RAFFLES INSTITUTION
2023 YEAR 6 PRELIMINARY EXAMINATION

CANDIDATE NAME

CLASS 23

MATHEMATICS 9758/01

Paper 1 **3 hours**

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and class on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

FOR EXAMINER'S USE						
Q1	Q2	Q3	Q4	Q5	Q6	Q7
4	4	4	5	8	9	10
Q8	Q9	Q10	Q11	Q12	Total	
10	10	11	12	13	100	

This document consists of ~~23~~ 6 printed pages ~~and 1 blank page~~.

RAFFLES INSTITUTION
Mathematics Department

1 Without using a calculator, solve the inequality

$$\frac{9}{1-x^2} < \frac{x+5}{x+1}.$$ [4]

2 Griffles popcorn is available in bags of 3 sizes: small, medium and large.

At the Griffles physical store, Amanda bought 3 small, 7 medium and 1 large bag of Griffles popcorn and paid a total of \$85.50 after enjoying a 5% discount off the usual selling price.

Beatrice found out that at the Griffles online store, for every 2 bags of the same size purchased at the usual selling price, a customer will receive an additional bag of the same size free. Beatrice paid \$18.50 less than Amanda at the Griffles online store and received 3 small bags, 7 medium bags and 1 large bag of Griffles popcorn, inclusive of the free bags.

Given that the usual selling price of a large bag of Griffles popcorn is 2.4 times that of a small bag of Griffles popcorn, write down and solve equations to find the usual selling price of a bag of Griffles popcorn for each of the sizes. [4]

3 The volume of a sphere with radius r is given by $\frac{4}{3}\pi r^3$ and the surface area of a sphere with radius r is given by $4\pi r^2$.

(a) A meteorite enters the Earth's atmosphere and starts burning up in such a way that its surface area is decreasing at a constant rate of 100 cm^2 per second. Assuming the meteorite is always spherical, find the rate at which the radius is changing when the radius is 5 cm. [2]

(b) Another meteorite enters the Earth's atmosphere and burns up at a rate such that its volume is decreasing at a rate proportional to its surface area. Assuming the meteorite is always spherical, show that the radius decreases at a constant rate. [2]

4 **(a)** Describe a sequence of transformations that will map the graph of $y = \mathrm{f}(x)$ onto the graph of $y = 1 - 2\mathrm{f}(x)$. [3]

(b) A point $P(a, b)$ is said to be R-invariant if it lies on both the graphs of $y = \mathrm{f}(x)$ and $y = \frac{1}{\mathrm{f}(x)}$.

Sketch the graph of a function $y = \mathrm{f}(x)$, which is defined for all real x, for which $\mathrm{f}'(x) < 0$ for all real x and there are no R-invariant points. [2]

5 The function f is defined by $\mathrm{f}: x \mapsto \dfrac{3x+2}{x-1}$, for $x \in \mathbb{R}$, $x > 1$.

(a) Find $\mathrm{f}^{-1}(x)$ and state its domain. [3]

(b) Sketch on the same diagram the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, giving the equations of any asymptotes. Hence find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [5]

6 Alice bakes a large pie and removes it from the oven at 1 pm. The temperature reading of the pie using a food thermometer is 200 °C at that instant.

She left the pie on the kitchen countertop to cool down. The temperature of the kitchen is kept at a constant 32 °C. The temperature of the pie t minutes after it is removed from the oven is θ °C. This temperature decreases at a rate proportional to the difference between the temperature of the pie and the temperature of the kitchen.

At 1.15 pm, Alice checks the temperature of the pie again and it is 180 °C.

(a) Find an exact expression for θ in terms of t, simplifying your answer. [5]

(b) Sketch a graph of θ against t. [2]

(c) Food safety experts caution that bacteria grow fastest in food within the temperature range of 5 °C to 60 °C, known as the temperature danger zone. Find the latest time to the nearest minute to safely store the pie to avoid this danger zone. [2]

7 **(a)** Find $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$.

(There is no need to express your answer as a single algebraic fraction.) [4]

(b) Explain why $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)}$ is a convergent series and state the sum to infinity. [2]

(c) Deduce $\displaystyle\sum_{r=4}^{n} \frac{1}{(2r+3)(2r+5)}$. [4]

[Turn over

8 **(a)** Use the substitution $u = 1 - \mathrm{e}^x$, or otherwise, to find $\displaystyle\int_1^2 \frac{\mathrm{e}^x}{\left(1-\mathrm{e}^x\right)^3}\,\mathrm{d}x$ in exact form.

(You do not need to simplify your answer.) [4]

(b) Find $\displaystyle\int_0^1 \tan^{-1} x \,\mathrm{d}x$ in exact form. [3]

Function f is defined by

$$\mathrm{f}(x) = \begin{cases} \tan^{-1} x, & \text{for } -1 \le x \le 1, \\ \dfrac{\mathrm{e}^x}{\left(1-\mathrm{e}^x\right)^3}, & \text{for } 1 < x < 2, \end{cases}$$

and $\mathrm{f}(x) = \mathrm{f}(x+3)$ for all real values of x.

(c) Find $\displaystyle\int_{-2}^2 \left|\mathrm{f}(x)\right| \mathrm{d}x$, leaving your answer in exact form. [3]

9 The curve C has equation

$$y = \frac{ax^2 + x}{x+b},$$

where a and b are positive constants such that $a \neq \dfrac{1}{b}$.

(a) Given that C has no stationary points, use differentiation to find the relationship between a and b. [4]

It is now given that $a = 2$ and $b = 1$.

(b) Sketch C, stating the equations of any asymptotes, coordinates of any turning points and of the points where C crosses the axes. [4]

(c) Hence find the volume of the solid formed when the finite region bounded by C and the x-axis is rotated completely about the x-axis. [2]

10 **Do not use a calculator in answering this question.**

(a) Solve the equation $z^3 - 4z^2 + 6z - 4 = 0$. [4]

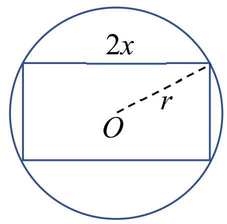
(b) Hence solve the equation $\mathrm{i}w^3 + 4w^2 - 6\mathrm{i}w - 4 = 0$, giving the roots in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta < 2\pi$. [4]

(c) For the value of w found in part **(b)** with the largest θ, find the smallest positive integer n such that w^n is a positive real number, showing your working clearly. [3]

11 One geometric optimization problem concerns finding the largest area axis-parallel rectangle inside an object.

The largest area rectangle problem has many industrial applications. For example, consider a piece of metal with a certain number of flaws in it. The piece of metal is cut into triangles and circles of equal area which does not contain any of the flaws. This problem can then be applied to find the maximum area of a rectangular sheet that can be contained in a circle and a triangle.

(a) A rectangle with width $2x$ units is inscribed inside a circle with centre O and radius r units as shown in the diagram below.

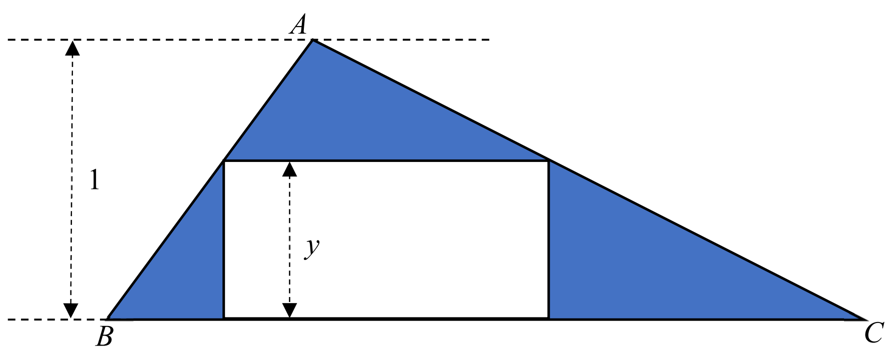


(i) Show that the area A units^2 of the rectangle can be expressed as $A = 4x\sqrt{r^2 - x^2}$. [1]

(ii) Using differentiation, determine the area of the largest rectangle that can be inscribed in a circle of radius r units. [5]

(iii) Hence state the area of the largest rectangle that can be inscribed in a circle of area π unit^2. [1]

(b) The diagram below shows a triangle ABC of height 1 unit.



Let y units be the height of the unshaded rectangle with 2 sides parallel to BC.
Let S denote the total areas of the three smaller shaded triangles and T denote the area of the triangle ABC.

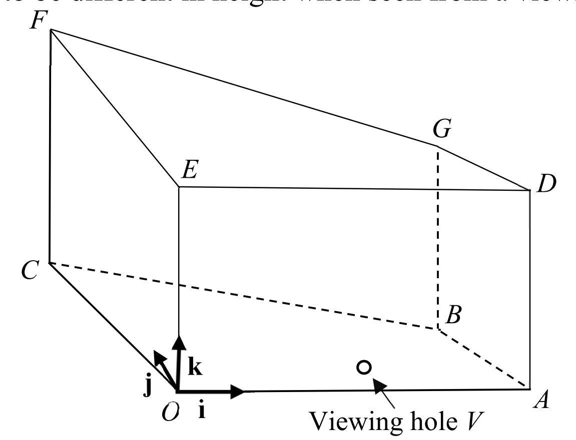
(i) Show that $S = \left(y^2 + (1-y)^2\right)T$. [2]

(ii) Determine the area of the largest rectangle that can be inscribed in a triangle of area π unit^2, showing your working clearly. [3]

[Turn over

12 Diagram 1 below shows a model created to study the Ames Room illusion, where identical objects are perceived to be different in height when seen from a viewing hole, V.

Diagram 1

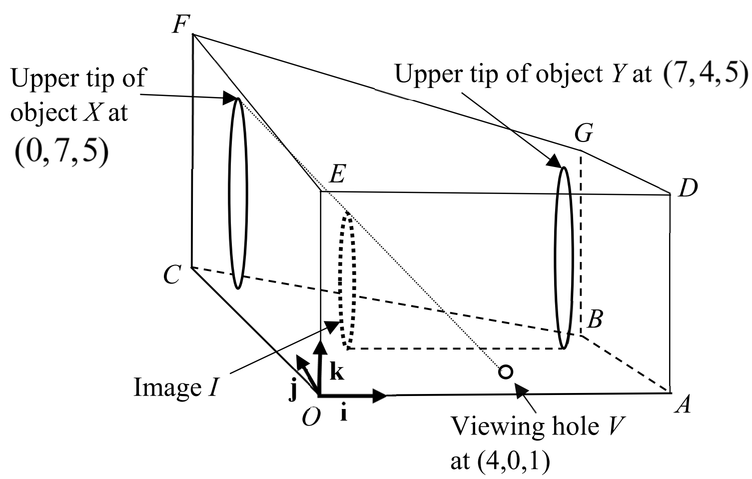


The front facing wall of the room is modelled by the rectangle $OADE$ which lies on the x-z plane, where O is the origin, and the floor $OABC$ lies on the x-y plane. OA and OE are parallel to the unit vectors $\mathbf{i}$ and $\mathbf{k}$ respectively, with $OA = 8$ units and $OE = 5$ units.
The square $ABGD$ is perpendicular to both $OABC$ and $OADE$, with $AB = 5$ units.

Given that $\overrightarrow{OC} = -\mathbf{i} + 8\mathbf{j}$, $\overrightarrow{OF} = -\mathbf{i} + 8\mathbf{j} + 8\mathbf{k}$ and $\overrightarrow{CF} = 8\mathbf{k}$,

(a) show that triangle EFG lies on the plane $5x - 8y + 23z = 115$. [4]

Diagram 2



A vertically upright object X is placed in the model with its upper tip at $(0, 7, 5)$. Object Y, also vertically upright and identical to X, is placed in the model with its upper tip at $(7, 4, 5)$. Their bottom tips touch the floor $OABC$ and the viewing hole V is at $(4, 0, 1)$.

Instead of seeing object X at its actual position, the illusion causes the viewer to see a corresponding image I which is also vertically upright and has its bottom tip touching the floor $OABC$ (shown in dotted outline in Diagram 2 above).
It is further given that the image I and object Y lie on the plane $y = 4$.

(b) Find a vector equation of the line of sight connecting V and the upper tip of object X and hence show that the ratio of the height of image I to the height of object Y is 23 : 35. [4]

(c) A point light source S is to be installed on edge BC, so that it is focused on the upper tip of object Y. Find the position vector $\overrightarrow{OS}$ so that the distance between S and the upper tip of object Y is minimised. [5]