# 4. Differentiation and its Applications (solutions)

## (I) Tangents and Normals (Normal/Implicit Differentiation)

**1**

**(a)**

Let $y=\dfrac{x}{\sin^{-1}(3x)}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\sin^{-1}(3x)-x\left(\dfrac{3}{\sqrt{1-9x^2}}\right)}{\left(\sin^{-1}(3x)\right)^2}=\frac{\sqrt{1-9x^2}\sin^{-1}(3x)-3x}{\left(\sin^{-1}(3x)\right)^2\sqrt{1-9x^2}}$$

**(b)**

$$y^{\cos 2x}=x^3$$

Taking ln on both sides, $(\cos 2x)(\ln y)=\ln x^3=3\ln x$

Differentiate w.r.t. *x*,

$$(-2\sin 2x)(\ln y)+(\cos 2x)\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{x} \quad \text{...........} \ (1)$$

At $x=\pi$, $y^{\cos 2\pi}=\pi^3\Rightarrow y=\pi^3$

Subst into (1): $(-2\sin 2\pi)\left(\ln \pi^3\right)+(\cos 2\pi)\dfrac{1}{\pi^3}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{3}{\pi}$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=3\pi^2$$

> **Note**
> There is no need to simplify $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ expression.
> We need the corresponding *y*-coordinate to evaluate the gradient to the curve at $x=\pi$.

**2**

**(i)**

$y=xe^{-x} \quad\Rightarrow\quad \dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{-x}-x\mathrm{e}^{-x}=\mathrm{e}^{-x}(1-x)$

Graph is decreasing: $\dfrac{\mathrm{d}y}{\mathrm{d}x}<0$

$\mathrm{e}^{-x}(1-x)<0 \quad\Rightarrow\quad x>1$

[Also accept $x\geq 1$]

**(ii)**

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{e}^{-x}(-1)-\mathrm{e}^{-x}(1-x)=\mathrm{e}^{-x}(x-2)$$

Graph is concave downwards: $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$

$\mathrm{e}^{-x}(x-2)<0 \quad\Rightarrow\quad x<2$

> **Need** $x>1$ and $x<2$

Therefore, for graph to be decreasing **and** concave downwards: $1<x<2$.
[Also accept $1\leq x<2$, $1<x\leq 2$, or $1\leq x\leq 2$ ]

**(iii)**

Consider gradient at $(a,b)$ = $\mathrm{e}^{-a}(1-a)$:

$$\frac{h-b}{0-a}=\mathrm{e}^{-a}(1-a)$$

Since $b=ae^{-a}$

$$h-ae^{-a}=-ae^{-a}(1-a)$$

$$h=ae^{-a}-ae^{-a}(1-a)=a^2\mathrm{e}^{-a}$$

> **Alternative**
> Tangent at $(a,b)$: $y-b=\mathrm{e}^{-a}(a-2)(x-a)$
>
> Subst $R(0,h)$ into tangent and $b=ae^{-a}$ :
> $h=ae^{-a}-ae^{-a}(1-a)=a^2\mathrm{e}^{-a}$

$$\frac{\mathrm{d}h}{\mathrm{d}a} = 2a\mathrm{e}^{-a} - a^2\mathrm{e}^{-a} = a\mathrm{e}^{-a}(2-a)$$

At max/min point: $\frac{\mathrm{d}h}{\mathrm{d}a} = 0$

$a\mathrm{e}^{-a}(2-a) = 0$

$a = 2$ or $a = 0$

| $a$ | $2^-$ | $2$ | $2^+$ |
|---|---|---|---|
| $\frac{\mathrm{d}h}{\mathrm{d}a}$ | + | 0 | - |
| Slope | / | — | \ |

| $a$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\frac{\mathrm{d}h}{\mathrm{d}a}$ | - | 0 | + |
| Slope | \ | — | / |

**Alternative : Second derivative Test**

$$\frac{\mathrm{d}^2h}{\mathrm{d}a^2} = \mathrm{e}^{-a}\left(2-4a+a^2\right) < 0 \text{ when } a = 2$$

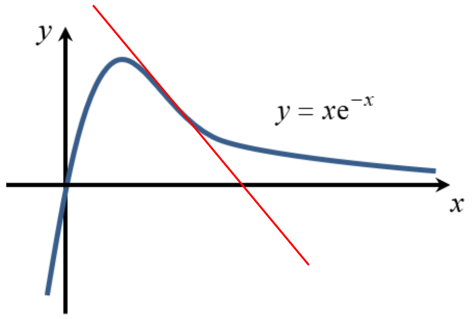
Greatest possible $h = 4\mathrm{e}^{-2}$ at $a = 2$.

**Alternative :**

**From the graph, greatest *h* occurs when $\frac{\mathrm{d}y}{\mathrm{d}x}$ is most negative (i.e. where $\frac{\mathrm{d}y}{\mathrm{d}x}$ is min, in this case).**

**Thus need** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0 \Rightarrow x = 2$

**Greatest possible** $h = 2^2\mathrm{e}^{-2} = \frac{4}{\mathrm{e}^2}$



**3. (i)**

$y = \mathrm{f}(x)$ strictly increasing $\Rightarrow$ $\mathrm{f}'(x) > 0$

$y = \mathrm{f}(x)$ concave downwards $\Rightarrow$ $\mathrm{f}''(x) < 0$ [so the gradient function $\mathrm{f}'(x)$ decreases]

We observe from $y = \mathrm{f}'(x)$ graph that $x > 1$ (ans)

**(ii)**

For stationary points on graph of $y = \mathrm{f}(x)$,
Need $\mathrm{f}'(x) = 0 \Rightarrow x = -3, 0$

| $x$ | $(-3)^-$ | $-3$ | $(-3)^+$ |
|---|---|---|---|
| $\mathrm{f}'(x)$ | –ve | 0 | –ve |
| tang | \ | — | \ |

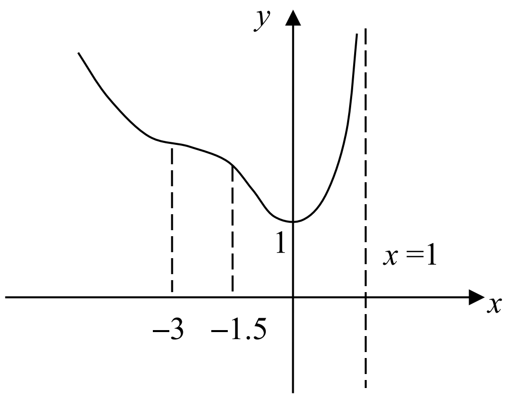
Point of inflexion at $x = -3$

| $x$ | $0^-$ | $0$ | $0^+$ |
|---|---|---|---|
| $\mathrm{f}'(x)$ | –ve | 0 | +ve |
| tang | \ | — | / |

Minimum point at $x = 0$.

[We observe the signs from $y = \mathrm{f}'(x)$ graph]

**(iii)**



**4**

**(i)** Differentiating $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$ w.r.t. $x$, $\quad -\frac{1}{x^2}-\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{y^2}{x^2}$$

Implicit differentiation right away!

Since $x\neq 0, y\neq 0,\ x^2>0$ and $y^2>0$.

It follows that $\frac{\mathrm{d}y}{\mathrm{d}x}<0$ and thus $y$ is a decreasing function.

**(ii)** From **(i)**, $\frac{\mathrm{d}y}{\mathrm{d}x}<0$ at all points $(x,y)$. Thus there are no points on the curve such that $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

There are no stationary points on the curve.

**(iii)** Gradient at $(2a,2a)=-\frac{(2a)^2}{(2a)^2}=-1$

Equation of tangent at $(2a,2a)$: $\quad y-2a=-1(x-2a)\quad$ i.e. $y=-x+4a$

Solving $\begin{cases}\frac{1}{x}+\frac{1}{y}=\frac{1}{a}\\ y=-x+4a\end{cases}$

$$\frac{1}{x}+\frac{1}{4a-x}=\frac{1}{a}$$

$$\Rightarrow x^2-4ax+4a^2=0$$

$$\Rightarrow (x-2a)^2=0 \qquad \Rightarrow x=2a$$

$\Rightarrow x=2a$, which is the point where we construct the tangent

With no other intersection points, the tangent does not meet the curve again.

**5(a)** $\mathrm{f}(x)=\tan\left(\mathrm{e}^{\mathrm{g}(x)}\right)\Rightarrow \mathrm{f}\,'(x)=\sec^2\left(\mathrm{e}^{\mathrm{g}(x)}\right)\mathrm{g}\,'(x)\mathrm{e}^{\mathrm{g}(x)}$

$$\mathrm{f}\,'(5)=\left[\sec^2(\mathrm{e}^1)\right](-3)\mathrm{e}^1=\frac{-3\mathrm{e}}{\cos^2(\mathrm{e})}=-9.81$$

**(b)**

$$y=\frac{x^2}{2x-1}\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-1)(2x)-x^2(2)}{(2x-1)^2}=\frac{2x(x-1)}{(2x-1)^2}$$

Function is increasing $\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}>0\Rightarrow\frac{2x(x-1)}{(2x-1)^2}>0$, where $x\neq\frac{1}{2}$

Thus inequality reduces to $x(x-1)>0$

$$\therefore x<0 \quad\text{or}\quad x>1$$

[Accept $x\leq 0$ or $x\geq 1$]

**6(i)**

$$3x^2-4xy+2y^2-2=0$$

Differentiate with respect to $x$ :

$$6x-4\left[x\frac{\mathrm{d}y}{\mathrm{d}x}+y\right]+2\left(2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)-0=0$$

$$3x-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2y+2y\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3x-2y}{2x-2y} \quad \text{(shown)}$$

**(ii)**

For tangents to the curve parallel to $x$-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\frac{3x-2y}{2x-2y}=0$$

$$\Rightarrow y=\frac{3}{2}x$$

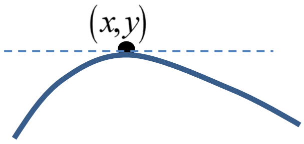
Solving $\begin{cases}3x^2-4xy+2y^2-2=0\\ y=\frac{3}{2}x\end{cases}$,

$$3x^2-4x\left(\frac{3}{2}x\right)+2\left(\frac{3}{2}x\right)^2-2=0 \Rightarrow 3x^2-4=0$$

$$x=\frac{2}{\sqrt{3}}, y=\sqrt{3}$$

$$x=-\frac{2}{\sqrt{3}}, y=-\sqrt{3}$$

The points are $\left(\frac{2}{\sqrt{3}},\sqrt{3}\right)$ and $\left(-\frac{2}{\sqrt{3}},-\sqrt{3}\right)$



Such points $(x,y)$ must satisfy 2 conditions:

1. $\frac{\mathrm{d}y}{\mathrm{d}x}=0$
2. Lie on the curve $3x^2-4xy+2y^2-2=0$

**(iii)**

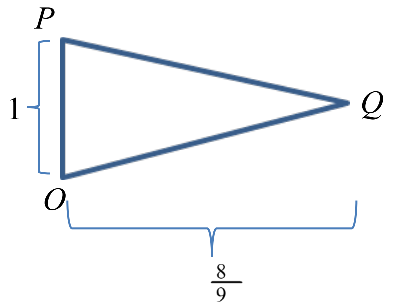
At $P(0,1)$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3(0)-2(1)}{2(0)-2(1)}=1$

Gradient of normal $=-1$

Equation of normal at *P*: $y-1=-1(x-0)$ i.e. $y=1-x$

Solving $\begin{cases}3x^2-4xy+2y^2-2=0\\ y=1-x\end{cases}$,

$$3x^2-4x(1-x)+2(1-x)^2-2=0$$

| | |
|---|---|
| | $9x^2-8x=0$<br>$x(9x-8)=0$<br>$x=\frac{8}{9}$ or $0$<br>At point $Q$, $x=\frac{8}{9}$<br>Area of triangle $OPQ=\frac{1}{2}(1)\left(\frac{8}{9}\right)=\frac{4}{9}$<br> <br>**Note**: Wherever the $y$-coord point $Q$, triangle $OPQ$ has base of length 1 and height $\frac{8}{9}$ . |
| **7(i)** | $\sqrt{x}+\sqrt{y}=\sqrt{a}$ ,  for $x>0, y>0$ , $a$ positive constant<br>Differentiate with respect to $x$ :<br>$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x}=0$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{\sqrt{y}}{\sqrt{x}}\neq 0$  since $y\neq 0$<br>Hence $C$ has no stationary points. |
| **(ii)** | As $x\to 0$ , $\frac{\mathrm{d}y}{\mathrm{d}x}\to -\infty$ . The tangent to $C$ approaches the line $x=0$ (the $y$-axis). |
| **8(i)** | $(4x-y)^2+16y=48$ ----- (1)<br>Differentiate with respect to $x$ :<br>$2(4x-y)\left(4-\frac{\mathrm{d}y}{\mathrm{d}x}\right)+16\frac{\mathrm{d}y}{\mathrm{d}x}=0$<br>$8(4x-y)-2(4x-y)\frac{\mathrm{d}y}{\mathrm{d}x}+16\frac{\mathrm{d}y}{\mathrm{d}x}=0$<br>$\left[2(4x-y)-16\right]\frac{\mathrm{d}y}{\mathrm{d}x}=8(4x-y)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4(4x-y)}{(4x-y)-8}$ |
| **(ii)** | Tangent // $x$-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow 4x-y=0$<br>Same concept as **Q6(ii).**<br>Substitute into eqn (1) : $0^2+16y=48\Rightarrow y=3$<br>$\Rightarrow$ eqn of tangent is $y=3$.  Coordinates of $P=\left(\frac{3}{4},3\right)$ |
| **(iii)** | Tangent // $y$-axis $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined $\Rightarrow (4x-y)-8=0\therefore y=4x-8$<br>Substitute into eqn (1) : $8^2+16(4x-8)=48\Rightarrow x=\frac{7}{4}$<br>Eqn of tangent is $x=\frac{7}{4}$.  Coordinates $Q=\left(\frac{7}{4},-1\right)$ |

| | |
|---|---|
| **(iv)** | Coordinates $R\left(\frac{7}{4},3\right)$<br>Area of triangle $PQR$ = $\frac{1}{2}\left[3-(-1)\right]\left(\frac{7}{4}-\frac{3}{4}\right)=2$ units$^2$<br>[Note that triangle *PQR* is a right angled triangle!!!] |

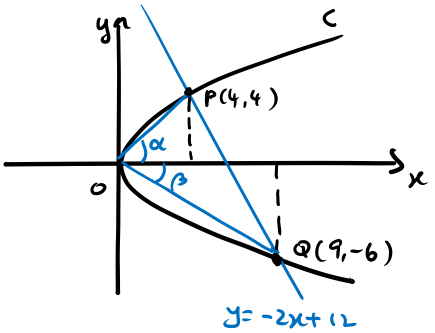
### (II) Tangents and Normals (Parametric Equations)

| | |
|---|---|
| **9(i)** | $\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t\sin t+\mathrm{e}^t\cos t;\qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-\mathrm{e}^{-t}\cos t+\mathrm{e}^{-t}(-\sin t)$<br>$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\mathrm{e}^{-t}\cos t+\mathrm{e}^{-t}(-\sin t)}{\mathrm{e}^t\sin t+\mathrm{e}^t\cos t}$<br>$=\frac{-\mathrm{e}^{-t}(\cos t+\sin t)}{\mathrm{e}^t(\sin t+\cos t)}=-\mathrm{e}^{-2t}$<br>Gradient of normal at $t=p$ is $-\frac{1}{-\mathrm{e}^{-2p}}=\mathrm{e}^{2p}$.<br>Equation of normal is $y-\mathrm{e}^{-p}\cos p=\mathrm{e}^{2p}(x-\mathrm{e}^p\sin p)$<br>$y=\mathrm{e}^{2p}(x-\mathrm{e}^p\sin p)+\mathrm{e}^{-p}\cos p$ |
| **(ii)** | For $p=\frac{\pi}{2}$, equation of normal becomes $y=\mathrm{e}^{\pi}(x-\mathrm{e}^{\frac{\pi}{2}})$<br>At $x=0$, $y=-\mathrm{e}^{\frac{3\pi}{2}}$.<br>At $y=0$, $x=\mathrm{e}^{\frac{\pi}{2}}$.<br>Area of Triangle $OAB$ = $\frac{1}{2}\left(\mathrm{e}^{\frac{3\pi}{2}}\right)\left(\mathrm{e}^{\frac{\pi}{2}}\right)=\frac{1}{2}\mathrm{e}^{2\pi}$ unit$^2$ |
| **10(i)** | $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=2\times\frac{1}{2t}=\frac{1}{t}$<br>At $P$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{p}$ and gradient of normal = $-p$<br>Equation of normal to $C$: $y-2p=-p(x-p^2)$, i.e. $y=-px+p^3+2p$ |
| **(ii)** | For $p=2$, equation of normal becomes<br>$y=-2x+8+2(2)$<br>$y=-2x+12$<br>When this normal cuts $C$,<br>$2t=-2t^2+12$<br>$t^2+t-6=0$<br>$(t-2)(t+3)=0$<br>$t=2$ or $t=-3$<br>At $P$, $t=2$. Hence, at $Q$, $t=-3$. $Q\equiv(9,-6)$<br>Solve simultaneous eqn<br>$\begin{cases}x=t^2, y=2t\\ y=-2x+12\end{cases}$ |

Since $P \equiv (4,4)$, $\measuredangle\alpha = \tan^{-1}\left(\frac{4}{4}\right) = \frac{\pi}{4}$

Since $Q \equiv (9,-6)$, $\measuredangle\beta = \tan^{-1}\left(\frac{6}{9}\right) = \tan^{-1}\left(\frac{2}{3}\right)$

Hence the angle $QOP = \measuredangle\alpha + \measuredangle\beta = \frac{\pi}{4} + \tan^{-1}\left(\frac{2}{3}\right)$ (shown)



[**Note** : Support your proof with a diagram]

**11(i)**

$x = 3(1-t)$, $y = \frac{1}{t^3}$ for $t \neq 0$.

$$\frac{dy}{dx} = \frac{-3t^{-4}}{-3} = \frac{1}{t^4}$$

For $t \neq 0$, $t^4 > 0$ and hence $\frac{dy}{dx} = \frac{1}{t^4} > 0$.

The curve is an increasing function.

**(ii)**

Equation of tangent $L_1$ at $\left(3-3t, \frac{1}{t^3}\right)$:

$$y - \frac{1}{t^3} = \frac{1}{t^4}\left[x-(3-3t)\right]$$

$$y = \frac{1}{t^4}x - \frac{3}{t^4} + \frac{3}{t^3} + \frac{1}{t^3}$$

$$y = \frac{1}{t^4}x - \frac{3}{t^4} + \frac{4}{t^3}$$

Sub $x = 0$, $y = 0$,

$$0 = \frac{1}{t^4}(0) - \frac{3}{t^4} + \frac{4}{t^3}$$

$$\frac{3}{t^4} = \frac{4}{t^3}$$

$3 = 4t$ (since $t \neq 0$ )

Thus $t = \frac{3}{4}$

Coordinates of $P = \left(\frac{3}{4}, \frac{64}{27}\right)$

**(iii)**

Gradient of $L_1 = \frac{1}{\left(\frac{3}{4}\right)^4} = \frac{256}{81}$

$$\therefore \frac{1}{t^4} = \frac{256}{81}$$

$$t^4 = \frac{81}{256}$$

It is clear that $t = -\frac{3}{4}$ is the only distinct $t$ value to obtain the same gradient.

$$t=\frac{3}{4}\ \text{(rejected $\because$ it's point $P$)} \quad \text{or} \quad t=-\frac{3}{4}$$

Equation of $L_2$:

$$y=\frac{256}{81}x-\frac{3}{\left(-\frac{3}{4}\right)^4}+\frac{4}{\left(-\frac{3}{4}\right)^3} \quad \text{, i.e.} \quad y=\frac{256}{81}x-\frac{512}{27}$$

**(iv)** Coordinates of $Q=\left(0,-\frac{512}{27}\right)$

Area of triangle $OPQ=\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{512}{27}\right)=\frac{64}{9}$ unit$^2$

Similar to **Q6(iii)** :
Find area of triangle *OPQ*.

**12(i)**

$$x=u^{\frac{1}{2}} \qquad y=\frac{1}{u^2}-2u$$

$$\frac{\mathrm{d}x}{\mathrm{d}u}=\frac{1}{2\sqrt{u}} \qquad \frac{\mathrm{d}y}{\mathrm{d}u}=-\frac{2}{u^3}-2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}=\left(-\frac{2}{u^3}-2\right)\times2\sqrt{u}=-4u^{\frac{-5}{2}}\left(1+u^3\right)$$

**(ii)** At $x=1, u=1$. Thus $y=-1$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=-8$.

Equation of tangent: $y+1=-8(x-1)\Rightarrow y=-8x+7$.

When $y=0, x=\frac{7}{8}$ and when $x=0, y=7$.

The tangent meets the axes at $A\left(\frac{7}{8},0\right)$ and $C(0,7)$.

Equation of normal: $y+1=\frac{1}{8}(x-1)\Rightarrow y=\frac{1}{8}x-\frac{9}{8}$.

When $y=0, x=9$ and when $x=0, y=-\frac{9}{8}$.

The normal meets the axes at $B(9,0)$ and $D\left(0,-\frac{9}{8}\right)$.

Finally $AB=9-\frac{7}{8}=\frac{65}{8}$ and $CD=7+\frac{9}{8}=\frac{65}{8}$

$\therefore\ AB=CD$ (shown)

A nice technique to acquire!

**(iii)** Let $G$ be $\frac{\mathrm{d}y}{\mathrm{d}x}$. Thus $G=-4u^{-\frac{5}{2}}\left(1+u^3\right)$

$$\Rightarrow \frac{\mathrm{d}G}{\mathrm{d}u}=-4\left(-\frac{5}{2}u^{\frac{-7}{2}}+\frac{1}{2}u^{\frac{-1}{2}}\right)=-2u^{\frac{-7}{2}}\left(u^3-5\right)$$

When $u=2$, $\frac{\mathrm{d}G}{\mathrm{d}t}=\frac{\mathrm{d}G}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}t}=-2\left(2^{\frac{-7}{2}}\right)\left(2^3-5\right)\times(0.5)=-\frac{3}{8\sqrt{2}}$

$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing at $\frac{3}{8\sqrt{2}}$ units per second.

**Note:**
$\frac{\mathrm{d}G}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$,
the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$.

**13(i)** $\frac{\mathrm{d}x}{\mathrm{d}t}=\mathrm{e}^t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=1-\frac{1}{t}=\frac{t-1}{t}$ $\quad\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t-1}{t\mathrm{e}^t}$

**For tangents parallel to *y*-axis,** $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined.

$$\Rightarrow \quad t\mathrm{e}^t=0$$

Since $t>0$ for the curve $C$, $t\mathrm{e}^t>0$ and thus there is no solution.

Hence there are no tangents parallel to $y$-axis. (Shown)

**For tangents parallel to *x*-axis,** $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$\Rightarrow \quad t-1=0$$
$$\Rightarrow \quad t=1$$

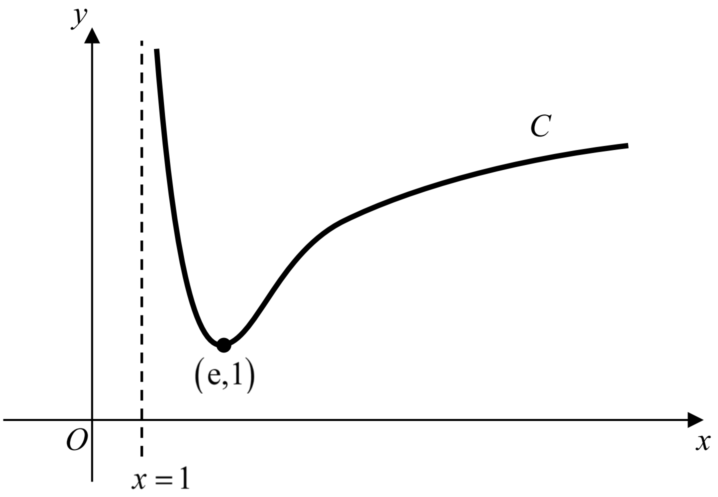
When $t=1$, $y=1-\ln 1=1$.

Thus equation of the tangent that is parallel to the $x$-axis is $y=1$.

**(ii)** As $t\to 0$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{t-1}{t\mathrm{e}^t}\to-\infty$ and $x\to 1^+$

Thus tangents to $C$ will tend to the vertical line $x=1$.

**(iii)**



**14**

$$\frac{\mathrm{d}x}{\mathrm{d}t}=-2\sin t,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\cos t \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\cos t}{2\sin t}$$

Equation of tangent at $P$ ($t=\theta$):

$$y-\sin\theta=\frac{-\cos\theta}{2\sin\theta}(x-2\cos\theta)$$
$$(2\sin\theta)y-2\sin^2\theta=(-\cos\theta)x+2\cos^2\theta$$
$$(2\sin\theta)y+(\cos\theta)x=2\left(\sin^2\theta+\cos^2\theta\right)$$
$$(\cos\theta)x+(2\sin\theta)y=2 \text{ (shown)}$$

Keep to $\sin\theta$ and $\cos\theta$ (Observe the final equation to show)

Equation of normal at $P$ ($t=\theta$):

$$y-\sin\theta=\frac{2\sin\theta}{\cos\theta}(x-2\cos\theta)$$
$$(\cos\theta)y-\sin\theta\cos\theta=(2\sin\theta)x-4\sin\theta\cos\theta$$
$$(2\sin\theta)x-(\cos\theta)y=3\sin\theta\cos\theta \text{ (shown)}$$

**(i)**

Solving $\begin{cases} x=2\cos t, y=\sin t \\ (\cos\theta)x+(2\sin\theta)y=2 \end{cases}$

$$(\cos\theta)(2\cos t)+(2\sin\theta)(\sin t)=2$$
$$\cos\theta\cos t+\sin\theta\sin t=1$$
$$\cos(t-\theta)=1$$
$$t-\theta=0$$
$$t=\theta$$

Check MF26
Cosine addition formulae

Since the tangent at point $P$ intersects the curve only at $t=\theta$, the tangent does not cut the curve again.

**(ii)**

At $A$, $y=0$

$$(2\sin\theta)x=3\sin\theta\cos\theta$$
$$x=\frac{3}{2}\cos\theta$$

At $B$, $x=0$

$$-(\cos\theta)y=3\sin\theta\cos\theta$$
$$y=-3\sin\theta$$

mid-point of $AB$: $\left(\frac{3}{4}\cos\theta,-\frac{3}{2}\sin\theta\right)$

As $\theta$ varies, the mid-point traces a curve with $x$- and $y$- coordinates always satisfying the parametric equations : $x=\frac{3}{4}\cos\theta, y=-\frac{3}{2}\sin\theta$

Yes, this is the parametric eqn of the curve traced by the mid-point.

Converting the parametric equations to cartesian equation:

$$\cos^2\theta+\sin^2\theta=1$$

$$\frac{16x^2}{9}+\frac{4y^2}{9}=1$$

$$\frac{x^2}{\left(\frac{3}{4}\right)^2}+\frac{y^2}{\left(\frac{3}{2}\right)^2}=1$$

The curve is a vertical ellipse, centre at $(0,0)$, with major axis $3$ units and minor axis $\frac{3}{2}$ unit.

**OR**

The curve is a vertical ellipse, centre at $(0,0)$, with **semi**-major axis $\frac{3}{2}$ units and **semi**-minor axis $\frac{3}{4}$ units.

**15(i)** As $t\to-\infty$, $x\to\sin t$ and $y\to-\cos t$.

Using trigonometric identity $\sin^2 t+\cos^2 t=1$, the cartesian equation of *C* is $x^2+y^2=1$.

The shape of *C* tends to a circle with centre at the origin and unit radius as $t\to-\infty$.

[We call this the unit circle with centre $(0,0)$]

**(ii)** $\frac{dx}{dt}=e^t+\cos t$ , $\frac{dy}{dt}=e^t+\sin t$. $\therefore$ $\frac{dy}{dx}=\frac{e^t+\sin t}{e^t+\cos t}$

At *P*, gradient of normal is $-\frac{e^\theta+\cos\theta}{e^\theta+\sin\theta}$.

$\therefore$ equation of normal is $y-\left(e^\theta-\cos\theta\right)=-\frac{e^\theta+\cos\theta}{e^\theta+\sin\theta}\left[x-\left(e^\theta+\sin\theta\right)\right]$

$$\Rightarrow y=-\frac{e^\theta+\cos\theta}{e^\theta+\sin\theta}\left[x-\left(e^\theta+\sin\theta\right)\right]+\left(e^\theta-\cos\theta\right)\Rightarrow y=-\frac{e^\theta+\cos\theta}{e^\theta+\sin\theta}x+2e^\theta$$

**(iii)** Using equation of normal found in **(ii)**, point $D$ is $\left(0,\,2e^\theta\right)$.

*E* is a point on *C*.

From $y=e^t-\cos t$ , when $y=0$, $\Rightarrow$ $e^t=\cos t$

By inspection, $t=0$. $\therefore$ $x=e^0+\sin 0=1$. Hence point $E$ is $(1,0)$.

**(iv)** The mid-point of *DE* is $\left(\frac{1}{2},e^\theta\right)$. As $\theta$ varies through positive values, the *x*-coord is fixed (at $\frac{1}{2}$) and the *y*-coord = $e^\theta>1$,

$\therefore$ The path is part of a vertical line with equation $x=\frac{1}{2}$ and $y>1$.

[Part of a straight line is called a half-line]

## (III) Rate of change and Maximisation/Minimisation problems

**16(i)** Let $V$ m$^3$ be the volume of water in the pond at time $t$ sec.

$$V=\pi(4.5)^2 h=20.25\pi h \quad \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}h}=20.25\pi$$

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}$$

$$-0.8=20.25\pi\times\frac{\mathrm{d}h}{\mathrm{d}t}$$

Water is pumped out of the pond. Thus $V$ decreases and $\frac{\mathrm{d}V}{\mathrm{d}t}=-0.8$.

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-0.8}{20.25\pi}=-0.012575=-0.0126$$

$\therefore$ rate of change of depth $= -\ 0.0126$ m/s (3 s.f.)

**(ii)** Time required $=\dfrac{1.9-1.2}{0.012575}=55.7\text{s}$ (3 s.f.)

We can just do $\dfrac{\text{depth}}{\text{rate}}$ since the depth is decreasing at a constant rate.

**17.** Given $V=\dfrac{1}{k}p^n$,

$$\frac{\mathrm{d}V}{\mathrm{d}p}=\frac{1}{k}np^{n-1}=\frac{1}{k}n\left(\frac{p^n}{p}\right)=\frac{n}{p}\left(\frac{1}{k}p^n\right)=\frac{nV}{p}=-2.3\left(\frac{V}{p}\right) \text{, given } n=-2.3$$

Note Using this method, we won't need to calculate the value of $k$.

When $V=32, p=105, \dfrac{\mathrm{d}p}{\mathrm{d}t}=0.2$:

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}p}\times\frac{\mathrm{d}p}{\mathrm{d}t}$$

$$=-2.3\left(\frac{32}{105}\right)\times 0.2 = -\ 0.140 \text{ (to 3 s.f.)}$$

Thus the rate of decrease of volume at the instant is $0.140\ \text{cm}^3\text{s}^{-1}$.

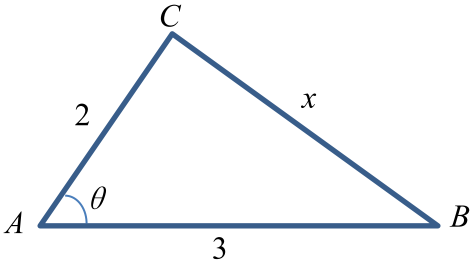
**18 (a)** Let $BC=x$ cm and $\angle BAC=\theta$ radians.

By cosine rule, $x^2=3^2+2^2-2(3)(2)\cos\theta$

$x^2=13-12\cos\theta$

Differentiate with respect to $t$,

$$2x\frac{\mathrm{d}x}{\mathrm{d}t}=12\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$$



When $\theta=\dfrac{\pi}{3}$, $x=\sqrt{7}$.

$$2\sqrt{7}\frac{\mathrm{d}x}{\mathrm{d}t}=12\left(\frac{\sqrt{3}}{2}\right)(0.1) \quad \left[\text{Given } \frac{\mathrm{d}\theta}{\mathrm{d}t}=0.1 \text{ always}\right]$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{0.3\sqrt{3}}{\sqrt{7}}=0.196 \text{ cm/s (3 s.f.)}$$

**(b)**

$$x^2 - xy = p^2 + y^2$$

Differentiate with respect to $x$ : $2x - x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y - 2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x-y}{x+2y}$$

Normal parallel to the $x$-axis $\Rightarrow$ tangent parallel to the $y$-axis.
i.e. $x + 2y = 0 \Rightarrow x = -2y$

Substitute back to the original equation:

$$(-2y)^2 - y(-2y) = p^2 + y^2$$

$$5y^2 = p^2$$

$$y = \pm\frac{p}{\sqrt{5}}$$

$$y = \frac{p}{\sqrt{5}},\ x = -\frac{2p}{\sqrt{5}}.$$

$$y = -\frac{p}{\sqrt{5}},\ x = \frac{2p}{\sqrt{5}}.$$

Same concept as **Q6(ii).**

Thus the coordinates are $\left(-\dfrac{2p}{\sqrt{5}}, \dfrac{p}{\sqrt{5}}\right)$ and $\left(\dfrac{2p}{\sqrt{5}}, -\dfrac{p}{\sqrt{5}}\right)$.

**19.**

Volume, $V = \pi r^2 h + \dfrac{1}{2}\left(\dfrac{4}{3}\pi r^3\right) = 500$

$$\pi r^2\left(h + \frac{2}{3}r\right) = 500 \Rightarrow h = \frac{500}{\pi r^2} - \frac{2}{3}r$$

Surface area, $S = \pi r^2 + 2\pi rh + \dfrac{1}{2}\left(4\pi r^2\right)$

$$= 3\pi r^2 + 2\pi r\left(\frac{500}{\pi r^2} - \frac{2}{3}r\right)$$

Make $S$ in terms of a single variable, $r$.

$$= \frac{5}{3}\pi r^2 + \frac{1000}{r}$$

$$\frac{\mathrm{d}S}{\mathrm{d}r} = \frac{10}{3}\pi r - \frac{1000}{r^2} = \frac{10\pi r^3 - 3000}{3r^2}$$

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 0 \quad \Rightarrow 10\pi r^3 - 3000 = 0$$

$$\Rightarrow r = \sqrt[3]{\frac{300}{\pi}}$$

$\dfrac{\mathrm{d}^2S}{\mathrm{d}r^2} = \dfrac{10}{3}\pi + \dfrac{2000}{r^3} > 0$ when $r = \sqrt[3]{\dfrac{300}{\pi}}$. Thus $S$ is min when $r = \sqrt[3]{\dfrac{300}{\pi}}$.

$$\text{Minimum surface area } = \frac{\frac{5}{3}\pi\left(\frac{300}{\pi}\right)+1000}{\sqrt[3]{\frac{300}{\pi}}} = 1500\sqrt[3]{\frac{\pi}{300}} = 328.1715$$

Cost of box with minimum surface area $=(328.1715)(0.015)=\$4.92$ (2 d.p.)

**20.**

$V=(\text{Base Area})\times\text{height}$

$$2\sqrt{3}=\left(\frac{1}{2}x^2\sin 60°\right)\times h$$

$$2\sqrt{3}=\left(\frac{1}{2}x^2\frac{\sqrt{3}}{2}\right)\times h$$

$$h=\frac{8}{x^2} \quad \text{(shown)}$$

Total cost of constructing prism,

$$\begin{aligned}C&=1\times(\text{edges})+2\sqrt{3}\,(2\times\text{triangles})+2(3\times\text{rectangles})\\&=(3h+6x)+2\sqrt{3}\left(2\left(\frac{1}{2}x^2\frac{\sqrt{3}}{2}\right)\right)+2(3xh)\\&=\left(3\left(\frac{8}{x^2}\right)+6x\right)+3x^2+2\left(3x\left(\frac{8}{x^2}\right)\right)\\&=\frac{24}{x^2}+6x+3x^2+\frac{48}{x}\\&=3x^2+6x+48x^{-1}+24x^{-2} \quad \text{(shown)}\end{aligned}$$

$$\frac{\mathrm{d}C}{\mathrm{d}x}=6x+6-\frac{48}{x^2}-\frac{48}{x^3}=0$$

For stationary $C$, $\frac{\mathrm{d}C}{\mathrm{d}x}=0$

$$\begin{aligned}6x+6-\frac{48}{x^2}-\frac{48}{x^3}&=0\\x^4+x^3-8x-8&=0\\x^3(x+1)-8(x+1)&=0\\(x^3-8)(x+1)&=0\\x^3=8 \quad \text{or} \quad x&=-1\,\text{(rejected)}\\x&=2\end{aligned}$$

$\frac{\mathrm{d}^2C}{\mathrm{d}x^2}=6+\frac{96}{x^3}+\frac{144}{x^4}>0$ when $x=2$. Thus $C$ is min when $x=2$.

Min $C=3(2)^2+6(2)+48(2)^{-1}+24(2)^{-2}=54$

Minimum cost is \$54.

**21(i)**

$$k=\left(\frac{1}{2}x^2\sin\frac{\pi}{3}+hx\right)(3x)=\frac{3\sqrt{3}}{4}x^3+3hx^2$$

$$\therefore h=\frac{1}{3x^2}\left(k-\frac{3\sqrt{3}}{4}x^3\right)=\frac{k}{3x^2}-\frac{\sqrt{3}}{4}x$$

$$\begin{aligned}A&=2\left(\frac{1}{2}x^2\sin\frac{\pi}{3}\right)+2(3x^2)+2(hx)+2(3hx)\\&=\frac{\sqrt{3}}{2}x^2+6x^2+8hx\\&=\frac{\sqrt{3}}{2}x^2+6x^2+8x\left(\frac{k}{3x^2}-\frac{\sqrt{3}}{4}x\right)\\&=\frac{\sqrt{3}}{2}x^2+6x^2+\frac{8k}{3x}-2\sqrt{3}x^2\\&=6x^2-\frac{3\sqrt{3}}{2}x^2+\frac{8k}{3x}\ \text{(shown)}\end{aligned}$$

$$\therefore\frac{\mathrm{d}A}{\mathrm{d}x}=12x-3x\sqrt{3}-\frac{8k}{3x^2}$$

For stationary $A$, $\frac{\mathrm{d}A}{\mathrm{d}x}=12x-3x\sqrt{3}-\frac{8k}{3x^2}=0$

$$\begin{aligned}12x-3x\sqrt{3}-\frac{8k}{3x^2}&=0\\9x^3\left(4-\sqrt{3}\right)&=8k\\x^3&=\frac{8k}{9\left(4-\sqrt{3}\right)}\\\therefore x&=\sqrt[3]{\frac{8k}{9\left(4-\sqrt{3}\right)}}\end{aligned}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2}=12-3\sqrt{3}+\frac{16k}{3x^3}>0\text{ since }x^3>0,k>0,\ 12-3\sqrt{3}>0$$

<u>Alternative</u>

$$\begin{aligned}\frac{\mathrm{d}^2A}{\mathrm{d}x^2}&=12-3\sqrt{3}+\frac{16k}{3x^3}=12-3\sqrt{3}+\frac{16k}{3}\left(\frac{9\left(4-\sqrt{3}\right)}{8k}\right)\\&=12-3\sqrt{3}+6\left(4-\sqrt{3}\right)=36-9\sqrt{3}>0\end{aligned}$$

$\therefore$ $A$ is a minimum when $x=\sqrt[3]{\frac{8k}{9\left(4-\sqrt{3}\right)}}$.

**(ii)** Given $k = 360, A = 300$, we have

$$300 = 6x^2 - \frac{3\sqrt{3}}{2}x^2 + \frac{8(360)}{3x}$$

$$600x = 12x^3 - 3\sqrt{3}x^3 + 1920$$

$$\left(12 - 3\sqrt{3}\right)x^3 - 600x + 1920 = 0$$

From GC, since $x > 0$, $x = 3.8442$ or $x = 6.8587$

$$x = 3.8442,\ h = \frac{360}{3(3.8442)^2} - \frac{\sqrt{3}}{4}(3.8442) = 6.46.$$

$$x = 6.8587,\ h = \frac{360}{3(6.8587)^2} - \frac{\sqrt{3}}{4}(6.8587) = -0.419 \ \ (\text{rej. } \because\ h > 0)$$

$$\therefore\ x = 3.84,\ h = 6.46.$$

**22(i)** By Pythagoras Theorem, $AC = \sqrt{100^2 + h^2}$

Time-taken to swim from $A$ to $C = \dfrac{\sqrt{100^2 + h^2}}{v}$

$CD = 300 - h$

Time-taken to run from $C$ to $D$ $= \dfrac{300 - h}{4v}$

$\therefore$ Time taken from $A$ to $D$, $t = \dfrac{\sqrt{100^2 + h^2}}{v} + \dfrac{300 - h}{4v}$ (shown)

**(ii)**

$$\frac{\mathrm{d}t}{\mathrm{d}h} = \frac{h}{v\sqrt{100^2 + h^2}} - \frac{1}{4v}$$

$$\frac{\mathrm{d}t}{\mathrm{d}h} = 0,\ \frac{h}{v\sqrt{100^2 + h^2}} - \frac{1}{4v} = 0$$

$$\sqrt{100^2 + h^2} = 4h$$

$$15h^2 = 100^2$$

$$h = \frac{100}{\sqrt{15}} \quad (h > 0)$$

| $h$ | $\left(\frac{100}{\sqrt{15}}\right)^-$ | $\frac{100}{\sqrt{15}}$ | $\left(\frac{100}{\sqrt{15}}\right)^+$ |
|---|---|---|---|
| Sign of $\frac{\mathrm{d}t}{\mathrm{d}h}$ | – ve | 0 | + ve |
| Tangent | \ | — | / |

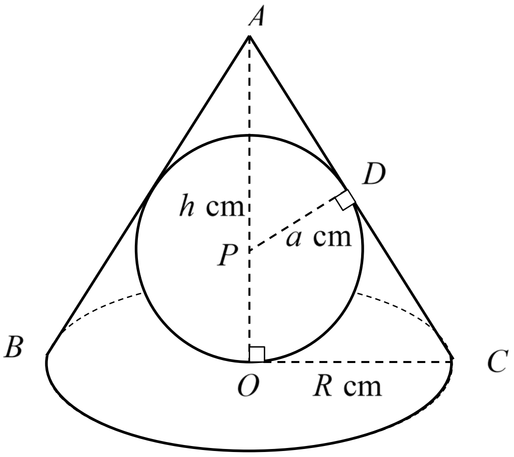
Thus time taken is the shortest when $h = \dfrac{100}{\sqrt{15}}$.

**23(i)** Let centre of the sphere be $P$.

$AC^2 = h^2 + R^2 \quad ----(1)$

Since $\Delta APD \sim \Delta ACO$ , $\dfrac{R}{a} = \dfrac{AC}{h-a}$

$$AC = \frac{Rh - Ra}{a} \quad ----(2)$$



From (1) and (2),

$$\left(\frac{Rh-Ra}{a}\right)^2 = h^2 + R^2$$

$$R^2h^2 - 2R^2ha + R^2a^2 = h^2a^2 + R^2a^2$$

$$R^2\left(h^2 - 2ha\right) = h^2a^2$$

$$\therefore R = \frac{ha}{\sqrt{\left(h^2 - 2ha\right)}}$$

**Alternative**

Let centre of the sphere be $P$.

$AC^2 = h^2 + R^2 \quad ----(1)$

$$AD = \sqrt{\left(AP^2 - PD^2\right)}$$

$$= \sqrt{\left((h-a)^2 - a^2\right)}$$

$$= \sqrt{\left(h^2 - 2ha\right)}$$

Using congruent triangles $PCO$ and $PCD$, $CO = CD = R$ cm.

$AC = AD + DC$

$$= \sqrt{\left(h^2 - 2ha\right)} + R \quad ---(2)$$

Equating (1) and (2),

$$h^2 + R^2 = h^2 - 2ha + 2R\sqrt{\left(h^2 - 2ha\right)} + R^2$$

$$\therefore R = \frac{ha}{\sqrt{\left(h^2 - 2ha\right)}}$$

**(ii)** Volume of cone, $V$

$$= \tfrac{1}{3}\pi R^2 h$$

$$= \tfrac{1}{3}\pi\left(\frac{h^2a^2}{h^2 - 2ha}\right)h$$

$$= \tfrac{1}{3}\pi a^2\left(\frac{h^2}{h - 2a}\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\tfrac{1}{3}\pi a^2\left[\frac{2h(h-2a)-h^2}{(h-2a)^2}\right]$$

$$=\tfrac{1}{3}\pi a^2\left[\frac{h^2-4ha}{(h-2a)^2}\right]$$

$$=\tfrac{1}{3}\pi a^2\,\frac{h(h-4a)}{(h-2a)^2}$$

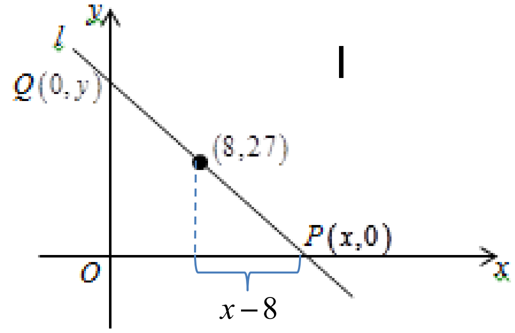
$$\frac{\mathrm{d}V}{\mathrm{d}h}=0\Rightarrow h=4a \text{ or } h=0 \text{ (reject } \because h>0)$$

| $h$ | $(4a)^-$ | $4a$ | $(4a)^+$ |
|---|---|---|---|
| Sign of $\frac{\mathrm{d}V}{\mathrm{d}h}$ | – ve | 0 | + ve |
| Tangent | ╲ | — | ╱ |

Thus, $V$ is minumum when $h=4a$.

$$\therefore \text{ minimum } V=\tfrac{1}{3}\pi a^2\cdot\frac{(4a)^2}{4a-2a}=\tfrac{8}{3}\pi a^3 \text{ cm}^3$$

**24.**



Using similar triangles, $\dfrac{27}{y}=\dfrac{x-8}{x}$

$$y=\frac{27x}{x-8}$$

Let $h$ be the hypotenuse ($PQ$).

$$PQ^2=h^2=x^2+y^2$$

$$h^2=x^2+\left(\frac{27x}{x-8}\right)^2 \quad \text{..... (1)}$$

$$PQ=\sqrt{x^2+\left(\frac{27x}{x-8}\right)^2}$$

Differentiate eqn (1) implicitly wrt $x$,

$$2h\frac{\mathrm{d}h}{\mathrm{d}x}=2x+2\left(\frac{27x}{x-8}\right)\left[\frac{(x-8)27-27x}{(x-8)^2}\right]$$

$$=2x-2\left[\frac{27x(216)}{(x-8)^3}\right]$$

At stationary points, $\frac{\mathrm{d}h}{\mathrm{d}x}=0$

$$0=x-\left[\frac{5832x}{(x-8)^3}\right]$$

$$x(x-8)^3-5832x=0$$

$$x\left[(x-8)^3-5832\right]=0$$

$$(x-8)^3=5832 \text{ (shown)} \quad \text{or} \quad x=0 \text{ (rejected)}$$

*l* won't have negative gradient in this case.

Thus $x_1$ satisfies the equation $(x-8)^3-5832=0$ and

$$x_1=\sqrt[3]{5832}+8=26$$

$$y=\frac{27(26)}{26-8}=39$$

| $x$ | $26^-$ | $26$ | $26^+$ |
|---|---|---|---|
| $\frac{\mathrm{d}h}{\mathrm{d}x}=\frac{x-\left(\frac{5832x}{(x-8)^3}\right)}{h},(h>0)$ | − ve | 0 | +ve |
| Tangent | \ | — | / |

Therefore, hypotenuse $PQ$ is minimum when $x = 26$ units

Area of triangle required $=\frac{1}{2}(39)(26)=507\text{ unit}^2$

**25(i)** Observe that the arc length of the sector in Diagram 1 is also the circumference of the circle in Diagram 2.

Thus we have $a\theta = 2\pi r$, i.e., $r = \dfrac{a\theta}{2\pi}$.

**Alternative**
Observe that the area of the sector is the curved surface area of the cone.
$$\pi ra = \tfrac{1}{2}a^2\theta$$
$$r = \frac{a\theta}{2\pi}$$

**(ii)**
$$V = \frac{1}{3} \times \text{base area} \times \text{height}$$
$$= \frac{1}{3}\left(\pi r^2\right)\sqrt{a^2 - r^2}$$
$$= \frac{1}{3}\left[\pi\left(\frac{a\theta}{2\pi}\right)^2\right]\sqrt{a^2 - \left(\frac{a\theta}{2\pi}\right)^2}$$
$$= \frac{a^2\theta^2}{12\pi}\sqrt{\left(\frac{a}{2\pi}\right)^2\left(4\pi^2 - \theta^2\right)}$$
$$= \frac{a^3\theta^2}{24\pi^2}\sqrt{\left(4\pi^2 - \theta^2\right)}.$$
$$V^2 = \frac{a^6\theta^4}{576\pi^4}\left(4\pi^2 - \theta^2\right)$$

**(iii)** When $a = 2$, we have $V^2 = \dfrac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)$

Sub $a = 2$ in at the first instance.
It is easier to work with numbers.

Differentiating $V^2 = \dfrac{1}{9\pi^4}(4\pi^2\theta^4 - \theta^6)$ w.r.t. $\theta$
$$2V\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{1}{9\pi^4}\left(16\pi^2\theta^3 - 6\theta^5\right)$$
$$= \frac{2\theta^3}{9\pi^4}\left(8\pi^2 - 3\theta^2\right)$$

When $\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = 0$, $\theta^3\left(8\pi^2 - 3\theta^2\right) = 0$, i.e., we have

$\theta = 0$ (N.A.) or $\theta = \pm\sqrt{\dfrac{8\pi^2}{3}}$ (reject –ve).

Thus, $\theta = \sqrt{\dfrac{8\pi^2}{3}}$ and max $V = \dfrac{8\pi^2}{3}\left(\dfrac{1}{3\pi^2}\right)\sqrt{\left(4\pi^2 - \dfrac{8\pi^2}{3}\right)}$.
$$= \frac{8}{9}\sqrt{\frac{4\pi^2}{3}}$$
$$= \frac{16\pi\sqrt{3}}{27}\ \text{cm}^3$$

| | |
|---|---|
| **26(i)** | $L=\sqrt{a^2+x^2}+\sqrt{b^2+(m-x)^2}$ |
| **(ii)** | $\frac{\mathrm{d}L}{\mathrm{d}x}=\frac{x}{\sqrt{a^2+x^2}}-\frac{m-x}{\sqrt{b^2+(m-x)^2}}$<br>$\frac{\mathrm{d}L}{\mathrm{d}x}=\left(\frac{1}{2}\right)\left(a^2+x^2\right)^{-\frac{1}{2}}(2x)+\left(\frac{1}{2}\right)\left(b^2+(m-x)^2\right)^{-\frac{1}{2}}\left(2(m-x)\right)(-1)$<br>When $\frac{\mathrm{d}L}{\mathrm{d}x}=0$,<br>$\frac{x}{\sqrt{a^2+x^2}}=\frac{m-x}{\sqrt{b^2+(m-x)^2}}$<br>$x\sqrt{b^2+(m-x)^2}=(m-x)\sqrt{a^2+x^2}$<br>$x^2\left(b^2+(m-x)^2\right)=(m-x)^2\left(a^2+x^2\right)$<br>$x^2b^2+x^2(m-x)^2=(m-x)^2a^2+x^2(m-x)^2$<br>Observe that there is a common term "$x^2(m-x)^2$" in both LHS and RHS<br>$x^2b^2=(m-x)^2a^2$<br>$xb=\pm a(m-x)$<br>$xb=ma-xa$ or $-ma+xa$<br>$x=\frac{ma}{a+b}$ or $\frac{ma}{a-b}$ (rejected as $0<x<m$)<br>Thus $x=\frac{ma}{a+b}$ is the only value that gives a stationary value of $L$. |
| **(iii)** | $\tan\theta_i=\frac{a}{\left(\frac{ma}{b+a}\right)}=\frac{b+a}{m}$<br>Note: $\frac{a(b+a)}{ma}=\frac{b(b+a)}{mb}$<br>$\tan\theta_r=\frac{b}{\left(m-\frac{ma}{b+a}\right)}=\frac{b}{\left(\frac{mb}{b+a}\right)}=\frac{b+a}{m}$<br>Since $\theta_i$ and $\theta_r$ are acute and $\tan\theta_i=\tan\theta_r\Rightarrow\theta_i=\theta_r$. |
| **(iv)** | For the fire at $B$ when $a=1, b=2, m=5$, $x=\frac{5\times 1}{1+2}=\frac{5}{3}$<br>$\because\tan\theta_i=\frac{a}{x}=\frac{3}{5}\Rightarrow$ largest $\theta_i=\tan^{-1}\left(\frac{3}{5}\right)$<br>For the other end of the fire, $m=6$ (since the length of the fire is 1), $x=\frac{6\times 1}{1+2}=2$<br>$\because\tan\theta_i=\frac{a}{x}=\frac{1}{2}\Rightarrow$ smallest $\theta_i=\tan^{-1}\left(\frac{1}{2}\right)$<br>$\therefore$ range of $\theta_i$ values : $\tan^{-1}\left(\frac{1}{2}\right)\le\theta_i\le\tan^{-1}\left(\frac{3}{5}\right)$ |