ANDERSON SERANGOON JUNIOR COLLEGE

H2 MATHEMATICS **9758**

JC2 Prelim Paper 1 (100 marks)

9 Sept 2024

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME	

CLASS			/		

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
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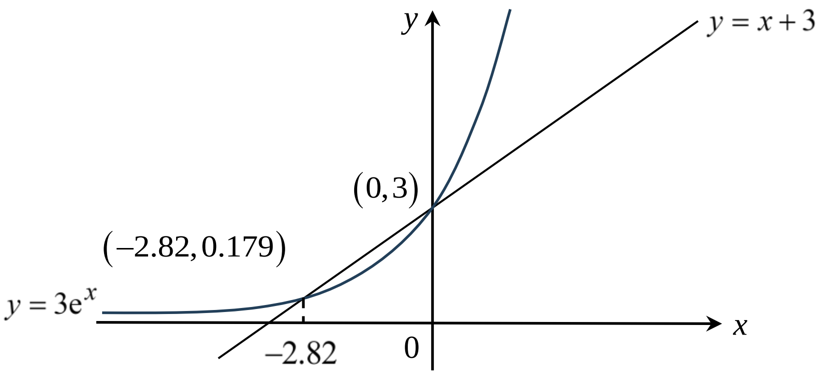
7	
8	
9	
10	
11	
Total	

1 **(a)**



From the graphs, $x < -2.82$ or $x > 0$.

$\int_{-2}^{2} \left|3e^x - x - 3\right| \, dx$

$= \int_{-2}^{0} -(3e^x - x - 3)dx + \int_{0}^{2} (3e^x - x - 3)dx$

$= -\left[3e^x - \frac{x^2}{2} - 3x\right]_{-2}^{0} + \left[3e^x - \frac{x^2}{2} - 3x\right]_{0}^{2}$

$= -[3 - (3e^{-2} - 2 + 6)] + [(3e^2 - 2 - 6) - 3]$

$= 3e^{-2} + 3e^2 - 10$

2 **(i)** $\frac{d}{dx}\left(e^{\sin^{-1}x}\sqrt{1-x^2}\right) = e^{\sin^{-1}x} - \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}} \cdots\cdots (1)$

(ii) $\int x \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \, dx = xe^{\sin^{-1}x} - \int e^{\sin^{-1}x} dx$

From (1), $e^{\sin^{-1}x}\sqrt{1-x^2} + \int \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}} \, dx + c = \int e^{\sin^{-1}x} dx$

So $\int x \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \, dx = xe^{\sin^{-1}x} - e^{\sin^{-1}x}\sqrt{1-x^2} - \int \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}} \, dx - c$

$\therefore \int x \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}} \, dx = \frac{1}{2}\left(xe^{\sin^{-1}x} - e^{\sin^{-1}x}\sqrt{1-x^2}\right) + D$

3 **(i)**

$\frac{dx}{dt} = 2 + \frac{2}{t^3}$

$\frac{dy}{dt} = 2 - \frac{1}{t^2}$

$\frac{dy}{dx} = \frac{2 - \frac{1}{t^2}}{2 + \frac{2}{t^3}} = \frac{t\left(2t^2 - 1\right)}{2t^3 + 2}$

When $t = 1$,

$\dfrac{dy}{dx}=\dfrac{1}{4}$, $x=1$ and $y=3$.

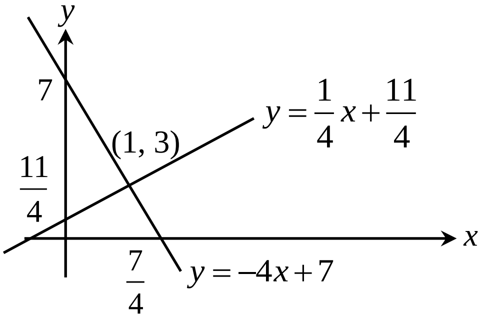
Eqn of tangent at P: $y-3=\dfrac{1}{4}(x-1)$

$y=\dfrac{1}{4}x+\dfrac{11}{4}$

Eqn of normal at P: $y-3=-4(x-1)$

$y=-4x+7$

(ii)



Area of $OAPB$ = $\dfrac{1}{2}\left(\dfrac{7}{4}\right)(7)-\dfrac{1}{2}\left(7-\dfrac{11}{4}\right)(1)=4\,\text{units}^2$

4 **(a)** $u_r-u_{r-1}=r^3+\left(\dfrac{1}{2}\right)^r$

$$\sum_{r=1}^{n}(u_r-u_{r-1})=\sum_{r=1}^{n}\left[r^3+\left(\frac{1}{2}\right)^r\right]$$

$$\left(\begin{array}{c} u_1\cancel{-u_0} \\ +\cancel{u_2}-\cancel{u_1} \\ +\cancel{u_3}-\cancel{u_2} \\ + \\ \vdots \\ +\cancel{u_{n-1}}-\cancel{u_{n-2}} \\ +u_n-\cancel{u_{n-1}} \end{array}\right)=\sum_{r=1}^{n}r^3+\sum_{r=1}^{n}\left(\frac{1}{2}\right)^r$$

$$u_n-u_0=\frac{n^2(n+1)^2}{4}+\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$\therefore u_n=\frac{n^2(n+1)^2}{4}+\left(1-\left(\frac{1}{2}\right)^n\right)+2$$

$$=\frac{n^2(n+1)^2}{4}-\left(\frac{1}{2}\right)^n+3$$

(b) $\displaystyle\sum_{r=9}^{n}\left((r+2)^3+\left(\frac{1}{2}\right)^{r+2}\right)$

$$= \sum_{r=11}^{r=n+2}\left(r^3+\left(\frac{1}{2}\right)^r\right) \ (\because \text{ Replace } r \text{ by } r-2)$$

$$= u_{n+2} - u_{10}$$

$$= \left[\frac{(n+2)^2(n+3)^2}{4} - \left(\frac{1}{2}\right)^{n+2} + 3\right] - \left[\frac{(10)^2(11)^2}{4} - \left(\frac{1}{2}\right)^{10} + 3\right]$$

$$= \frac{(n+2)^2(n+3)^2}{4} - \left(\frac{1}{2}\right)^{n+2} - 3025 + \frac{1}{1024}$$

5 **(a)** $|\mathbf{a}+\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$

$$\Rightarrow |\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}-\mathbf{b}|^2$$

$$\Rightarrow (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})$$

$$\Rightarrow |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2 = |\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2$$

$$\Rightarrow 4\mathbf{a}\cdot\mathbf{b} = 0$$

$$\Rightarrow 4|\mathbf{a}||\mathbf{b}|\cos\theta = 0$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors, then $\theta = 90^\circ$, thus $\mathbf{a} \perp \mathbf{b}$

(b)(i) Since $\mathbf{a}\times(\mathbf{b}+\mathbf{c})$, $\mathbf{b}\times(\mathbf{c}+\mathbf{a})$ and $\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ are all vectors, the addition of these vectors will lead to a resultant vector.

(ii) $\mathbf{a}\times(\mathbf{b}+\mathbf{c}) + \mathbf{b}\times(\mathbf{c}+\mathbf{a}) + \mathbf{c}\times(\mathbf{a}+\mathbf{b})$

$$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} + \mathbf{b}\times\mathbf{a} + \mathbf{c}\times\mathbf{a} + \mathbf{c}\times\mathbf{b}$$

$$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{c} + \mathbf{b}\times\mathbf{c} - \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} - \mathbf{b}\times\mathbf{c} \ (\because \mathbf{b}\times\mathbf{a} = -\mathbf{a}\times\mathbf{b} \text{ and } \mathbf{c}\times\mathbf{a} = -\mathbf{a}\times\mathbf{c})$$

$$= \mathbf{0}$$

will not be given if the student wrote it as a scalar quantity

6 **(i)**

$$(x+y)\frac{dy}{dx} + ky = 2 \qquad \cdots(1)$$

Differentiating (1) w.r.t. x:

$$(x+y)\frac{d^2y}{dx^2} + \left(1+\frac{dy}{dx}\right)\frac{dy}{dx} + k\frac{dy}{dx} = 0$$

$$(x+y)\frac{d^2y}{dx^2} + (1+k)\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \cdots(2)$$

(ii) $\sin\left(2x+\frac{\pi}{2}\right) = \sin 2x \cos\frac{\pi}{2} + \cos 2x \sin\frac{\pi}{2} = \cos 2x$

$$\frac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)} = \frac{1}{\cos^2 2x}$$

$$\approx \left(1 - \frac{(2x)^2}{2}\right)^{-2}$$

$=\left(1-2x^2\right)^{-2}$

$=1+4x^2+\ldots$

$x=0,\quad y=1:\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=2-k$

$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=3k-6$

$4=2\left(\dfrac{3k-6}{2}\right)$

$k=\dfrac{10}{3}$

(iii)
Differentiating (2) w.r.t. x:

$$(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)=0$$

$$(x+y)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+\left(3+3\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$$

When $x=0,\quad y=1,\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=1,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-3,\ \dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}=18$

$\therefore y=1+x-\dfrac{3}{2}x^2+3x^3+\ldots$

7 **(a)** $9y^2-54y-x^2-2x+79=0$

$\Rightarrow 9\left(y^2-6y\right)-\left(x^2+2x\right)+79=0$

$\Rightarrow 9\left[\left(y-3\right)^2-9\right]-\left[\left(x+1\right)^2-1\right]+79=0$

$\Rightarrow 9\left(y-3\right)^2-\left(x+1\right)^2=1$

$\Rightarrow \left(3y-9\right)^2-\left(x+1\right)^2=1$

$y^2-x^2=1$

Replace x by $x+1$ $\quad\downarrow\quad$ Translation of 1 unit in the negative x direction

$y^2-\left(x+1\right)^2=1$

Replace y by $y-9$ $\quad\downarrow\quad$ Translation of 9 units in the positive y direction

$\left(y-9\right)^2-\left(x+1\right)^2=1$

Replace y by $3y$ $\quad\downarrow\quad$ Scaling parallel to the $y-$axis by a factor of $\dfrac{1}{3}$

$\left(3y-9\right)^2-\left(x+1\right)^2=1$

(b)(i) Consider the line $y=k$. To find the range of y where curve C cannot lie,

$\Rightarrow\left(3k-9\right)^2-\left(x+1\right)^2=1$ has no real roots

$\Rightarrow x^2+2x+2-(3k-9)^2=0$ has no real roots.

$\Rightarrow 4-4\left[2-(3k-9)^2\right]<0$

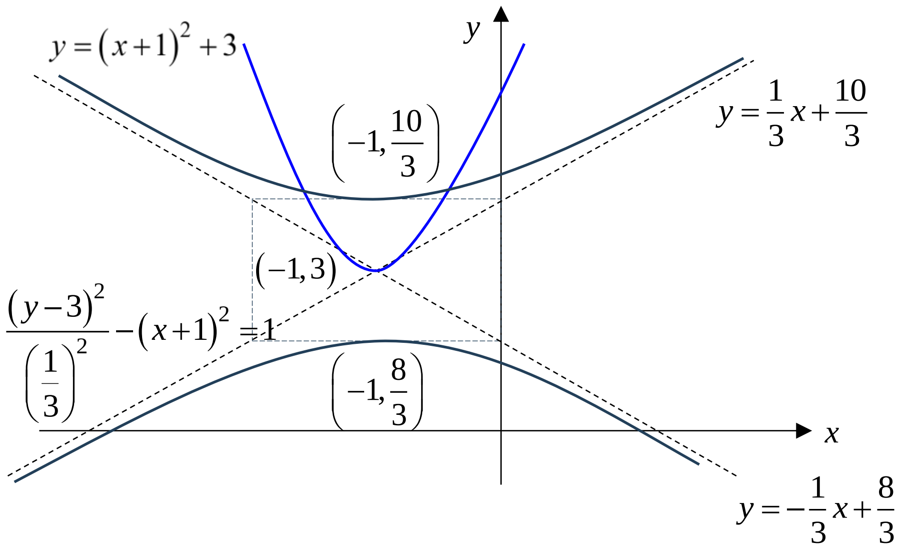
$(3k-9)^2-1<0$

$(3k-9+1)(3k-9-1)<0$

$\frac{8}{3}<k<\frac{10}{3}$

Thus y cannot lie between $\frac{8}{3}$ and $\frac{10}{3}$.

(ii)



(iii) By adding the graph of $y=(x+1)^2+3$, since there are 2 intersections between the 2 curves, so the equation $9\left[(x+1)^2+3\right]^2-54\left[(x+1)^2+3\right]-x^2-2x+79=0$ will have 2 real roots.

8 **(a)** $y=\left|4+2x-x^2\right|,\ x\in\mathbb{R},\ x\geq 3.5$

$\therefore y=-\left(4+2x-x^2\right)\quad (\because 4+2x-x^2<0 \text{ for } x\geq 3.5)$

$y=x^2-2x-4$

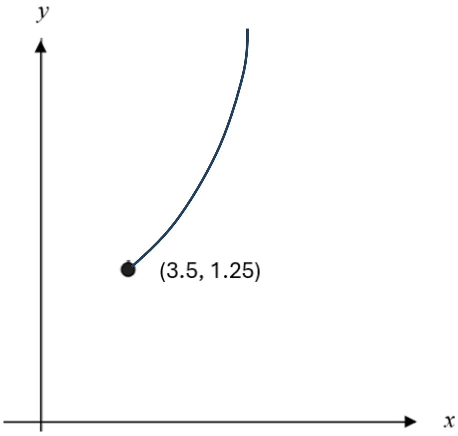
$y=(x-1)^2-1-4$

$y=(x-1)^2-5$

$x=1+\sqrt{y+5}$ or $x=1-\sqrt{y+5}$ $(\text{rej } \because x\geq 3.5)$

$x=1+\sqrt{y+5}$

$f^{-1}(x)=1+\sqrt{x+5}$



$R_f = [1.25, \infty)$

$\therefore D_{f^{-1}} = [1.25, \infty)$

(b) $f^{-1}(x) = f(x),\ x \in D_f \cap D_{f^{-1}}$

$x = f(x),\ x \in [3.5,\ \infty)$

$x = x^2 - 2x - 4$

If equate f^{-1} found in **(a)** to f, can also give the

$x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = -1$ (rej $\because x \geq 3.5$) or $x = 4$

(c) $R_g = \left[4 + e^{-a}, \infty\right)$

$D_f = [3.5, \infty)$

Since $e^{-a} > 0$, $\therefore 4 + e^{-a} > 3.5$

$\therefore R_g \subset D_f$, fg exists.

$R_g = \left[4 + e^{-a}, \infty\right)$

$$f(4 + e^{-a}) = (4 + e^{-a})^2 - 2(4 + e^{-a}) - 4$$
$$= 16 + 8e^{-a} + e^{-2a} - 8 - 2e^{-a} - 4$$
$$= 4 + 6e^{-a} + e^{-2a}$$

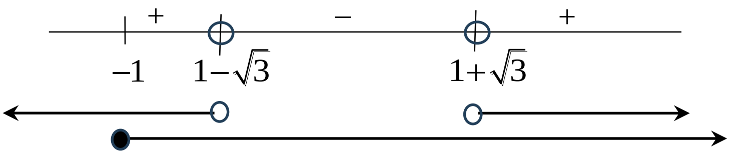
$R_{fg} = \left[4 + 6e^{-a} + e^{-2a}, \infty\right)$

(d) $\dfrac{g(x)}{x^2 - 2x - 2} \geq 0$ **AND** $x \in D_g$

$\dfrac{4 + e^{ax}}{(x-1)^2 - 3} \geq 0$ **AND** $x \geq -1$

Since $4 + e^{ax} > 0,\ \forall x \in \mathbb{R}$

$\therefore \dfrac{1}{(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})} \geq 0$ **AND** $x \geq -1$



$\left(x < 1 - \sqrt{3} \text{ or } x > 1 + \sqrt{3}\right)$ **AND** $x \geq -1$

$-$ for $x < 1-\sqrt{3}$ or $x > 1+\sqrt{3}$

$\therefore -1 \le x < 1-\sqrt{3}$ **OR** $x > 1+\sqrt{3}$

9 **(ai)** $z_1 = \left(\frac{1+\mathrm{i}}{1-\mathrm{i}}\right)\times\left(\frac{1+\mathrm{i}}{1+\mathrm{i}}\right) = \frac{1}{2}(1+2\mathrm{i}-1) = \mathrm{i} = \mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}$

$z_2 = \cos\left(\frac{\pi}{4}\right) + \mathrm{i}\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\mathrm{i}$

$z_1 + z_2 = \mathrm{e}^{\frac{\pi}{2}\mathrm{i}} + \mathrm{e}^{\frac{\pi}{4}\mathrm{i}} = \mathrm{e}^{\frac{1}{2}\left(\frac{\pi}{2}\mathrm{i}+\frac{\pi}{4}\mathrm{i}\right)}\left(\mathrm{e}^{\frac{\pi}{8}\mathrm{i}} + \mathrm{e}^{-\frac{\pi}{8}\mathrm{i}}\right)$

$= 2\cos\frac{\pi}{8} \times \mathrm{e}^{\frac{3\pi}{8}\mathrm{i}}$

(ii) $z_1 + z_2 = \frac{1}{\sqrt{2}} + \left(1+\frac{1}{\sqrt{2}}\right)\mathrm{i}$

$\tan\frac{3\pi}{8} = \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}$

$= \frac{\frac{\sqrt{2}+1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}+1$ (shown)

(bi) Method 1

$w = \cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\theta\mathrm{i}}$

$1-w^2 = 1-\mathrm{e}^{2\theta\mathrm{i}} = \mathrm{e}^{0\mathrm{i}} - \mathrm{e}^{2\theta\mathrm{i}}$

$= \mathrm{e}^{\theta\mathrm{i}}\left(\mathrm{e}^{-\theta\mathrm{i}} - \mathrm{e}^{\theta\mathrm{i}}\right)$

$= w(\cos\theta - \mathrm{i}\sin\theta - \mathrm{i}\sin\theta - \cos\theta)$

$= w(-2\mathrm{i}\sin\theta)$

$= -2\mathrm{i}w\sin\theta$

Method 2

$1-w^2 = 1-(\cos\theta+\mathrm{i}\sin\theta)^2$

$= 1-\cos^2\theta + \sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$

$= 2\sin^2\theta - 2\mathrm{i}\sin\theta\cos\theta$

$= 2\sin\theta(\sin\theta - 2\mathrm{i}\cos\theta)$

$= -2\mathrm{i}\sin\theta(\cos\theta + \mathrm{i}\sin\theta)$

$= -2\mathrm{i}w\sin\theta$

(ii)

$|1-w^2| = |-2\mathrm{i}w\sin\theta| = |-2\sin\theta|\,|\mathrm{i}|\,|w|$

$= 2\sin\theta$

$$\arg\left(1-w^2\right)=\arg\left(-2\mathrm{i}w\sin\theta\right)$$
$$=\arg\left[\left(-2\sin\theta\right)\mathrm{i}\right]+\arg\left(w\right)$$
$$=-\frac{\pi}{2}+\theta$$

(iii) Method 1

$$\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=\left(\frac{2\sin\theta\mathrm{e}^{\left(-\frac{\pi}{2}+\theta\right)\mathrm{i}}}{\mathrm{e}^{\left(\frac{\pi}{2}-\theta\right)\mathrm{i}}}\right)^n=\left(2\sin\theta\right)^n\mathrm{e}^{n\left(-\pi+2\theta\right)\mathrm{i}}$$

$\because \sin\theta>0$ when $\theta=\frac{\pi}{5}$,

$$\therefore \arg\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=n\left(-\pi+2\theta\right)$$

Method 2

$$\arg\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n=n\left[\arg\left(1-w^2\right)-\arg\left(\mathrm{i}\right)-\arg w*\right]$$
$$=n\left(\theta-\frac{\pi}{2}-\frac{\pi}{2}-\left(-\theta\right)\right)$$
$$=n\left(2\theta-\pi\right)$$

Since $\left(\frac{1-w^2}{\mathrm{i}w*}\right)^n$ is real and negative, and sub in $\theta=\frac{\pi}{5}$

$$\therefore n\left(2\left(\frac{\pi}{5}\right)-\pi\right)=\pi+2k\pi, k\in\mathbb{Z}$$
$$-\frac{3}{5}n=\left(2k+1\right), k\in\mathbb{Z}$$
$$n=-\frac{5}{3}\left(2k+1\right), k\in\mathbb{Z}$$

From GC tables, when $k=-2,-5,-8$,
Smallest n = 5, 15, 25

10 **(a)** $\frac{n}{2}\left[10+\left(n-1\right)0.65\right]>\frac{7\left(1.04^n-1\right)}{1.04-1}$

$$\frac{n\left(9.35+0.65n\right)}{2}-175\left(1.04^n-1\right)>0$$

$n>13.396$

Least number of days = 14

(b) Total dist covered by A $=\frac{28}{2}\left[10+27\left(0.65\right)\right]=385.7$

Total dist covered by B $=\frac{7\left(1.04^{28}-1\right)}{1.04-1}=349.77$

Total number of bowls $=385+349=734$

(c) Let a be the distance athlete A will need to cover on the 1st day

$$\frac{28}{2}\left[2a+27(0.65)\right]\geq 400$$

$a\geq 5.5107$

Hence athlete A will need to run at least 5511 m

(d) Let n be the number of days where the distance covered is at most 10 km.

$$7(1.04)^{n-1}\leq 10$$

$n\leq 10.094$

$$\text{Total dist covered by B} = \frac{7\left(1.04^{10}-1\right)}{1.04-1}+10(18)=264.04$$

Number of bowls contributed by A from the run = 264

11 **(i)** $\dfrac{dx}{dt}=kx$

$$\int\frac{1}{x}\,dx=\int k\ dt$$

$$\ln|x|=kt+C$$

$$x=Ae^{kt},\ A=\pm e^{C}$$

When t = 0, x = 5

$\Rightarrow A=5$

When t = 30, x = 5120

$\Rightarrow 5120=5e^{30k}$

$$k=\frac{1}{30}\ln 1024=\frac{1}{3}\ln 2$$

$$x=5e^{\frac{t}{3}\ln 2}=5(2)^{\frac{t}{3}}$$

(ii) $\dfrac{dy}{dp}=a\left(6400y-y^2\right)$

$$\int\frac{1}{6400y-y^2}\,dy=\int a\ dp$$

$$\int\frac{1}{3200^2-(y-3200)^2}\,dy=\int a\ dp$$

$$\frac{1}{6400}\ln\left|\frac{3200+(y-3200)}{3200-(y-3200)}\right|=ap+C$$

$$\ln\left|\frac{y}{6400-y}\right|=6400ap+6400C$$

$$\frac{y}{6400-y}=Be^{6400ap}$$

$$y=\frac{6400Be^{6400ap}}{1+Be^{6400ap}}=\frac{6400}{1+De^{-6400ap}}$$

When p = 0, y = 5120

$$5120=\frac{6400}{1+D}$$

$$D = \frac{1}{4} \cdots\cdots (1)$$

When p = 30, y = 3200

$$3200 = \frac{6400}{1 + \frac{1}{4}e^{-192000a}}$$

$$a = \frac{-1}{192000}\ln 4$$

$$\therefore y = \frac{6400}{1 + \frac{1}{4}e^{\frac{p}{30}\ln 4}} = \frac{6400}{1 + 2^{-2}\left(2^{\frac{p}{15}}\right)}$$

$$y = \frac{6400}{1 + 2^{\left(\frac{p}{15} - 2\right)}} \text{, where } H = -2$$

(iii) As $p \to \infty, 2^{\left(\frac{p}{15} - 2\right)} \to \infty$ and $y \to 0$

Since eventually no one in the town will be infected by the virus, the vaccine produced is therefore effective.