2 - 6ry^2 - 4y^3\right)$$

$$= \frac{2y\left(2r^2 - 3ry - 2y^2\right)}{\sqrt{2r^2y^2 - 2ry^3 - y^4}}$$

When $\frac{dA}{dy} = 0$, $y\left(2r^2 - 3ry - 2y^2\right) = 0$

$2r^2 - 3ry - 2y^2 = 0$ or $y = 0$ (na)

$(r - 2y)(2r + y) = 0$

$r - 2y = 0$ or $y = -2r\,(na)$

$y = \frac{r}{2}$

When $y = \frac{r}{2}$, $x^2 = 2r^2 - 2r\left(\frac{r}{2}\right) - \left(\frac{r}{2}\right)^2$

$$= \frac{3r^2}{4}$$

$$x = \frac{\sqrt{3}}{2}r$$

teachmejcmath.sg@gmail.com

	$\left(\frac{r}{2}\right)^-$	$\left(\frac{r}{2}\right)$	$\left(\frac{r}{2}\right)^+$
$\frac{dA}{dy}$	>0	0	<0

The greatest $A = 2\left(\frac{r}{2}\right)\left(\frac{\sqrt{3}r}{2}\right)$

$= \frac{\sqrt{3}r^2}{2}$

14 (a)

$$\int_0^{\frac{3}{a}} \sqrt{9-(ax)^2}\,dx$$

Let $ax = 3\sin\theta$.
$\frac{dx}{d\theta} = \frac{3}{a}\cos\theta$

$$= \frac{3}{a}\int_0^{\frac{\pi}{2}} 3\cos^2\theta\, d\theta$$

$$= \frac{9}{a}\int_0^{\frac{\pi}{2}} \frac{\cos 2\theta + 1}{2} d\theta$$

$$= \frac{9}{2a}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{9\pi}{4a}$$

So $a = 2$

(b)

$y^2 + 2xy + 2x^2 = 1$

$(y+x)^2 + x^2 = 1$

$(y+x)^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2} - x$

We reject $y = -\sqrt{1-x^2} - x$ since $y \ge 0$

Hence $y = \sqrt{1-x^2} - x$.

$\frac{dy}{dx} = \frac{-x}{\sqrt{1-x^2}} - 1 < 0$ for all positive values of x on the curve

When $x = y$, $5x^2 = 1$, so $x = \frac{1}{\sqrt{5}}$.

$$V = \pi\int_0^{\frac{1}{\sqrt{5}}} \left[\left(\sqrt{1-x^2} - x\right)^2 - x^2\right] dx \text{ or } \pi\int_0^{\frac{1}{\sqrt{5}}} \left(\sqrt{1-x^2} - x\right)^2 dx - \frac{1}{3}\pi\left(\frac{1}{\sqrt{5}}\right)^2 \frac{1}{\sqrt{5}} = 0.716$$

(by GC)

SAINT ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2 **9740/02**
Paper 2

Wednesday **12 Sep 2007** **3 hours**

Additional materials : Answer paper
List of Formulae(MF15)
Cover Sheet

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of **6** printed pages including this page.
[Turn over

Section A (40 marks)

1 Find the three complex numbers, z_0, z_1, z_2 that satisfy the equation $z^3 - 4\sqrt{2} + 4\sqrt{2}i = 0$. Give your answers in the form $re^{i\theta}$, where θ is in terms of π. [4]

Hence show that $w = z_0^6 + z_1^6 + z_2^6$ is an imaginary number. Find $\text{Im}(w)$. [3]

2 When Mrs Wong retired in 2006, she put a sum of $5000 into a fund that has a constant rate of return of 5 % per annum. Starting in 2006, she withdraws $400 each year and gives the money to her granddaughter as a birthday gift. Denote the amount of money Mrs Wong has at time t years by $$x$.

(i) The differential equation relating x and t is in the form $\frac{\mathrm{d}x}{\mathrm{d}t} = kx + c$. State the values of k and c. . [1]

(ii) Solve the differential equation and find the amount of money Mrs Wong has after 15 years. Give your answer to the nearest integer. [4]

(iii) In which year will the granddaughter receive her last $400? [2]

Comment on whether the model can be regarded as a good model of the situation in the real world. [1]

3 The parametric equations of a curve are given by $x = a\sec\theta$, $y = a\tan\theta$ where a is a real constant.

(i) Show that $\frac{dy}{dx} = \operatorname{cosec}\theta$. [2]

(ii) Find the rate of change of y at $\theta = \frac{\pi}{6}$ when x is increasing at a constant rate of 2 units per second. [2]

(iii) Find the equation of the tangent to the curve at the point $R(a\sec\alpha, a\tan\alpha)$ and show that the equation of the normal to the curve is $y = 2a\tan\alpha - x\sin\alpha$. [2]

(iv) The tangent and normal at R meet the x-axis at T and N respectively.
By finding the x-coordinates of T and N in the simplest form, show that $|ON||OT| = 2a^2$. [5]

[Turn over

4 The points A, B and C have position vectors, relative to the origin, $\mathbf{i} - 2\mathbf{k}$, $3\,\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $-5\,\mathbf{j} - 7\,\mathbf{k}$ respectively.

(i) Find, in the form $\mathbf{r} \cdot \mathbf{n} = p$, an equation of the plane Π_1 which contains A, B and C. [3]

The line l_1 passes through the points A and B. The point R on l_1 is such that CR is perpendicular to l_1. The line l_2 passes through C and R and the point Q on l_2 is such that $3\overrightarrow{CR} = 2\overrightarrow{CQ}$.

(ii) Find the position vector of R. [4]

(iii) Deduce an equation, in the form $\mathbf{r} \cdot \mathbf{n} = p$, for the plane Π_2 which contains the line l_2 and is perpendicular to Π_1. [3]

(iv) Find the area of the triangle BCQ, leaving your answer in surd form. [4]

Section B (60 marks)

5 There are 10 members in the Project Work committee who sit around a table. There is 1 seat designated for the Chairman only. There are 4 members who constitute a Written Report subcommittee. Find the number of ways the 10 members can be seated around the table if

(a) the members of the Written Report subcommittee have to sit together; [2]
(b) no two members of the Written Report subcommittee sit next to each other. [3]

6 The heights of a new species of sunflower are normally distributed with a mean of 180 cm and a variance of 90 cm.

(i) Find the least value of n such that the probability that the average height of a random sample of n sunflowers exceeds 181.5 cm is less than 1%. [3]

(ii) A random sample of 20 sunflowers is taken and the average height is found. Find the probability that the sample mean is greater than 178 cm. [2]

A hundred such samples, each of 20 sunflowers, are taken. In how many of these would you expect the sample mean to be greater than 178 cm? [1]

[Turn over

7 Of the bicycles parked in the void deck of a HDB flat, 60% belongs to males and the rest to females. 90% of the bicycles belonging to the males are racers as are 70% of the bicycles belonging to females. Dynamos are fitted to 5% of the non-racing and 1% of the racing bicycles irrespective of whether the bicycle is owned by a male or a female.

If a bicycle is chosen at random find the probability that it is

(i) a racer with a dynamo belonging to a male, [2]
(ii) a racer without dynamo [3]

A bicycle is chosen at random. If it is not a racer, find the probability that it belongs to a female. [3]

8 Measurements of the relative humidity and moisture content of samples of a certain type of raw material on 10 days yielded the following results :

Relative Humidity(x%)	46	53	37	42	34	29	60	44	41	48
Moisture Content(y%)	12	14	11	13	10	8	17	12	10	15

(i) Draw a scatter diagram for the data. Calculate the product-moment correlation coefficient. Explain what they reveal about the relationship between relative humidity and moisture content. [3]

(ii) It is required to estimate the moisture content given that the relative humidity is 58%. Find the equation of a suitable least squares regression line. Use your equation to obtain the required estimate. [3]

(iii) Use your equation in (ii) to estimate the moisture content when relative humidity is 10%. Comment on the reliability of the estimation. [2]

(iv) It was found out that the moisture content of the raw material was measured wrongly due to a fault in the measuring instrument. The actual moisture content is 10% more than the measured values. Will this affect the product-moment correlation coefficient? Explain your answer. [1]

[Turn over

9 Calls to a chatline service are received at random times at an average rate of 9 per hour. The service is offered for 10 hours in one day.

(i) Show the probability that more than 9 calls are made in a randomly chosen one-hour period is 0.413. [2]

k one hour periods are chosen at random, where $k \geq 100$. The number of calls received in each period is recorded. Given the probability that more than 9 calls are made in at least 20 out of the k periods exceeds 0.9, use a Normal approximation to show that k satisfy the(approximate) inequality

$$k - 47.2 > 1.53\sqrt{k}$$ [4]

(ii) By using a suitable approximation, find also the probability that the total number of calls made in one day is at least 90 but less than 100. [3]

10 A certain brand of butter cookies are packed in two sizes : small and large. For each size, the mass, in grams, of a randomly chosen packet is normally distributed with mean and standard deviation as follows :

	Mean	**Standard deviation**
Small packet	720	9
Large packet	970	10

(a) Find the probability that, of 2 randomly chosen large packets, one contains more than 960 g of cookies and one contains less than 960 g of cookies. [2]

(b) The cookies are delivered to supermarkets in boxes of 20 packets of the same size. Find the probability that a box of 20 small packets contains more than 4 packets whose contents are each less than 710 g. [3]

(c) Find the probability that 3 randomly chosen large packets contain at least 10 g more than 4 randomly chosen small packets. [3]

(d) The label on a small packet of cookies reads “Net mass : 715 grams”. A trading standard officer insists that at least 90% of such packets should contain cookies with a mass of at least 715 g. Assuming that the standard deviation remains unchanged, find the least value of the new mean mass of a small packet of cookies that is consistent with this requirement. [3]

[Turn over

11(a) A department store reported that over the past six months, the average amount spent per customer has been \$215 with standard deviation of \$100. The store carried out a sales promotion for a week on a range of products. In order to test, at 5% level of significance, whether or not the sales promotion has increased the average amount spent per customer, a random sample of 100 customers visiting the store during the sales promotion week was taken. The amount spent per customer was recorded.

State, with a reason, whether, in performing the above test, it is necessary to assume that the amount spent per customer follows a normal distribution. [1]

Write down the null and alternative hypotheses under test. Show that the null hypothesis is rejected if the sample mean is greater than \$231.45. [3]

(b) Based on past results, the average score in a round obtained by a golfer was 88 points. In order to improve his skills in golf (i.e. **make his score smaller**), he attended a series of golf lessons. For each of the 8 rounds after the lessons, he calculated his score, x. The results are summarized by :

$$\sum(x-10)=615,\ \ \sum(x-10)^2=47343.$$

(i) Calculate unbiased estimates of the population mean and variance.

Carry out an appropriate test to establish whether or not the lessons had significantly produced, at the 10% level, a significant improvement in his skills in golf (i.e. **had made his scores smaller).**
State an assumption made in using the test that you have chosen. [6]

(ii) Suppose the data from the 8 rounds had been such that the unbiased estimate of population variance as found in (i) was larger, but without changing the sample mean. State, with a reason, whether this would have an effect on the conclusion of the test. [2]

End of Paper 2

[Turn over

[Turn over

2007 H2 Maths Prelim Exam(Paper 2) Solutions
Section A

1
$$4\sqrt{2}-i4\sqrt{2}=8e^{-i\frac{\pi}{4}}$$

$$z^3=4\sqrt{2}-i4\sqrt{2}$$
$$z^3=8e^{i\left(2k\pi-\frac{\pi}{4}\right)},\ k=0,\ 1,\ 2(\text{or }-1).$$

$$z=2e^{\frac{i}{3}\left(2k\pi-\frac{\pi}{4}\right)}$$
$$z=2e^{i\left(\frac{(8k-1)\pi}{12}\right)},\ k=0,\ 1,\ 2(\text{or }-1)$$
$$z=2e^{-i\frac{\pi}{12}},\ 2e^{i\frac{7\pi}{12}}\text{ and }2e^{i\frac{5\pi}{4}}\ (\text{or }2e^{-i\frac{3\pi}{4}})$$

$$w=z_0^6+z_1^6+z_2^6$$
$$=(2e^{-i\frac{\pi}{12}})^6+(2e^{i\frac{7\pi}{12}})^6+(2e^{i\frac{5\pi}{4}})^6$$
$$=2^6\left[e^{-i\frac{\pi}{2}}+e^{i\frac{7\pi}{2}}+e^{i\frac{15\pi}{2}}\right]$$
$$=2^6\left[e^{-i\frac{\pi}{2}}+e^{i\frac{\pi}{2}}+e^{i\frac{\pi}{2}}\right]$$
$$=2^6\left[(-i)+(-i)+(-i)\right]$$
$$=-192i\ \text{(Shown)}$$

Hence Im(w) = -192

2(i) $\dfrac{dx}{dt}=0.05x-400$, $k=0.05$, $c=-400$

(ii) $$\int\frac{1}{0.05x-400}\,dx=\int dt$$
$$\frac{1}{0.05}\ln\left|0.05x-400\right|=t+C$$
$$\ln\left|0.05x-400\right|=0.05t+C_1$$
$$0.05x-400=Ae^{0.05t}$$
$$0.05x=Ae^{0.05t}+400$$
$$x=Be^{0.05t}+8000$$
When $t=0$, $x=5000 \Rightarrow 5000=B+8000$
$\Rightarrow B=-3000$
Hence $x=-3000e^{0.05t}+8000$
When $t=15$,
$$x=-3000e^{0.75}+8000=-6351.0+8000$$
$$\approx 1649$$

(iii) When $x = 0$, $e^{0.05t} = \frac{8}{3} \Rightarrow 0.05t = 0.9808$

$\Rightarrow t = 19.62$

On the 19th year, the granddaughter will receive her last \$400.
It will be in year (2006+19-1) = 2024

Possible answers for last part:
Most rate of returns of fund fluctuates from year to year, so probably not a good model.
The returns are normally compounded over discrete time interval (usually years) rather than continuously. So probably not a good model.
When $x = 0$, the model will no longer valid. So may not be a good model.

3(i) $x = a\sec\theta$ $\qquad$ $y = a\tan\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$ $\qquad$ $\frac{dy}{d\theta} = a\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx}$$
$$= \frac{a\sec^2\theta}{a\sec\theta\tan\theta}$$
$$= \cos ec\theta$$

(ii) $$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$
$$= (\cos ec\theta)(2)$$

At $\theta = \frac{\pi}{6}, \frac{dy}{dt} = 2\cos ec\frac{\pi}{6} = 4$

(iii) Equation of tangent at point R :

$$y - a\tan\alpha = \cos ec\alpha\left(x - a\sec\alpha\right)$$
$$y = a(\tan\alpha - \cos ec\alpha\sec\alpha) + x\cos ec\alpha$$

Equation of normal at point R :

$$y - a\tan\alpha = -\frac{1}{\cos ec\alpha}\left(x - a\sec\alpha\right)$$
$$y = a\tan\alpha - (\sin\alpha)\left(x - \frac{a}{\cos\alpha}\right)$$
$$y = 2a\tan\alpha - x\sin\alpha$$

(iv) At point T, $y = 0$

$$0 = a(\tan\alpha - \cos ec\alpha\sec\alpha) + x\cos ec\alpha$$
$$x = \frac{a(\tan\alpha - \cos ec\alpha\sec\alpha)}{-\cos ec\alpha}$$
$$= a\sec\alpha(1 - \sin^2\alpha)$$
$$= a\cos\alpha$$

At point N, $y = 0$

$$0 = 2a\tan\alpha - x\sin\alpha$$

$$x = \frac{2a\tan\alpha}{\sin\alpha}$$

$$= 2a\sec\alpha$$

$$|ON||OT| = |a\cos\alpha||2a\sec\alpha|$$

$$= 2a^2$$

4(i) Given $\overrightarrow{OA} = \begin{pmatrix}1\\0\\-2\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}3\\1\\-1\end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix}0\\-5\\-7\end{pmatrix}$,

so $\overrightarrow{AB} = \begin{pmatrix}2\\1\\1\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}-1\\-5\\-5\end{pmatrix}$

A normal to Π_1 is $\mathbf{n}_1 = \begin{pmatrix}2\\1\\1\end{pmatrix} \times \begin{pmatrix}-1\\-5\\-5\end{pmatrix} = \begin{pmatrix}0\\9\\-9\end{pmatrix} = 9\begin{pmatrix}0\\1\\-1\end{pmatrix}$

Vector eqn of Π_1 is $\mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$$\mathbf{r}\cdot\begin{pmatrix}0\\1\\-1\end{pmatrix} = 2$$

(ii) An equation of line l_1 is $\mathbf{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix}$

so $\overrightarrow{OR} = \begin{pmatrix}1+2\lambda\\\lambda\\-2+\lambda\end{pmatrix}$, and $\overrightarrow{CR} = \begin{pmatrix}1+2\lambda\\\lambda+5\\5+\lambda\end{pmatrix}$

$$\overrightarrow{CR} \perp \overrightarrow{AB} \Rightarrow \begin{pmatrix}1+2\lambda\\\lambda+5\\5+\lambda\end{pmatrix} \bullet \begin{pmatrix}2\\1\\1\end{pmatrix} = 0$$

$$\Rightarrow 6\lambda + 12 = 0 \Rightarrow \lambda = -2$$

Hence $\overrightarrow{OR} = \begin{pmatrix}-3\\-2\\-4\end{pmatrix}$

(iii)

A normal to Π_2 is $\mathbf{n_2} = \overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Alternative mtd : A normal to Π_2

$= \underset{\sim}{n_1} \times \overrightarrow{QR}$ or $\overrightarrow{QC}$ or $\overrightarrow{RC}$

Vector eqn of Π_2 is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \\ -7 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = -12$$

(iv) $3\overrightarrow{CR} = 2\overrightarrow{CQ} \Rightarrow \overrightarrow{CQ} = \frac{3}{2}\overrightarrow{CR}$

$$\overrightarrow{CQ} = \frac{3}{2}\left[\begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ -7 \end{pmatrix}\right] = \frac{9}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BR} = \begin{pmatrix} -3 \\ -2 \\ -4 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -6 \\ -3 \\ -3 \end{pmatrix} = -3\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$CQ = \frac{9}{2}\sqrt{3}$ $\quad$ $, BR = 3\sqrt{6}$

Area of $\Delta BCQ = \frac{1}{2} CQ \times BR = \frac{1}{2}(\frac{9}{2}\sqrt{3})(3\sqrt{6}) = \frac{81}{4}\sqrt{2}$

Alternative method :

$$\overline{CB} = 3\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Area $= \frac{1}{2}\left|\overline{CB} \times \overline{CQ}\right|$

$$= \frac{1}{2}\left|3\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \frac{9}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right| = \frac{1}{2} \cdot \frac{27}{2}\left|\begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}\right| \quad = \frac{27}{4}\sqrt{18} = \frac{81}{4}\sqrt{2}$$ (ans. in surd form)

Section B

5(a) No. of ways the Chairman(only 1 way), the block of 4 subcommittee members and 5 others can be seated = 6!
No. of ways the 4 subcommittee members can be seated within the block = 4!
By Multiplication Principle, total no of ways = 6!4! = 17280

5(b) No. of ways the members(excluding the subcommittee) can be seated = 5!
No. of ways the 4 subcommittee members can be seated between the other 6 members = ${}^6P_4$
By Multiplication Principle, total no of ways = 5! ${}^6P_4$ = 43200

6(i) Let X be r.v. "height of sunflower in cm".

$\overline{X} \sim N(180, \frac{90}{n})$

$P(\overline{X}_n > 181.5) < 0.01 \Rightarrow 1 - P(\overline{X}_n \le 181.5) < 0.01 \Rightarrow P(\overline{X}_n \le 181.5) > 0.99$

$$P\left(Z \le \frac{181.5-180}{\sqrt{90/n}}\right) > 0.99$$

$$\frac{181.5-180}{\sqrt{90/n}} > 2.326$$

$n > 216.4$

Least n = 217

(ii) $\overline{X}_{20} \sim N(180, \frac{90}{20})$

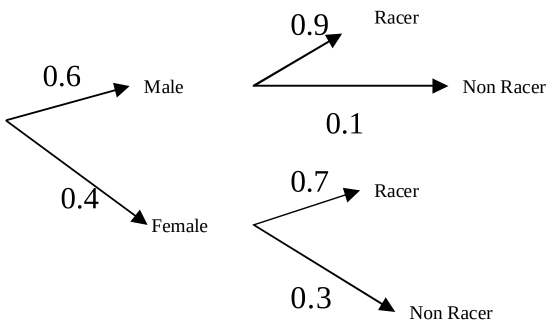
$P(\overline{X}_{20} > 178) = 0.8271$

Let Y be "no. of samples with mean > 178 out of 100 sample"

$Y \sim$ B(100, 0.8271)

E(Y) = $100 \times 0.8271 \approx 83$

7



(i) P(a racer with a dynamo belonging to a male) $= 0.9 \times 0.6 \times 0.01$
$= 0.0054$

(ii) P(a racer without a dynamo) $= (0.9 \times 0.6 + 0.7 \times 0.4)0.99$
$= 0.8118$

(iii) P(it belongs to a girl if it is not a racer)

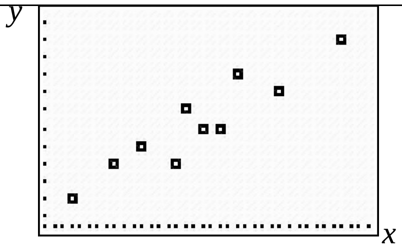
$= P\left(\text{belongs to female} \setminus \text{not a racer}\right)$

$= P\left(\dfrac{\text{belongs to female and is not a racer}}{\text{not a racer}}\right)$

$= \dfrac{0.3 \times 0.4}{0.1 \times 0.6 + 0.3 \times 0.4}$

$= 0.667$

8 (i)



$r = 0.932$

The scatter diagram and the value of r suggests a strong positive linear correlation between x and y.

(ii) Regression line of y on x : $y = 0.348 + 0.273x$

When $x = 58$, $y = 0.348 + 0.273(58) = 16.2$

So moisture content is 16.2%.

(iii) When $x = 10$, $y = 0.348 + 0.273(10) = 3.08$. So moisture content is 3.08%.

The estimation may be unreliable as $x = 10$ is outside the range of the given data.

(iv) No. r will be the same because the value of r is not affected by a linear transformation on y.

9 (i) Let X be the number of calls made in a one-hour period.

$X \sim Po(9)$

$P(X > 9) = 1 - P(X \le 9) = 0.413$ (to 3 sf) (Shown)

Let Y be the number of periods where more than 9 calls are made out of k periods.

$Y \sim B(k,\ 0.413)$

$Y \sim N(0.413k,\ \ 0.2424k)$ approximately

Given that $P(Y \ge 20) > 0.9$

$P(Y \ge 19.5) > 0.9$

$P(Y < 19.5) < 0.1$

$$P\left(Z < \frac{19.5 - 0.413k}{\sqrt{0.2424k}}\right) < 0.1$$

$$\frac{19.5 - 0.413k}{\sqrt{0.2424k}} < -1.282$$

$$19.5 - 0.413k < -1.282\sqrt{0.2424k}$$

$$k - 47.2 > 1.53\sqrt{k}$$

(ii) Let T be the total number of calls made in one day.

$T \sim Po(90)$

Since $\lambda > 10$, $T \sim N(90,90)$ approximately.

$P(90 \le T < 100) = P(89.5 \le T < 99.5)$ after continuity correction.

Probability required = normalcdf($(89.5, 99.5, 90, \sqrt{90})$

= 0.363 (to 3 sf)

10(a) Let S be r.v. "mass of a small packet in grams" and L be "mass of a large packet in grams".

S ~ N(720, 9^2) ; L ~ N(970, 10^2)

(a) Required probability = 2P(L>960)P(L<960)

=2(0.8413)(1 – 0.8413)

=0.267

(b) Let X be r.v. "no. of small packets of mass less than 710 g in a box of 20".

P(S < 710) = 0.1333

X ~ B(20, 0.1333)

P(X≥5) = 1 – P(X≤4) = 0.117

(c) $(L_1 + L_2 + L_3) - (S_1 + S_2 + S_3 + S_4) \sim N(30, 624)$

$P(L_1 + L_2 + L_3 - (S_1 + S_2 + S_3 + S_4) \ge 10) = 0.788$

(d)

P(S≥715) ≥ 0.9

$$P\left(Z < \frac{715 - \mu}{9}\right) \le 0.1$$

$$\frac{715 - \mu}{9} \le -1.282$$

$\mu \ge 726.5$

Least value of the mean mass of a small packet = 727 g

11(a) It is not necessary to assume that the amount spent per customer follows a normal distribution because the sampling distribution of means is approximately normal by Central Limit Theorem (since sample size is large).

Let X be the amt. spent per customer.
Test $H_0 : \mu = 215$ vs $H_1 : \mu > 215$

Under $H_0, \bar{X} \sim N(215, \frac{100^2}{100}) = N(215,100)$

At 5% level, the critical $\bar{x}$-value is invNorm(0.95, 215, 10) = 231.45
So H_0 is rejected when $\bar{x} > 231.45$.

Alternative method :
At 5% level, the critical z-value is invNormal(0.95) = 1.645.
If $\bar{x} > 231.45$,

$$\frac{\bar{X} - 215}{10} > \frac{231.45 - 215}{10}$$

$z > 1.645$
So H_0 is rejected at 5% level.

11(b) (i) Unbiased estimate of population mean = $\bar{x} = \frac{615}{8} + 10 = 86.875 \approx 86.9$

Unbiased estimate of population variance

$$= \frac{1}{7}\left(47343 - \frac{615^2}{8}\right)$$

$= 9.268$
≈ 9.27

Test $H_0 : \mu = 88$ vs $H_1 : \mu < 88$

GC : T-test, Stats, $\mu_0 = 88, \bar{x} = 86.875, s_x = \sqrt{9.268} = 3.044$,
$n = 8$, $\mu < \mu_0$. Calculate.

Since p = 0.165 > 0.1 (or t = - 1.045>invT(0.1,7)= -1.41)

Do not reject H_0. There is insufficient evidence at 10% level of significance that the lessons had made an improvement in his golf.

Assumption : The score in a round follows a Normal distribution.

(ii) The conclusion would be the same(i.e. H_0 will not be rejected).

This is because the new test statistic value, $t' = \frac{86.875 - 88}{s/\sqrt{8}} = \frac{-1.125}{s/\sqrt{8}}$ > critical value (or t' is bigger than t(the test statistic value found in part i)) when s^2 >9.27, hence it will still not lie in the critical region.

OR(comparing p-values): $P(T < t') > P(T < t) = 0.165 > 0.1$