Solutions (Sequencs and Series)

1(a)	First term =1, common difference = d S_5, S_{10}, S_{20} form a GP . $\Rightarrow \dfrac{S_{10}}{S_5} = \dfrac{S_{20}}{S_{10}}$ $\Rightarrow \left(\dfrac{10}{2}(2+9d)\right)^2 = \dfrac{5}{2}(2+4d).\dfrac{20}{2}(2+19d)$ $\Rightarrow (2+9d)^2 = (2+4d).(2+19d)$ $\Rightarrow 5d^2 - 10d = 0$ $\Rightarrow d = 2$ or $d = 0$ (rejected as AP is increasing) $\dfrac{2S_n}{S_{n+1} - 100} > 1$ $\dfrac{2n^2}{(n+1)^2 - 100} > 1$ $\dfrac{2n^2 - (n+1)^2 + 100}{(n+1)^2 - 100} > 0$ $\dfrac{n^2 - 2n + 99}{(n+1)^2 - 100} > 0$ Since $n^2 - 2n + 99 > 0$ as discriminant < 0 and coefficient of n^2 is +ve. $(n+1)^2 - 100 > 0$ $(n+1+10)(n+1-10) > 0$ $(n+11)(n-9) > 0$ $n > 9$ $\therefore$ least n is 10.
(b)	

nth month	Outstanding amt owed at the start of the month (in hundreds)	Outstanding amt owed at the end of the nth month (in hundreds)
1	34	34
2	(34)2	(34)2
3	$(34)2^2$	$34(2)^2$-70
4	$2\left(34(2)^2 - 70\right)$	$34(2)^3 - 70(2) - 70$
5	$2\left(34(2)^3 - 70(2) - 70\right)$	$34(2)^4 - 70(2)^2 - 70(2) - 70$
n		$34(2)^{n-1} - 70(2)^{n-3} - \dots\dots\dots - 70(2) - 70$

Total amount of money owed at the end of nth month

	$= 100(34(2)^{n-1} - 70(2)^{n-3} - .... - 70(2) - 70)$ $= 100\left(34(2)^{n-1} - \dfrac{70((2)^{n-2} - 1)}{2-1}\right)$ $= 100\left(70 - 2^{n-1}\right)$ To be free from debt, $70 - 2^{n-1} \le 0$ $\Rightarrow 2^n \ge 140$ $\Rightarrow n \ge \dfrac{\ln 140}{\ln 2} = 7.129$ Least $n = 8$ Earliest month to be free from debt = Jan 2014

2(i)	Length of film in the 1st layer = 42π Length of film in the 2nd layer = $(42 + 2x)\pi$ Length of film in the 3rd layer = $(42 + 4x)\pi$ $\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$ Length of film in the nth layer = $[42 + 2x(n - 1)]\pi = 124\pi$ $2x(n-1) = 82$ $\therefore \quad x(n-1) = 41$ (shown)
2(ii)	$S_n = 16766\pi$ mm, $a = 42\pi$ mm, $l = 124\pi$ mm Using $S_n = \dfrac{n}{2}(a+l)$ $n = \dfrac{2S_n}{a+l} = \dfrac{2(16766\pi)}{42\pi + 124\pi} = 202$ Using (i), $x(n-1) = 41$, $x = \dfrac{41}{n-1} = 0.20398$ $x = 0.204$ (to 3 s.f.)
2(iii)	The sum of the first 10 terms = k (the sum of the first 5 terms) $\dfrac{a(1-r^{10})}{1-r} = k \times \dfrac{a(1-r^5)}{1-r}$ $1 - r^{10} = k(1-r^5)$ $(1-r^5)(1+r^5) = k(1-r^5)$

	Since $r \neq 1,\ 1-r^5 \neq 0$ Therefore, $k=1+r^5$ (shown)
2(iv)	Since k =33 and a = 50, $1+r^5=33$ $r^5=32$ $r=2$ Suppose the roll of film is cut into n pieces. The sum of the first n terms ≤ total length of the film $\dfrac{50(2^n-1)}{2-1} \leq 52580$ $2^n-1 \leq 1051.6$ $2^n \leq 1052.6$ $n\ln 2 \leq \ln 1052.6$ $n \leq 10.03974$ $\Rightarrow$ the largest possible value of n is 10. Hence the largest number of pieces is 10 pieces.

3	$U_{2n-1}=ar^{2n-2}$.
(i)	Given that $U_1+U_3+\cdots+U_{2n-1}=U_{2n}+U_{2n+1}+\cdots$ ie $a+ar^2+\cdots+ar^{2n-2}=ar^{2n-1}+ar^{2n}+ar^{2n+1}+\cdots$ $\Rightarrow \dfrac{a\left(1-\left(r^2\right)^n\right)}{1-r^2}=\dfrac{ar^{2n-1}}{1-r}$ $\Rightarrow \dfrac{1-r^{2n}}{(1+r)(1-r)}=\dfrac{r^{2n-1}}{1-r}$ $\Rightarrow 1-r^{2n}=r^{2n-1}(1+r)$ $\therefore\ 2r^{2n}+r^{2n-1}-1=0$ [Shown] When $n=5$, we have $2r^{10}+r^9-1=0$ So $r=-1$ (NA convergent series) or 0.892
(ii)	Now $V_n=\dfrac{1}{a}\left(\dfrac{1}{r}\right)^{n-1}$. Letting the nth term of the new sequence be T_n, we have $T_n=\ln\left(\dfrac{U_n}{V_n}\right)=\ln(U_n)-\ln(V_n)$

	$=\left(\ln a+(n-1)\ln r\right)-\left(-\ln a+(n-1)\ln\left(\frac{1}{r}\right)\right)$ $=2\ln a+2(n-1)\ln r$ $T_n-T_{n-1}=2\ln a+2(n-1)\ln r-2\ln a-2(n-2)\ln r$ $=2\ln r$, a constant Hence the new sequence is an arithmetic progression
	Alternative: $T_n=\ln\left(\frac{U_n}{V_n}\right)=\ln(U_n)-\ln(V_n)=\ln(U_n)-\left[-\ln(U_n)\right]=2\ln(U_n)$ $T_n-T_{n-1}=2\ln(U_n)-2\ln(U_{n-1})=2\left[\ln\left(\frac{ar^{n-1}}{ar^{n-2}}\right)\right]=2\ln r=\text{constant}$

4	(i) Total volume: 850 m^3 Volumes (in m^3) filled: 4, 4.5, 5, 5.5, 6, … … Assume that the container is completely filled at the *n*th pumping action. For the AP above, $S_n=\frac{n}{2}\left[2(4)+(n-1)(0.5)\right]\ge 850$ $n^2+15n-3400\ge 0$ ----- (*) Considering $n^2+15n-3400=0$, we have $n=\frac{-15\pm\sqrt{15^2-4(1)(-3400)}}{2(1)}$, i.e., $n\approx 51.2899$ or $n\approx -66.2899$ (or use G.C.) Hence from (*), $(n-51.2899)(n+66.2899)\ge 0$ $\Rightarrow n\le -66.2899$ (NA, since $n>0$) or $n\ge 51.2899$ $\therefore$ at the 52th pumping action, the container would be completely filled, with some liquid overflowing. $S_{52}=\frac{52}{2}\left[2(4)+51(0.5)\right]$ $=871$ $\therefore$ volume of liquid that overflows $=(871-850)$ $m^3=21 m^3$ **<u>Alternative</u>** $S_n=\frac{n}{2}\left[2(4)+(n-1)(0.5)\right]\ge 850$ Using G.C.,

n	$\frac{n}{2}[2(4)+(n$
51	841.5
52	871
53	901

$\therefore$ volume of liquid that overflows
$=(871-850)\ \text{m}^3$
$=21\,\text{m}^3$

(ii)
For machine A,
$$S_{50}=\frac{50}{2}\left[2(4)+49(0.5)\right]$$
$$=812.5$$

Volumes (in m^3) filled by B: $5,\ 5\left(\frac{5}{6}\right),\ 5\left(\frac{5}{6}\right)^2,\ \ldots$

For the GP above,
$$S_{10}=\frac{5\left(1-\left(\frac{5}{6}\right)^{10}\right)}{1-\frac{5}{6}}$$
$$=\frac{156941}{6239}\approx 25.1548\ \text{m}^3$$
$\therefore$ total volume filled after 10$^{\text{th}}$ pumping action by B
$=(812.5+25.1548)\ \text{m}^3$
$\approx 837.6548\ \text{m}^3$
$\approx 838\ \text{m}^3$

(b)
(For the GP above, since $|r|=\frac{5}{6}<1,$ sum to infinity exists.)

$$\text{Sum to infinity}=\frac{5}{\left(1-\frac{5}{6}\right)}$$
$$=30$$

Volume of container to be filled by machine B
$=(850-812.5)\ \text{m}^3=37.5\ \text{m}^3$
Since the theoretical maximum volume that B alone can fill is only 30 m^3, which is less than 37.5 m^3,
the container would never be completely filled.

	<u>Alternative:</u> Since the container can only be filled to a theoretical maximum total volume of 812.5 + 30 = 842.5 m^3, which is less than 850 m^3, the container would never be completely filled.

5(a) **(i)**	Note that the progression of **A** is 1,3,5,7,9,11, … (consists of all positive odd numbers) *n*th term of **G**, $T_n = \frac{a}{2}\left(3^n - 1\right) - \frac{a}{2}\left(3^{n-1} - 1\right)$ $= \frac{a}{2}\left(3^n - 3^{n-1}\right) = \frac{a}{2}(3^{n-1})(3-1) = a(3^{n-1})$ 3^{n-1} is odd for all $n \geq 1$. Hence, for $a(3^{n-1})$ to be a term in **A**, then $a = 2m-1$, $m \in Z^+$. That is, *a* may be any positive odd integer.
(ii)	Let the new progression be **H** : $2^{t_1}, 2^{t_2}, 2^{t_3}, 2^{t_4}, \ldots$ Let *d* be the common difference of **A**. Consider, $\frac{(n+1)\text{th term of } \mathbf{H}}{(n)\text{th term of } \mathbf{H}}$ $= \frac{H_{n+1}}{H_n} = \frac{2^{t_{n+1}}}{2^{t_n}} = 2^{t_{n+1} - t_n} = 2^d = 2^2 = 4$, which is a constant. Hence **H** is a geometric progression.
(b)(i)	Value of the machine after 5 years $= 70000(0.9)^5$ = \$41334.30.
(ii)	Maximum revenue that could possibly be generated by the machine $= \frac{14000}{1-(0.93)} = \$200000$.
	Consider at the nth year of operation, Total revenue + prevailing value − cost of new machine ≥ 40000 $\frac{14000(1-(0.93)^n)}{1-(0.93)} + 70000(0.9)^n - 70000 \geq 40000$ $20(1-(0.93)^n) + 7(0.9)^n - 7 \geq 4$ $20(0.93)^n - 7(0.9)^n \leq 9$ From G.C., $\begin{array}{c\|c} n & a_n \\ \hline \ldots & \ldots \\ 5 & 9.7803 \\ 6 & 9.2197 \\ 7 & 8.6859 \end{array}$ $\therefore n \geq 7$ Hence, the number of years the machine is in operation is 7.

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13) Let the AP be $a,\ a+d,\ \dots,\ a+4d$

$\frac{5}{2}(2a+4d) = 105 \Rightarrow 5a+10d = 105$

$a+2d = 21$ — (1)

$\frac{a+2d}{a} = \frac{a+3d}{a+2d}$ — (2)

Subt (1) into (2):

$\frac{21-2d+2d}{21-2d} = \frac{21-2d+3d}{21-2d+2d}$

$21^2 = (21-2d)(21+d)$

$-2d^2 + 21(-d) + 21^2 = 21^2$

$d(-2d-21) = 0$

$\therefore d = 0$ (rejected) or $-\frac{21}{2}$ # shown

$\therefore$ common ratio $r = \frac{21}{21-2(-\frac{21}{2})} = \frac{1}{2}$

since $|r|<1$, $\therefore$ GP is convergent.

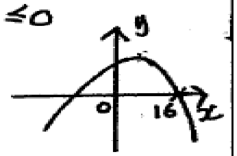
By (1): $a = 21-2(-\frac{21}{2}) = 42$. Hence, $S = \frac{42}{1-\frac{1}{2}} = 84$.

$A_n = \frac{n}{2}[2(42) + (n-1)(-\frac{21}{2})]$

$7\times 84 + \frac{n}{2}[84 + (n-1)(-\frac{21}{2})] \le 0$

$588 + 42n - \frac{21n^2}{4} + \frac{21}{4}n \le 0$

$-\frac{21}{4}n^2 + 47.25n + 588 \le 0$



From GC, least $n = 16$.

7(i)(a)

Total amount paid = 600 + 650 + 700 + ….

$a = 600, d = 50$

$S_n = \frac{n}{2}\left[2(600) + (n-1)50\right]$

$= n(600 + 25n - 25)$

$= 25n^2 + 575n$ (Shown)

Total paid after nth payment $= \$\left(25n^2 + 575n\right)$

(i)(b)

To complete fully repay her study loan,

Amount paid $\geq 30000 + 2500$

$25n^2 + 575n - 32500 \geq 0$

From GC, $n \leq -49.345$ or $n \geq 26.345$

It will take Emma 27 payments to fully repay her study loan.

(ii)(a)

Mth	Amount owed at start of the month	Amount owed at end of the month
Jan (n=1)	$30000 - x$	$30000 - x$

Feb (n=2)	$30000-2x$	$30000(1.01)-2x(1.01)$
Mar (n=3)	$30000(1.01)-2x(1.01)-x$	$30000(1.01^2)-2x(1.01^2)$ $-x(1.01)$
Apr	$30000(1.01^2)-2x(1.01^2)$ $-x(1.01)$ $-x$	$30000(1.01^3)-2x(1.01^3)$ $-x(1.01^2)$ $-x(1.01)$
…	…	…
n	$30000(1.01^{n-2})-2x(1.01^{n-2})$ $-x(1.01^{n-3})$ $-x(1.01^{n-4})$ ... $-x$	

Amount owed after 4 payments

$$=30000(1.01^2)-2x(1.01^2)-x(1.01)-x$$

(ii)(b) After n^{th} payments, amount owed

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-(1.01)^{n-3}x-(1.01)^{n-4}x-...-x$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-x\left[(1.01)^{n-3}+(1.01)^{n-4}+(1.01)^{n-5}+...+(1.01)^{0}\right]$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-x\left[\frac{1.01^{n-2}-1}{1.01-1}\right]$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-100x(1.01^{n-2}-1)$$

Alternative

After n^{th} payments, amount owed

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-(1.01)^{n-3}x-(1.01)^{n-4}x-...-(1.01)^{1}x-x$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-\left[(1.01)^{n-3}x+(1.01)^{n-4}x+...+(1.01)^{1}x\right]-x$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-x\left[\frac{1.01(1.01^{n-3}-1)}{1.01-1}\right]-x$$

$$=30000(1.01)^{n-2}-2x(1.01)^{n-2}-101x(1.01^{n-3}-1)-x$$

(ii)(c) For Emma to clear the debt, the amount that she owed after n^{th} payments ≤ 0

When $n=32$,

	$30000(1.01)^{n-2}-2x(1.01)^{n-2}-100x\left(1.01^{n-2}-1\right)\le 0$ $30000(1.01)^{32-2}-2x(1.01)^{32-2}-100x\left(1.01^{32-2}-1\right)\le 0$ $30000(1.01)^{30}-2x(1.01)^{30}-100x\left(1.01^{30}-1\right)\le 0$ $\Rightarrow x\ge \dfrac{30000(1.01)^{30}}{2(1.01)^{30}+100\left(1.01^{30}-1\right)}$ $x\ge 1078.837557$ Least value of x is \$1078.84
(iii)	Under plan A, total amount repaid $=30000+2500=32500$ Under plan B, total amount repaid $=1078.84\times 32=34522.88$ Therefore, it is cheaper for Emma to take up Plan A

8	**<u>Question 8(i)</u>** $\displaystyle\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=2^3-1^3$ $+3^3-2^3$ $+4^3-3^3$ $\vdots$ $+\quad (n)^3-(n-1)^3$ $+(n+1)^3-(n)^3$ $=(n+1)^3-1$ **<u>Question 8(ii)</u>** $(r+1)^3-r^3=r^3+3r^2+3r+1-r^3$ $=3r^2+3r+1$ $\displaystyle\sum_{r=1}^{n}\left((r+1)^3-r^3\right)=\sum_{r=1}^{n}\left(3r^2+3r+1\right)$ $\displaystyle(n+1)^3-1=3\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$ $\displaystyle n^3+3n^2+3n=3\sum_{r=1}^{n}r^2+3\left(\frac{n}{2}\right)(1+n)+n$

	$3\sum_{r=1}^{n} r^2 = n^3+3n^2+3n-\frac{3}{2}n-\frac{3}{2}n^2-n$ $\sum_{r=1}^{n} r^2 = \frac{1}{6}\left(2n^3+3n^2+n\right)$ $= \frac{n}{6}(n+1)(2n+1) = \frac{n}{6}\left(2n^2+3n+1\right)$ **<u>Question 8(iii)</u>** $\sum_{r=5}^{N}(r+2)^2 = \sum_{r=7}^{N+2} r^2$ $= \sum_{r=1}^{N+2} r^2 - \sum_{r=1}^{6} r^2$ $= \frac{N+2}{6}\left((N+2)+1\right)\left(2(N+2)+1\right)-\frac{6}{6}(6+1)\left(2(6)+1\right)$ $= \frac{1}{6}(N+2)(N+3)(2N+5)-91$ Alternatively $\sum_{r=5}^{N}(r+2)^2 = \sum_{r=5}^{N}(r^2+4r+4)$ $= \sum_{r=5}^{N} r^2 + \sum_{r=5}^{N}(4r+4)$ $= \sum_{r=1}^{N} r^2 - \sum_{r=1}^{4} r^2 + \sum_{r=5}^{N}(4r+4)$ $= \frac{N}{6}(N+1)(2N+1)-\frac{4}{6}(5)(9)+\frac{N-4}{2}\left(24+4N+4\right)$ $= \frac{N}{6}(N+1)(2N+1)-30+(N-4)(14+2N)$ $= \frac{N}{6}(2N^2+3N+1)-30+14N+2N^2-56$ $= \frac{N^3}{3}+\frac{5}{3}N^2+\frac{37}{6}N-86$

9(a)	$\mathrm{f}(r-1)-\mathrm{f}(r) = \frac{1}{(r-1)^2}-\frac{1}{r^2}, \quad r \neq 1$ $\frac{1}{(r-1)^2}-\frac{1}{r^2} = \frac{r^2-r^2+2r-1}{r^2(r-1)^2} = \frac{2r-1}{r^2(r-1)^2}$

(i)	$\dfrac{3}{2^2(1)^2}+\dfrac{5}{3^2(2)^2}+\dfrac{7}{4^2(3)^2}+\ldots\ldots=\sum_{r=2}^{\infty}\dfrac{2r-1}{r^2(r-1)^2}$ $\sum_{r=2}^{n} \mathrm{f}(r-1)-\mathrm{f}(r)=\dfrac{1}{1^2}-\dfrac{1}{2^2}$ $+\dfrac{1}{2^2}-\dfrac{1}{3^2}$ $+\dfrac{1}{3^2}-\dfrac{1}{4^2}$ $+\ldots$ $+\dfrac{1}{(n-2)^2}-\dfrac{1}{(n-1)^2}$ $+\dfrac{1}{(n-1)^2}-\dfrac{1}{n^2}$ $=1-\dfrac{1}{n^2}$ $\sum_{r=2}^{\infty}\dfrac{2r-1}{r^2(r-1)^2}=1$ as $n\to\infty$
(ii)	Since $\dfrac{2r-1}{r^2(r+1)^2}<\dfrac{2r-1}{r^2(r-1)^2}$, $\sum_{r=2}^{n}\dfrac{2r-1}{r^2(r+1)^2}<\sum_{r=2}^{n}\dfrac{2r-1}{r^2(r-1)^2}=1-\dfrac{1}{n^2}<1$ (since $0<\dfrac{1}{n^2}<1$) $\sum_{r=2}^{n}\dfrac{2r-1}{r^2(r+1)^2}<1$

10 (a) (i)	$v_n - v_{n+1}$ $=\dfrac{(n+3)}{n(n+1)(n+2)(n+3)}-\dfrac{(n+4)}{(n+1)(n+2)(n+3)(n+4)}$ $=\dfrac{1}{n(n+1)(n+2)}-\dfrac{1}{(n+1)(n+2)(n+3)}$ $=\dfrac{(n+3)-n}{n(n+1)(n+2)(n+3)}=\dfrac{3}{n(n+1)(n+2)(n+3)}=3u_n$, So $k=3$.
(a) (ii)	$\sum_{n=1}^{N}u_n=\dfrac{1}{3}\sum_{n=1}^{N}(v_n-v_{n+1})=\dfrac{1}{3}(v_1-v_2$ $+v_2-v_3$

	$+v_3 - v_4$ $+\ldots$ $+v_N - v_{N+1})$ $= \frac{1}{3}\left(v_1 - v_{N+1}\right)$ $\sum_{n=1}^{N} u_n = \frac{1}{3}\left((1+3)u_1 - (N+4)u_{N+1}\right)$ $= \frac{1}{3}\left(\frac{4}{(1)(2)(3)(4)} - \frac{(N+4)}{(N+1)(N+2)(N+3)(N+4)}\right)$ $= \frac{1}{3}\left(\frac{1}{(1)(2)(3)} - \frac{1}{(N+1)(N+2)(N+3)}\right)$ $= \frac{1}{3}\left(\frac{1}{3!} - \frac{N!}{(N+3)!}\right)$

11	$u_{r-1} - u_{r+1} = \frac{1}{(r-1)^3} - \frac{1}{(r+1)^3} = \frac{(r+1)^3 - (r-1)^3}{\left[(r-1)(r+1)\right]^3}$ $= \frac{(r^3+3r^2+3r+1) - (r^3-3r^2+3r-1)}{(r^2-1)^3}$ $= \frac{6r^2+2}{(r^2-1)^3}$ (shown)

	$\sum_{r=2}^{n}\frac{3r^2+1}{\left(r^2-1\right)^3}=\frac{1}{2}\sum_{r=2}^{n}\frac{6r^2+2}{\left(r^2-1\right)^3}$ $=\frac{1}{2}\sum_{r=2}^{n}\left[u_{r-1}-u_{r+1}\right]$ $=\frac{1}{2}\begin{bmatrix} +u_1 & -\cancel{u_3} \\ +u_2 & -\cancel{u_4} \\ +\cancel{u_3} & -\cancel{u_5} \\ \vdots & \vdots \\ +\cancel{u_{n-3}} & -\cancel{u_{n-1}} \\ +\cancel{u_{n-2}} & -u_n \\ +\cancel{u_{n-1}} & -u_{n+1} \end{bmatrix}$ $=\frac{1}{2}\left[u_1+u_2-u_n-u_{n+1}\right]$ $=\frac{1}{2}\left[1+\frac{1}{8}-\frac{1}{n^3}-\frac{1}{\left(n+1\right)^3}\right]=\frac{1}{2}\left[\frac{9}{8}-\frac{1}{n^3}-\frac{1}{\left(n+1\right)^3}\right]$
	$\sum_{r=3}^{n}\frac{3(r-1)^2+1}{\left[(r-1)^2-1\right]^3}$ $=\sum_{s=2}^{n-1}\frac{3s^2+1}{\left(s^2-1\right)^3}$ Subt. $r-1=s$. When $r=3$, $s=2$.When $r=n$, $s=n-1$ $=\frac{1}{2}\left[\frac{9}{8}-\frac{1}{(n-1)^3}-\frac{1}{n^3}\right]$ As $n\to\infty$, $\frac{1}{n^3}\to 0$ and $\frac{1}{\left(n-1\right)^3}\to 0$. $\sum_{r=3}^{\infty}\frac{3(r-1)^2+1}{\left[(r-1)^2-1\right]^3}=\frac{1}{2}\left[\frac{9}{8}-0-0\right]=\frac{9}{16}$

12	$\frac{1}{r\left(r-2\right)}=\frac{1}{2}\left(-\frac{1}{r}+\frac{1}{r-2}\right)$ $=\frac{1}{2}\left(\frac{1}{r-2}-\frac{1}{r}\right)$. $A=-\frac{1}{2}$, $B=\frac{1}{2}$.

	$\sum_{r=3}^{n}\frac{1}{r(r-2)}=\frac{1}{2}\sum_{r=3}^{n}\left(\frac{1}{r-2}-\frac{1}{r}\right)$ $=\frac{1}{2}\left[\left(1-\cancel{\frac{1}{3}}\right)+\left(\frac{1}{2}-\cancel{\frac{1}{4}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{5}}\right)+\ldots+\left(\cancel{\frac{1}{n-4}}-\cancel{\frac{1}{n-2}}\right)+\left(\cancel{\frac{1}{n-3}}-\frac{1}{n-1}\right)+\left(\cancel{\frac{1}{n-2}}-\frac{1}{n}\right)\right]$ $=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n-1}-\frac{1}{n}\right)$. as $n\to\infty$, $\frac{1}{n-1}-\frac{1}{n}\to 0$. Hence $\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n-1}-\frac{1}{n}\right)\to\frac{3}{4}$. Therefore $\sum_{r=3}^{\infty}\frac{1}{r(r-2)}=\frac{3}{4}$. $\frac{1}{3\times5}+\frac{1}{4\times6}+\frac{1}{5\times7}+\ldots\ldots=\sum_{r=5}^{\infty}\frac{1}{r(r-2)}$ $=\sum_{r=3}^{\infty}\frac{1}{r(r-2)}-\frac{1}{3(1)}-\frac{1}{4(2)}$ $=\frac{3}{4}-\frac{1}{3(1)}-\frac{1}{4(2)}$ $=\frac{7}{24}$.

13(a)	Given $S_n=n^2-2n$. $S_n-S_{n-1}=n^2-2n-\left((n-1)^2-2(n-1)\right)$ $=n^2-2n-(n-1)^2+2n-2$ $=2n-3$ Since the progression is AP, common difference, $d=T_n-T_{n-1}$ $=2n-3-\left(2(n-1)-3\right)$ $=2$. OR $d=T_2-T_1$ $=1-(-1)$ $=2$
(b)(i)	Let d be the distance driven for every turn. $n=1,\ d=k\theta$

	$n=2$, $d=\left(\frac{8}{10}\right)k\theta$ or $0.8k\theta$ $n=3$, $d=\left(\frac{8}{10}\right)^2 k\theta$ or $0.64k\theta$
(ii)	Total distance driven into the wall after n turns $=k\theta+\left(\frac{8}{10}\right)k\theta+\left(\frac{8}{10}\right)^2 k\theta+...+\left(\frac{8}{10}\right)^{n-1} k\theta$ $=k\theta\left(1+\left(\frac{8}{10}\right)+\left(\frac{8}{10}\right)^2+...+\left(\frac{8}{10}\right)^{n-1}\right)$ $=k\theta\left(\frac{1-\left(\frac{8}{10}\right)^n}{1-\left(\frac{8}{10}\right)}\right)$ $=5k\theta\left(1-\left(\frac{4}{5}\right)^n\right)$ $a=5$, $b=\frac{4}{5}$
(iii)	Distance driven in the long run $=\lim\limits_{n\to\infty}5k\theta\left(1-\left(\frac{4}{5}\right)^n\right)$ $=5k\theta$ Given $k=2$, minimum length of the metal screw is 10θ unit.

14	(a) GP : a = a, r = $\frac{b}{a}$ The sum to infinity of the remaining terms, $S_\infty=\frac{ar^n}{1-r}$ $S_n=2S_\infty$ $\frac{a\left(1-r^n\right)}{1-r}=\frac{2ar^n}{1-r}$ $1-r^n=2r^n$ $3r^n=1$ $r^n=\frac{1}{3}$ $\left(\frac{b}{a}\right)^n=\frac{1}{3}$ $3b^n=a^n$

	(b) (i) $S_{10}=105+10u_5$ $\frac{10}{2}\left[2a+9d\right]=105+10\left[a+4d\right]$ $10a+45d=105+10a+40d$ $d=21$ (ii) $u_{25}\le 542$ and $u_{26}>542$ $a+24(21)\le 542$ and $a+25(21)>542$ $a\le 38$ and $a>17$ $\therefore 17<a\le 38$

15	$\frac{1}{4n^2-1}=\frac{1}{(2n-1)(2n+1)}$ $=\frac{1}{2(2n-1)}+\frac{1}{-2(2n+1)}=\frac{1}{2}\left[\frac{1}{2n-1}-\frac{1}{2n+1}\right]$ $S_N=\sum_{n=1}^{N}a_n=\sum_{n=1}^{N}\left(2^{1-n}-2^{2-n}+\frac{2}{4n^2-1}\right)$ $=\sum_{n=1}^{N}\left(2^{1-n}-2^{2-n}+\frac{1}{2r-1}-\frac{1}{2r+1}\right)$ $=$ $2^{-1}-2^{0}+\frac{1}{1}-\frac{1}{3}$ $+$ $2^{-2}-2^{-1}+\frac{1}{3}-\frac{1}{5}$ $+$ $2^{-3}-2^{-2}+\frac{1}{5}-\frac{1}{7}$ $+$ $\cdots$ $+$ $2^{-N}-2^{1-N}+\frac{1}{2N-3}-\frac{1}{2N-1}$ $+$ $2^{1-N}-2^{2-N}+\frac{1}{2N-1}-\frac{1}{2N+1}$ $=2^{1-N}-2^{1}+1-\frac{1}{2N+1}$ $=2^{1-N}-1-\frac{1}{2N+1}$ $\lim_{N\to\infty}S_N=0-1-0=-1$ since $2^{1-N}\to 0,\frac{1}{2N+1}\to 0$ as $N\to\infty$

<table>
<tr><td>16</td><td>Given $S_n = \frac{1}{a}\left[1-(a-1)^n\right]$,
$T_n = S_n - S_{n-1}$, $n \geq 2$
$= \frac{1}{a}\left[1-(a-1)^n\right] - \frac{1}{a}\left[1-(a-1)^{n-1}\right]$
$= -\frac{1}{a}(a-1)^n + \frac{1}{a}(a-1)^{n-1}$
$= \frac{1}{a}(a-1)^{n-1}(-a+1+1)$
$= \frac{1}{a}(a-1)^{n-1}(2-a)$
$T_n = \frac{1}{a}(a-1)^{n-1}(2-a), \quad n \in \mathbb{Z}^+$
$\frac{T_n}{T_{n-1}} = = \frac{\frac{1}{a}(a-1)^{n-1}(2-a)}{\frac{1}{a}(a-1)^{n-2}(2-a)} = (a-1) = \text{constant}$ (Shown)</td></tr>
<tr><td></td><td>(i) the total number of terms in the first n brackets
$=1+3+5+\cdots(1+(n-1)2)$
$=1+3+5+\cdots(2n-1)$
$= \frac{n}{2}(1+(2n-1)) = n^2$</td></tr>
<tr><td></td><td>(ii) number of terms in the 11th bracket = 1 + (11-1)2 = 21 terms
(middle term will be the 11th term)
Number of term from 1st bracket to 10th bracket = 10^2 = 100
$T_{111} = \frac{1}{a}(a-1)^{110}(2-a)$</td></tr>
<tr><td></td><td>(iii) For the sum to infinity of the series to exist, $|a-1|<1$
$0<a<2$

Method 1:
when $a = \frac{39}{20}$ (i.e. $|a-1|<1$),
sum to infinity of the series $= \frac{1}{a}\left[1-0\right] = \frac{1}{a}$

For the sum of all the terms in the first n brackets to be within 0.1% of the sum to infinity of the series,
$\left|S_{n^2} - S_\infty\right| < 0.1\% S_\infty$</td></tr>
</table>

	$\left\lvert \frac{1}{a}\left[1-(a-1)^{n^2}\right]-\frac{1}{a}\right\rvert < 0.1\% \frac{1}{a}$ $\frac{1}{a}\left\lvert -(a-1)^{n^2}\right\rvert < 0.001(\frac{1}{a})$ $(a-1)^{n^2} < 0.001 \quad \Rightarrow (\frac{19}{20})^{n^2} < 0.001$ Using GC, $n > 11.605$ At least 12 brackets. Method 2: (iii) when $a=\frac{39}{20}$, first term $=\frac{2-a}{a}=\frac{1}{39}$, $r=(a-1)=\frac{19}{20}$ sum to infinity of the series $=\frac{\frac{1}{39}}{1-\frac{19}{20}}=\frac{20}{39}$ the sum of all the terms in the first n brackets = the sum of n^2 terms in the GP $= \frac{1}{a}\left[1-(a-1)^{n^2}\right]=\frac{20}{39}-\frac{20}{39}(\frac{19}{20})^{n^2}$ For the sum of all the terms in the first n brackets to be within 0.1% of the sum to infinity of the series, $\left\lvert \frac{20}{39}-\frac{20}{39}(\frac{19}{20})^{n^2}-\frac{20}{39}\right\rvert < 0.1\% \frac{20}{39}$ $\left\lvert -\frac{20}{39}(\frac{19}{20})^{n^2}\right\rvert < 0.001\times\frac{20}{39}$ $(\frac{19}{20})^{n^2} < 0.001$ Using GC , $n > 11.605$ At least 12 brackets.

17(i)	common ratio = $\dfrac{a+4d}{a+8d}=\dfrac{a+d}{a+4d}$ $(a+4d)^2=(a+d)(a+8d)$ $a^2+8ad+16d^2=a^2+9ad+8d^2$ $8d^2-ad=0$ $d(8d-a)=0$ Since $d\neq 0,\ a=8d$. common ratio, r = $\dfrac{a+d}{a+4d}=\dfrac{9d}{12d}=\dfrac{3}{4}$ Since $\lvert r\rvert=\dfrac{3}{4}<1,$ the geometric series is convergent. Sum to infinity = $\dfrac{a+8d}{1-\frac{3}{4}}=\dfrac{2a}{\frac{1}{4}}=8a$
(ii)	$S_n<\dfrac{4}{5}S_\infty$ $\dfrac{2a\left(1-r^n\right)}{1-r}<\dfrac{4}{5}\cdot\dfrac{2a}{1-r}$ Since a > 0, 1 – r >0, $1-\left(\dfrac{3}{4}\right)^n<\dfrac{4}{5}$. $\left(\dfrac{3}{4}\right)^n>\dfrac{1}{5}$ $n<\dfrac{\ln(0.2)}{\ln(0.75)}=5.595$ Largest value of n is 5.

18	**(i)** common ratio r = $4-3x$ If series has a finite sum, $\lvert r\rvert<1\Rightarrow\lvert 4-3x\rvert<1$ $-1<4-3x<1$ $-5<-3x<-3$ $1<x<\dfrac{5}{3}$ (ii) From GC, minimum no of terms is 7

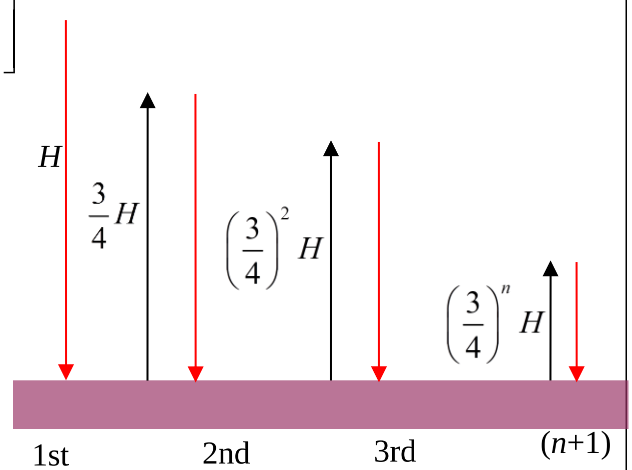
19	$\sum_{n=0}^{N} u_n = \sum_{n=0}^{N}\left(e^{-n} - e^{-n-1} + \frac{1}{n+2} - \frac{1}{n+1}\right)$ $= e^{-0} - e^{-1} + \frac{1}{2} - \frac{1}{1}$ $+ e^{-1} - e^{-2} + \frac{1}{3} - \frac{1}{2}$ $+ e^{-2} - e^{-3} + \frac{1}{4} - \frac{1}{3}$ +..... $+ e^{-N} - e^{-N-1} + \frac{1}{N+2} - \frac{1}{N+1} = \frac{1}{N+2} - e^{-N-1}$ $e^{N+1} = 1 + (N+1) + \frac{(N+1)^2}{2!} + ... > N+2$ $e^{-N-1} = \frac{1}{e^{N+1}} < \frac{1}{N+2} \Rightarrow \frac{1}{N+2} - e^{-N-1} > 0$ Hence $\sum_{n=0}^{N} u_n > 0$ Alternatively, sketch the graph of $y = \frac{1}{x+2} - e^{-x-1}$ for $x \geq 0$. The graph shows that $y > 0$ and hence $\sum_{n=0}^{N} u_n > 0$ for $N > 0$

20(a)	$P = \frac{n}{2}\left[2a + (n-1)d\right]$ $(2n+1)th$ term = 1st term of last n terms = $a + (2nd)$ $Q = \frac{n}{2}\left\{2\left[a + 2nd\right] + (n-1)d\right\}$ $Q - P = \frac{n}{2}\left[2a + 4nd + (n-1)d\right] - \frac{n}{2}\left[2a + (n-1)d\right] = 2n^2d$ ALTERNATIVE:

	$S_n = \frac{n}{2}\left[2a+(n-1)d\right] = P$ $S_{2n} = \frac{2n}{2}\left[2a+(2n-1)d\right]$ $S_{3n} = \frac{3n}{2}\left[2a+(3n-1)d\right]$ $Q-P = S_{3n} - S_{2n} - S_n$ $= 3an + \frac{3n}{2}(3n-1)d - \frac{2n}{2}\left[2a+(2n-1)d\right] - \frac{n}{2}\left[2a+(n-1)d\right]$ $= 2n^2 d$
(b)	When n =1: Fund has \$(1.035)(2500) – 150 n =2: \$(1.035)[(1.035)2500-150] – 150 $=(1.035)^2 2500-(1.035)150 - 150$ at nth year: $(1.035)^n 2500 - (1.035)^{n-1}150 - (1.035)^{n-2}150\ldots-150$ $=(1.035)^n 2500 - (150)[1+(1.035)+(1.035)^2+\ldots+(1.035)^{n-1}]$ $= (1.035)^n 2500 - 150\left(\frac{1.035^n - 1}{0.035}\right)$ $(1.035)^n 2500 - \frac{150}{0.035}(1.035)^n + \frac{30000}{7}$ $(1.035)^n\left[2500 - \frac{30000}{7}\right] + \frac{30000}{7}$ $= -\frac{12500}{7}(1.035)^n + \frac{30000}{7}$ $\frac{30000}{7} - \frac{12500}{7}(1.035)^n \ge 0$ $(1.035)^n \le \frac{30000}{12500}$ $n \le 25.44$ $n = 25 years$ Last year is 2035.

21i	$\frac{1}{r(r+1)(r+2)} = \frac{1}{2}\left\{\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right\}$ $P = \frac{1}{2}$, $Q = -\frac{1}{2}$ Sum of the series = $\sum_{r=4}^{n}\frac{1}{2}\left\{\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}\right\}$

	$=\frac{1}{2}\left\{\begin{array}{l}\left(\frac{1}{4.5}-\frac{1}{5.6}\right)+\left(\frac{1}{5.6}-\frac{1}{6.7}\right)+\ldots\\+\left(\frac{1}{(n-1)n}-\frac{1}{n(n+1)}\right)+\left(\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)}\right)\end{array}\right\}$ $=\frac{1}{2}\left\{\frac{1}{20}-\frac{1}{(n+1)(n+2)}\right\}$
ii	As $n\to\infty$, $S_\infty=\frac{1}{2}\left\{\frac{1}{20}-\frac{\frac{1}{n^2}}{1+\frac{3}{n}+\frac{2}{n}}\right\}=\frac{1}{2}.\frac{1}{20}=\frac{1}{40}$ $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots=\frac{1}{6}+\frac{1}{24}+\frac{1}{60}+\sum_{r=4}^{\infty}\frac{1}{r(r+1)(r+2)}=\frac{9}{24}+\frac{1}{40}=\frac{1}{4}$

22	S_n $=4+2\left(\frac{3}{4}\right)4+2\left(\frac{3}{4}\right)^2 4+2\left(\frac{3}{4}\right)^3 4+\ldots\ldots+2\left(\frac{3}{4}\right)^n 4$ $=4+2(4)\left[\left(\frac{3}{4}\right)+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\ldots\ldots+\left(\frac{3}{4}\right)^n\right]$ $=4+8\left(\frac{3}{4}\right)\frac{\left[1-\left(\frac{3}{4}\right)^n\right]}{1-\left(\frac{3}{4}\right)}$ $=4+24\left[1-\left(\frac{3}{4}\right)^n\right]$ $=28-24\left(\frac{3}{4}\right)^n$  **(ii)** $S_n=28-24\left(\frac{3}{4}\right)^n>24$ $\frac{1}{6}>\left(\frac{3}{4}\right)^n$ $n>\frac{\ln\left(\frac{1}{6}\right)}{\ln\left(\frac{3}{4}\right)}$

	$n > 6.23 \Rightarrow n = 7$ The ball must bounce at least 7 times for it to travel more than 24 m. **(iii)** $n \to \infty, \quad \left(\frac{3}{4}\right)^n \to 0, \quad S_\infty = 28$ Since sum to infinity is 28, the ball will not travel more than 28m.

Q23	**Suggested Solutions**
(i)	<table><tr><th>n</th><th>Length of n^{th} square</th></tr><tr><td>1</td><td>2</td></tr><tr><td>2</td><td>$\left(2^{\frac{1}{2}}\right)2 = 2^{\frac{3}{2}}$</td></tr><tr><td>3</td><td>$\left(2^{\frac{2}{2}}\right)2 = 2^{\frac{4}{2}}$</td></tr><tr><td>: .</td><td>: .</td></tr><tr><td>n</td><td>$2^{\frac{n+1}{2}}$</td></tr></table> The length of the n^{th} square is $2^{\frac{n+1}{2}}$ mm.
(ii)	$2^{\frac{n+1}{2}} < 210$ $\frac{n+1}{2} < \frac{\ln(210)}{\ln 2}$ $n < 14.428$ Hence maximum number of square is 14.
(iii)	From part **(i)**, length of the n^{th} square is $2^{\frac{n+1}{2}}$. Therefore, area of the n^{th} square $= \left(2^{\frac{n+1}{2}}\right)^2 = 2^{n+1}$. Area of the 1st square $= 2^2$ Area of the 4th square – Area of the 3rd square $= 2^5 - 2^4$ $= 2^4(2-1)$ $= 2^4$ Total shaded area in Figure 2 $= 2^2 + 2^4 = 20$
(iv)	<table><tr><th>n</th><th>Area of n^{th} square</th><th>Protruding area of n^{th} square</th></tr><tr><td>1</td><td>2^2</td><td>2^2</td></tr></table>

Q23	**Suggested Solutions**
	See below

2	2^3	$2^3-2^2=2^2(2-1)=2^2$	He will only shade up to the 28th square if he draws 30 squares.
3	2^4	$2^4-2^3=2^3(2-1)=2^3$	
4	2^5	2^4	
$\vdots$	$\vdots$	$\vdots$	
7	2^8	2^7	
$\vdots$	$\vdots$	$\vdots$	
28	2^{29}	2^{28}	

Total shaded area $=2^2+2^4+2^7+\ldots+2^{28}$

$$=4+\frac{2^4(2^{3(9)}-1)}{\left(2^3-1\right)}$$

$$=306,783,380\text{ mm}^2$$

$$=307\text{ m}^2\ \text{(3 s.f.)}.$$

24ai

$$r=\frac{a+d}{a+3d}=\frac{a+3d}{a+8d}$$

$\Rightarrow (a+d)(a+8d)=(a+3d)^2$

$\Rightarrow a^2+9ad+8d^2=a^2+6ad+9d^2$

$\Rightarrow d^2-3ad=0 \Rightarrow d(d-3a)=0$

$\Rightarrow d=0$ (rej.) or $d=3a$ (shown)

Alternatively, let b be the first term of the geometric series. Then

$$d=\frac{b-br}{5}=\frac{br-br^2}{2}$$

$\Rightarrow 2b-2br=5br-5br^2$

$\Rightarrow 5r^2-7r+2=0 \Rightarrow (5r-2)(r-1)=0$

$\Rightarrow r=\frac{2}{5}$ or $r=1$ (rej because otherwise $d=0$)

Hence

$$d=\frac{b-b\left(\frac{2}{5}\right)}{5}=-\frac{3}{25}b=-\frac{3}{25}(a+8d)$$

$25d=-3a-24d$

$d=3a$ (shown)

aii	$r=\dfrac{a+d}{a+3d}=\dfrac{a+3a}{a+9a}=\dfrac{4a}{10a}=\dfrac{2}{5}$. Since $\left\|\dfrac{2}{5}\right\|<1,$ the geometric series is convergent. Sum to infinity $=\dfrac{a+8d}{1-r}$ $=\dfrac{a+24a}{1-\frac{2}{5}}$ $=\dfrac{5}{3}(25a)$ $=\dfrac{125}{3}a$
bi	The distance the mountaineer climbs for each hour follows an arithmetic progression with first term 300 metres and common difference (– 10) metres. Total distance travelled after n hours $\le x$ $\dfrac{n}{2}\left[2(300)+(n-1)(-10)\right]\le x$ $\dfrac{n}{2}(600-10n+10)\le x$ $\dfrac{n}{2}(610-10n)\le x$ $n(305-5n)\le x$ $-5n^2+305n\le x$ (shown) $p=-5,\ q=305$
bii	If x = 2500, then $-5n^2+305n\le 2500$ $-5n^2+305n-2500\le 0$ $n\le 9.757$ or $n\ge 51.24$ Hence $n=9$.

25 (a)	$\mathrm{f}(r+1)-\mathrm{f}(r)=\dfrac{r+1}{2^{r+1}}-\dfrac{r}{2^r}$ $=\dfrac{r+1-2r}{2^{r+1}}$ $=\dfrac{1-r}{2^{r+1}}$

	$\sum_{r=1}^{n}\frac{1-r}{2^{r+1}} = \sum_{r=1}^{n}\left[\mathrm{f}(r+1)-\mathrm{f}(r)\right]$ $= \left[\begin{array}{l} \ \ \cancel{\mathrm{f}(2)}-\mathrm{f}(1) \\ +\cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(2)} \\ +\cancel{\mathrm{f}(4)}-\cancel{\mathrm{f}(3)} \\ \qquad \vdots \\ +\cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n-2)} \\ +\cancel{\mathrm{f}(n)}-\cancel{\mathrm{f}(n-1)} \\ +\mathrm{f}(n+1)-\cancel{\mathrm{f}(n)} \end{array}\right]$ $= \mathrm{f}(n+1)-\mathrm{f}(1)$ $= \frac{n+1}{2^{n+1}}-\frac{1}{2}$
b(i)	$a_n = \frac{2^n n^x}{3^n}$ $\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\sqrt[n]{\frac{2^n n^x}{3^n}}$ $= \lim_{n\to\infty}\left(\frac{2\sqrt[n]{n^x}}{3}\right)$ $= \frac{2}{3}\lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)$ $= \frac{2}{3}(1) \qquad \text{since } \lim_{n\to\infty}\left(\sqrt[n]{n^x}\right)=1$ $= \frac{2}{3} < 1$ Since $\lim_{n\to\infty}\sqrt[n]{a_n} = \frac{2}{3} < 1$, by the Cauchy Test, the series converges for all real values of x
b(ii)	$\sum_{r=0}^{\infty}\frac{2^r r}{3^r} = 0+1\left(\frac{2}{3}\right)+2\left(\frac{2}{3}\right)^2+3\left(\frac{2}{3}\right)^3+4\left(\frac{2}{3}\right)^4+\ldots$ $= \frac{2}{3}\left[1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots\right]$ $= \frac{2}{3}\left(1-\frac{2}{3}\right)^{-2} \qquad \text{since } 1+2\left(\frac{2}{3}\right)+3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+\ldots=\left(1-\frac{2}{3}\right)^{-2} \text{ with } y=\frac{2}{3}$ $= 6$

26	**4 (i)**	$a_{50} = 99a_1$ $a_1 + 49d = 99a_1$ $a_1 = \frac{1}{2}(0.15) = 0.075$ (Shown)	
	4 (ii)	$\sum_{n=1}^{50} a_n = \frac{50}{2}(0.075 + 99 \times 0.075) = 187.5$	
	4 (iii)	$b_k < a_{25}$ $(99 \times 0.075)(0.98)^{k-1} < (0.075) + 24(0.15)$ $k > 35.8$ least $k = 36$	
	4 (iv)	Consider $\frac{b_1(1-0.98^h)}{1-0.98} > 0.99\frac{b_1}{1-0.98}$ $0.98^h < 0.01$ $h > 227.9$ least $h = 228$	
	4 (v)	$\sum_{m=0}^{\infty} b_{1+3m}$ $= b_1 + b_4 + b_7 + \ldots$ $= \frac{99 \times 0.075}{1-(-0.98)^3}$ $= 3.82$ (3s.f.)	

27

(a) Number of weeks needed to sell off all the chickens $n=\dfrac{1000}{k}$

$$\begin{aligned}\text{Total cost } &= (0.5)\left[(1000)+(1000-k)+(1000-2k)+\cdots+(k)\right]\\ &= (0.5)\left\{\frac{1}{2}\left(\frac{1000}{k}\right)\left[1000+k\right]\right\}\\ &= \$\,\frac{250}{k}(1000+k)\ \text{(shown)}\end{aligned}$$

(b) $$\begin{aligned}\text{Total earnings } &= 12k\left[1+(0.95)+(0.95)^2+\cdots+(0.95)^{\frac{1000}{k}-1}\right]\\ &= 12k\left[\frac{1-(0.95)^{\frac{1000}{k}}}{1-0.95}\right]\\ &= 240k\left(1-(0.95)^{\frac{1000}{k}}\right)\end{aligned}$$

For Anson to make a profit,

Total earnings $>$ Total cost

$$240k\left(1-(0.95)^{\frac{1000}{k}}\right)>\frac{250}{k}(1000+k)$$

$$\frac{24k^2}{25}\left(1-(0.95)^{\frac{1000}{k}}\right)-1000-k>0$$

Plot the graph of $y=\dfrac{24x^2}{25}\left(1-(0.95)^{\frac{1000}{x}}\right)-1000-x$.

From GC, $k>38.2$

If k is a positive integer and 1000 is divisible by k,

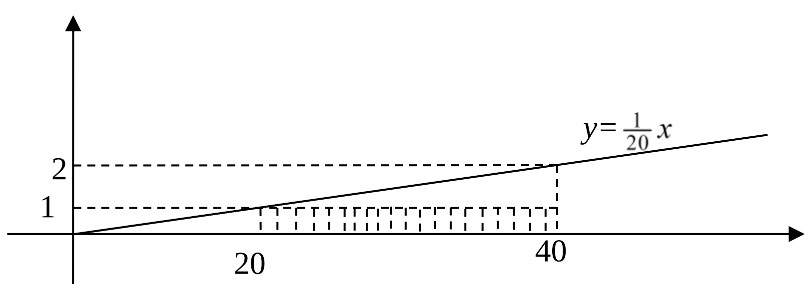
Smallest k = 40.

28	$\mathrm{f}(r)-\mathrm{f}(r+1)=\frac{1}{r}-\frac{1}{r+1}=\frac{1}{r(r+1)}$. $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}.....+\frac{1}{2n(2n+1)}=\sum_{r=1}^{2n}\frac{1}{r(r+1)}=\sum_{r=1}^{2n}[\mathrm{f}(r)-\mathrm{f}(r+1)]$ $= [\ \mathrm{f}(1)-\cancel{\mathrm{f}(2)}$ $+\ \cancel{\mathrm{f}(2)}-\cancel{\mathrm{f}(3)}$ $+ \ldots$ $+\ \cancel{\mathrm{f}(2n-1)}-\cancel{\mathrm{f}(2n)}$ $+\ \cancel{\mathrm{f}(2n)}-\mathrm{f}(2n+1)\ ]$ $=\mathrm{f}(1)-\mathrm{f}(2n+1)$ $=1-\frac{1}{2n+1}$ $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}\ldots(2n)^{\text{th}}\text{ term }=\sum_{r=1}^{2n}\frac{1}{(r+1)^2}$ Consider $\frac{1}{(r+1)^2}=\frac{1}{(r+1)(r+1)}<\frac{1}{r(r+1)}$ $\Rightarrow\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<\sum_{r=1}^{2n}\frac{1}{r(r+1)}$ $\Rightarrow\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<1-\frac{1}{2n+1}$ Since $\frac{1}{2n+1}>0$, $\sum_{r=1}^{2n}\frac{1}{(r+1)^2}<1$ (shown)

29 (i)	2 500 000 000 = 1000 000$(2)^{n-1}$ 2^{n-1} = 2500 n-1 = $\frac{\ln 2500}{\ln 2}$ = 11.2877 n = 12. 2877 His net worth will first exceed 2.5 billion when n = 13 The year 1993 + (13 -1)(1) = 2005 or 1993 + 13 – 1 = 2005
29 (ii)	100 000 + 1000 + 1000(1.5) + 1000(1.5^2) + ...(15 terms) = 100 000 + $\frac{1000(1.5^{15}-1)}{1.5-1}$ = $ 973 787.7808 = $ 973 788

30	$\frac{1}{3^r}\left(\frac{u_{r+1}}{3}-u_r\right)=2r$ $\sum_{r=1}^{n}\left(\frac{u_{r+1}}{3^{r+1}}-\frac{u_r}{3^r}\right)=2\sum_{r=1}^{n}r$ $\cancel{\frac{u_2}{3^2}}-\frac{u_1}{3^1}$ $+\cancel{\frac{u_3}{3^3}}-\cancel{\frac{u_2}{3^2}}$ $+\ldots$ $+\cancel{\frac{u_n}{3^n}}-\cancel{\frac{u_{n-1}}{3^{n-1}}}$ $+\frac{u_{n+1}}{3^{n+1}}-\cancel{\frac{u_n}{3^n}}=2\left(\frac{n}{2}(1+n)\right)$ $\frac{u_{n+1}}{3^{n+1}}-\frac{1}{3}=n(n+1)$ $u_{n+1}=3^{n+1}\left(n(n+1)+\frac{1}{3}\right)=3^n\left(3n^2+3n+1\right)$

31



Number of squares for $20 \le x \le 40$ is 20

Column	Range of x	Number of squares
1	$20 \le x \le 40$	20
2	$40 \le x \le 60$	40
3	$60 \le x \le 80$	60
	.	
	.	
n	$380 \le x \le 400$	

Number of columns = $n=\frac{380}{20}=19$ for $0 \le x \le 400$

OR $380 = 20 + (n-1)(20) \Rightarrow n=19$

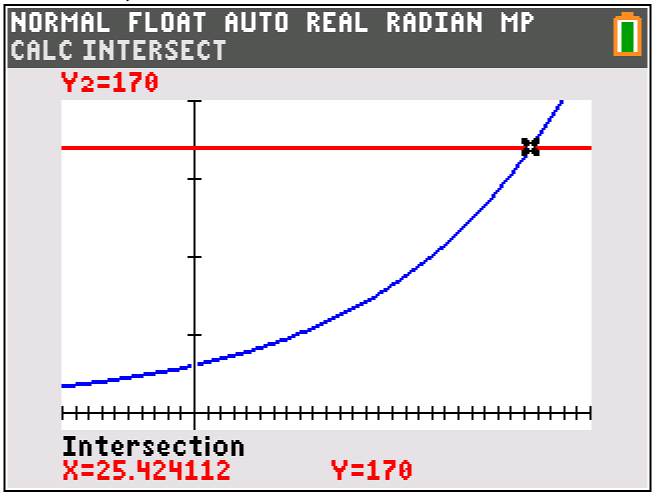
	OR $400 = 40 + (n\text{-}1)(20) \Rightarrow n = 19$ AP sequence : 20, 40, 60, …….. 19 terms Number of complete squares in region R for $0 \le x \le 400$ is $\dfrac{19}{2}\left[2(20)+18(20)\right] = 3800$ Region R is symmetrical about the line $x = 400$. Total number of squares in region R = 3800 x 2 = 7600

32 (i)	Let $\dfrac{4r-1}{3^{r+2}} = \dfrac{Ar}{3^r} - \dfrac{B(r+2)}{3^{r+2}} = \dfrac{(9A-B)r-2B}{3^{r+2}}$ Comparing constants: $B = 0.5$ Comparing coefficients of r: $4 = 9A - 0.5 \Rightarrow A = 0.5$ $\therefore \dfrac{4r-1}{3^{r+2}} = \dfrac{1}{2}\left[\dfrac{r}{3^r} - \dfrac{r+2}{3^{r+2}}\right]$
(ii)	$\displaystyle\sum_{r=1}^{n}\frac{4r-1}{3^{r+2}} = \frac{1}{2}\sum_{r=1}^{n}\left[\frac{r}{3^r} - \frac{r+2}{3^{r+2}}\right].$ $\displaystyle = \frac{1}{2}\left[\frac{1}{3} - \cancel{\frac{3}{3^3}}\right.$ $\displaystyle + \frac{2}{3^2} - \cancel{\frac{4}{3^4}}$ $\displaystyle + \cancel{\frac{3}{3^3}} - \cancel{\frac{5}{3^5}}$ . . . $\displaystyle + \cancel{\frac{n-2}{3^{n-2}}} - \cancel{\frac{n}{3^n}}$ $\displaystyle + \cancel{\frac{n-1}{3^{n-1}}} - \frac{n+1}{3^{n+1}}$ $\displaystyle \left. + \cancel{\frac{n}{3^n}} - \frac{n+2}{3^{n+2}}\right]$ $\displaystyle = \frac{1}{2}\left(\frac{5}{9} - \frac{n+1}{3^{n+1}} - \frac{n+2}{3^{n+2}}\right)$ (or equivalent)
(iii)	Series converges as $\displaystyle\lim_{n\to\infty}\frac{n+1}{3^{n+1}} = \lim_{n\to\infty}\frac{n+2}{3^{n+2}} = 0$. $\displaystyle\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}} = \frac{5}{18}$.

(iv)	$\sum_{r=1}^{\infty}\frac{1}{3^{r+2}}=\frac{1}{3^3}+\frac{1}{3^4}+\ldots=\frac{\frac{1}{27}}{1-\frac{1}{3}}=\frac{1}{18}$. Hence, $\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}}=\sum_{r=1}^{\infty}\left(\frac{4r}{3^{r+2}}-\frac{1}{3^{r+2}}\right)$ $\sum_{r=1}^{\infty}\frac{4r-1}{3^{r+2}}=\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}-\sum_{r=1}^{\infty}\frac{1}{3^{r+2}}$ From part (iii) and the sum just found, $\frac{5}{18}=\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}-\frac{1}{18}$ $\sum_{r=1}^{\infty}\frac{4r}{3^{r+2}}=\frac{5}{18}+\frac{1}{18}=\frac{1}{3}$ $\sum_{r=1}^{\infty}\frac{4r}{3^2\cdot 3^r}=\frac{1}{3}$ $\frac{4}{9}\sum_{r=1}^{\infty}\frac{r}{3^r}=\frac{1}{3}$ $\therefore\sum_{r=1}^{\infty}\frac{r}{3^r}=\frac{1}{3}\times\frac{9}{4}=\frac{3}{4}$

33 (i)	$S_{15}\text{ of B}=\frac{15}{2}\left(2(2.4)+14d\right)$ $=36+105d$
(ii)	$S_{15}\text{ of A}=\frac{2.4\left((1.2)^{15}-1\right)}{1.2-1}$ $=172.88$ >170 Yes, Adam can achieve his target.
(iii)	$U_{15}\text{ of A}>U_{15}\text{ of B}$ $U_{15}\text{ of A}-U_{15}\text{ of B}>0$ $2.4(1.2)^{14}-(2.4+14d)>0$ $d<2.02957$ $\max d=2.02(2\text{ dp})$ $2.4(1.2)^{14}-(2.4+14(2.02))=0.134$ (shown)
(iv)	$\text{New }S_{13}\text{ of A}=\frac{2.4\left(\left(1+\frac{r}{100}\right)^{13}-1\right)}{\left(1+\frac{r}{100}\right)-1}=170$

From GC,



x = 25.4%

34(i)

Hour	Start of hour	End of hour
1	1	1 + 3 = 4
2	4	4 + 12 = 16
3	16	16 + 48 = 64
4	…	…

$$4(4)^{n-1} > 200\,000$$

$$(4)^{n-1} > 50000$$

$$n-1 > 7.80482$$

$$n > 8.80842$$

Number of complete hours = 9

<u>Alternative Solution</u>

$$4(4)^{n-1} > 200\,000$$

n	Total
8	65536 < 200 000
9	262144 > 200 000
10	11048576 > 200 000

Number of complete hours = 9

(ii)

Day	Start of day	End of day
1	$200000 - x$	$1.02\left(200000 - x\right)$
2	$1.02\left(200000 - x\right)$ $-x$	$1.02[1.02\left(200000 - x\right)$ $-x]$ $= 1.02^2\left(200000\right)$ $-1.02x - 1.02^2 x$
3	…	…

At the end of day *n*, the number comments

$= 1.02^n\left(200000\right) - \left(1.02x + 1.02^2 x + ... + 1.02^n x\right)$ --- (*)

$$= 1.02^n\left(200000\right) - x\left(\frac{1.02\left(1.02^n - 1\right)}{0.02}\right)$$

$= 1.02^n\left(200000\right) - 51x\left(1.02^n - 1\right)$

(iii)

$1.02^{30}\left(200\,000\right) - 51x\left(1.02^{30} - 1\right) < 0$

$$x > \frac{1.02^{30}\left(200\,000\right)}{51\left(1.02^{30} - 1\right)}$$

$x \geq 8755$ (to nearest integer)

Day 1: no. of comments removed $= 15000$

Day 2: no. of comments removed $= 15000\left(0.9\right)$

Day 3: no. of comments removed $= 15000\left(0.9\right)^2$

As $n \to \infty$, no. of comments removed

$$= \frac{15000}{1 - 0.9} = 150\,000$$

Software Y is unable to remove all the comments because eventually it is only able to remove 150 000 comments.

35(i)	Let b be the first term of the arithmetic progression. $a = b + d \quad -(1)$ $ar^2 = b + 2d \quad -(2)$ $ar^7 = b + 4d \quad -(3)$ From equations (1), (2) and (3), $ar^2 - a = \dfrac{ar^7 - ar^2}{2} (= d)$ $2r^2 - 2 = r^7 - r^2$ $r^7 - 3r^2 + 2 = 0$
(ii)	Using GC, $r = 1$ or $r = 0.93725$ or $r = -0.77986$ (5 d.p.) (rejected since d is non-zero) Since $\lvert r \rvert < 1$, sum to infinity of geometric progression exists.
(iii)	The first even-numbered term of the A.P. is $a = 12$. $d = ar^2 - a = 12(0.93725^2 - 1) = 12(-0.12156) = -1.4587$ $\lvert E - S_\infty \rvert < 1000$ $\left\lvert \frac{n}{2}\left[2a + (n-1)(2d)\right] - \frac{a}{1-r} \right\rvert < 1000$ $\left\lvert \frac{n}{2}\left[2(12) + (n-1)(-2.9174)\right] - \frac{12}{1-0.93725} \right\rvert < 1000$ $\left\lvert \frac{n}{2}\left[24 + (n-1)(-2.9174)\right] - 191.23 \right\rvert < 1000$ Using GC, When $n = 28$, $\text{LHS} = 958 < 1000$ When $n = 29$, $\text{LHS} = 1028 > 1000$ Hence largest value of $n = 28$.

36a	Let the length of the first screwdriver be *b*, and the common ratio be *r*. $b + br + br^2 = 3(br^{12} + br^{13} + br^{14} + br^{15} + br^{16})$ $1 + r + r^2 - 3r^{12} - 3r^{13} - 3r^{14} - 3r^{15} - 3r^{16} = 0$ Using the GC, r = 0.882854 (6 s.f.) since $0 < r < 1$. There are 9 odd numbered screwdrivers, with common ratio of lengths being r^2.

	Total length = $\dfrac{b\left(1-\left(r^2\right)^9\right)}{1-r^2}=120$ $\dfrac{b\left(1-r^{18}\right)}{1-r^2}=120$ Using the GC, b = 29.6122 (6 s.f.) The total length is $\dfrac{b\left(1-r^{17}\right)}{1-r}=222.38$ cm $\quad$(to 2 d.p.)
36 (b)(i)	$\cos\left(u_{n+1}-u_n\right)=\cos u_{n+1}\cos u_n+\sin u_{n+1}\sin u_n$ $=\dfrac{1}{2}\cos^2 u_n-\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $+\dfrac{1}{2}\sin^2 u_n+\dfrac{\sqrt{3}}{2}\sin u_n\cos u_n$ $=\dfrac{1}{2}$ $\therefore u_{n+1}-u_n=\cos^{-1}\dfrac{1}{2}=\dfrac{\pi}{3}$ (since the difference is less than π) is a constant independent of *n*. Hence $\{u_n\}$ is an arithmetic progression. The common difference is $\dfrac{\pi}{3}$. **<u>Alternatively</u>** $\cos u_{n+1}=\dfrac{1}{2}\cos u_n-\dfrac{\sqrt{3}}{2}\sin u_n$ $=\cos u_n\cos\dfrac{\pi}{3}-\sin u_n\sin\dfrac{\pi}{3}$ $=\cos\left(u_n+\dfrac{\pi}{3}\right)$ $\therefore u_{n+1}=u_n+\dfrac{\pi}{3}$ since the difference is less than π. $u_{n+1}-u_n$ is a constant, therefore $\{u_n\}$ is an arithmetic progression. The common difference is $\dfrac{\pi}{3}$.
b(ii)	The total angle rotated over *n* twists is $\dfrac{n}{2}\left(2\left(\dfrac{2\pi}{3}\right)+(n-1)\dfrac{\pi}{3}\right)$.

<table>
<tr><td></td><td>

$\frac{n}{2}\left(\frac{4\pi}{3}+(n-1)\frac{\pi}{3}\right)\geq 25\times(2\pi)$

$\frac{2n\pi}{3}+\frac{\pi}{6}n(n-1)-50\pi\geq 0$

$4n+n(n-1)-300\geq 0$

$n^2+3n-300\geq 0$

Using the GC,

n	$n^2+3n-300$
15	-30
16	4
17	40

The minimum number of buttons presses required is 16.
</td></tr>
<tr><td>b(iii)</td><td>

Total angle rotated after 21 button presses is

$\frac{21}{2}\left(2\left(\frac{2\pi}{3}\right)+(20)\frac{\pi}{3}\right)$

$=14\pi+70\pi$

$=84\pi$

$84\pi k=144$, where k is a positive real constant.

The first button press rotates the screw by $\frac{2\pi}{3}$.

$d_1=ku_1=\frac{144}{84\pi}\left(\frac{2\pi}{3}\right)=\frac{8}{7}$

The first button press drills the screw in by $\frac{8}{7}$ mm (1.14mm).
</td></tr>
<tr><td>**37**(a)</td><td>

1st row: number of matches = 3
2nd row: number of matches = 6
3rd row: number of matches = 9

nth row: number of matches = $3+(n-1)(3)=3n$
</td></tr>
<tr><td></td><td>

1 row: total number of matches = 3
2 rows: total number of matches = 3 + 6
3 rows: total number of matches = 3 + 6 + 9

n rows: total number of matches

$=\frac{n}{2}(3+3n)$ OR $=\frac{n}{2}\left[2(3)+(n-1)(3)\right]$

$=\frac{3n(n+1)}{2}$ (shown) $=\frac{3n(n+1)}{2}$ (shown)
</td></tr>
</table>

	$\frac{3n(n+1)}{2} \le 2000$ Using GC, When $n=36$, $\frac{3n(n+1)}{2} = 1998 < 2000$ When $n=37$, $\frac{3n(n+1)}{2} = 2109 > 2000$ Maximum number of complete rows $= 36$.
(b)	Let $r = \frac{b}{a}$ $\frac{a}{1-r} = a + 2b$ $\frac{1}{1-r} = 1 + 2r$ $1 = 1 + r - 2r^2$ $2r^2 - r = 0$ $r(2r-1) = 0$ $r = 0$ (rejected $\frac{b}{a} \ne 0$) or $r = \frac{1}{2}$ $\therefore$ common ratio = $\frac{1}{2}$ $G_n = \frac{a\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$ $G_n = 2a\left(1-\left(\frac{1}{2}\right)^n\right)$ $\sum_{n=1}^{N} G_n = 2a\sum_{n=1}^{N}\left(1-\left(\frac{1}{2}\right)^n\right)$ $= 2a\left[N - \sum_{n=1}^{N}\left(\frac{1}{2}\right)^n\right]$ $= 2a\left[N - \frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}}\right]$

	$= 2a\left(N-\left(1-\left(\frac{1}{2}\right)^N\right)\right)$ $= 2aN - 2a\left(1-\left(\frac{1}{2}\right)^N\right)$ $= 2aN - G_N$

38(i)	$\frac{1}{(r-2)!}-\frac{1}{r!}=\frac{r(r-1)-1}{r!}=\frac{r^2-r-1}{r!}$ $\sum_{r=1}^{n}\frac{r^2-r-1}{r!}=\sum_{r=2}^{n}\left(\frac{1}{(r-2)!}-\frac{1}{r!}\right)+\frac{1-1-1}{1!}$ $= \left(\frac{1}{0!}-\cancel{\frac{1}{2!}}\right.$ $+\frac{1}{1!}-\cancel{\frac{1}{3!}}$ $+\cancel{\frac{1}{2!}}-\cancel{\frac{1}{4!}}$ $+\ldots$ $+\cancel{\frac{1}{(n-4)!}}-\cancel{\frac{1}{(n-2)!}}$ $+\cancel{\frac{1}{(n-3)!}}-\frac{1}{(n-1)!}$ $\left.+\cancel{\frac{1}{(n-2)!}}-\frac{1}{n!}\right) \quad -1$ $=1+1-\frac{1}{(n-1)!}-\frac{1}{n!}-1$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}$
(ii)	As $n\to\infty$, $-\frac{1}{(n-1)!}\to 0$ and $-\frac{1}{n!}\to 0$ $\therefore \sum_{r=1}^{n}\frac{r^2-r-1}{r!}\to 1$, a finite value $\therefore \sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}$ converges sum to infinity $=\sum_{r=1}^{\infty}\frac{r^2-r-1}{r!}=1$

(iii)	$\sum_{r=1}^{n}\frac{r-2}{(r-1)!}=\sum_{r=1}^{n}\frac{r^2-2r}{r!}=\sum_{r=1}^{n}\frac{r^2-r-r}{r!}$ $<\sum_{r=1}^{n}\frac{r^2-r-1}{r!}\quad\left(\because \frac{r}{r!}>\frac{1}{r!}\text{ for } r>1\right)$ $=1-\frac{1}{(n-1)!}-\frac{1}{n!}<1$
(iv)	$\sum_{r=4}^{n}\frac{r^2-5r+5}{(r-2)!}$ Replace r with $r + 2$. $\sum_{r=4}^{n}\frac{r^2-5r+5}{(r-2)!}=\sum_{r+2=4}^{r+2=n}\frac{(r+2)^2-5(r+2)+5}{(r+2-2)!}$ $=\sum_{r=2}^{n-2}\frac{r^2-r-1}{r!}$ $=\sum_{r=1}^{n-2}\frac{r^2-r-1}{r!}-\frac{1^2-1-1}{1!}$ $=1-\frac{1}{(n-2-1)!}-\frac{1}{(n-2)!}+1$ $=2-\frac{1}{(n-3)!}-\frac{1}{(n-2)!}.$

39	**(a)(i)** After one month, if she pays \$$x$ at the beginning of the month, she will owe the bank $\$(50000-x)\times(1.001)$ Hence $(50000-x)\times(1.001)=50000 \quad \Rightarrow \quad x=49.95$ Abbie needs to pay \$49.95 (to the nearest cent) a month. **(a)(ii)** One month after graduating, she owes $(50000-k)\times(1.00375)$. n mon ths after graduating, she will owe $1.00375^n(50000-k)-1.00375^{n-1}k-\ldots-1.00375k$ $=1.00375^n(50000)-k\left(1.00375^n+1.00375^{n-1}+\ldots+1.00375\right)$ $=1.00375^n(50000)-k\left[\dfrac{1.00375\left(1.00375^n-1\right)}{1.00375-1}\right]$ $=1.00375^n(50000)-\dfrac{803}{3}k\left(1.00375^n-1\right)$ (shown). **(a)(iii)** Sub n = 120, and k = 500: $1.00375^{120}(50000)-\dfrac{803}{3}(500)\left(1.00375^{120}-1\right)=2467.11>0$. No, she cannot. A monthly payment of \$500 is not enough. When n = 120, $1.00375^{120}(50000)-\dfrac{803}{3}k\left(1.00375^{120}-1\right)=0$ $\Rightarrow \quad k=516.26$ (nearest cent) She needs to pay \$516.26 per month. **(b)(i)** Oustanding amount upon graduation $=1.001^{36}(50000)$ $=51831.86$ Using Abbie's formula, but with a starting outstanding amount of \$51831.86, $1.00375^{120}(51831.86)-\dfrac{803}{3}k\left(1.00375^{120}-1\right)=0$ $\Rightarrow \quad k=535.17$ (nearest cent) He needs to pay \$535.17 per month.

	(b)(ii) $120\times535.17-50000=14220.43$ (to 2 d.p.) He paid \$14220.43 in interest altogether.

40	GP: $a=10000,\ r=1.07$
	$U_{15}=10000(1.07)^{14}$
	$U_{15}=25785.34\approx25785$
(i)	First three terms of G.P.: $a+6d$, $a+2d$, a $\Rightarrow\dfrac{a+2d}{a+6d}=\dfrac{a}{a+2d}$
	$a^2+4ad+4d^2=a^2+6ad$ $4d^2=2ad$ $d\neq0\Rightarrow d=\dfrac{a}{2}$
(ii)	Given $a+6d=3000$, where $d=\dfrac{a}{2}$ from (i) $a+3a=3000\Rightarrow a=\dfrac{3000}{4}=750$
	Total calories loss $S_n=\dfrac{n}{2}\left[2(750)+(n-1)(375)\right]$ $=\dfrac{375n}{2}(3+n)$
	G.P. $U_1=3000,\ r=\dfrac{a}{a+2\left(\dfrac{a}{2}\right)}=\dfrac{1}{2}$ Total calories gain $S_n=\dfrac{3000\left(1-\left(\dfrac{1}{2}\right)^n\right)}{1-\dfrac{1}{2}}=6000\left(1-\left(\dfrac{1}{2}\right)^n\right)$
	$\dfrac{375n}{2}(3+n)-6000\left(1-\left(\dfrac{1}{2}\right)^n\right)\geq200000$ $\dfrac{3n}{16}(3+n)-6\left(1-\left(\dfrac{1}{2}\right)^n\right)\geq200$
	From GC, $n\geq31.68$ (or 32) Least number of weeks $=32$.

41 (i)	$u_n = S_n - S_{n-1}$ $= n(2n+7) - (n-1)(2n+5)$ $= 2n^2 + 7n - (2n^2 + 3n - 5)$ $= 4n+5$ $u_1 = 4(1)+5 = 9$ $S_1 = 1[2(1)+7] = 9$ $u_1 = S_1$ $u_n - u_{n-1}$ $= (4n+5) - (4(n-1)+5)$ $= 4n+5-(4n+1)$ $= 4$ Since $u_n - u_{n-1} = 4$ (constant), the sequence is AP.
(ii)	$\sum_{n=1}^{N} \frac{1}{\sqrt{u_{n+1}} + \sqrt{u_n}}$ $= \sum_{n=1}^{N} \frac{\left(\sqrt{u_{n+1}} - \sqrt{u_n}\right)}{\left(\sqrt{u_{n+1}} + \sqrt{u_n}\right)\left(\sqrt{u_{n+1}} - \sqrt{u_n}\right)}$ $= \sum_{n=1}^{N} \frac{\left(\sqrt{u_{n+1}} - \sqrt{u_n}\right)}{u_{n+1} - u_n}$ $= \frac{1}{4}\sum_{n=1}^{N} \left(\sqrt{u_{n+1}} - \sqrt{u_n}\right)$ $= \frac{1}{4}\Big\{\left(\cancel{\sqrt{u_2}} - \sqrt{u_1}\right) +$ $\left(\cancel{\sqrt{u_3}} - \cancel{\sqrt{u_2}}\right) +$ $\vdots$ $\left(\cancel{\sqrt{u_N}} - \cancel{\sqrt{u_{N-1}}}\right) +$ $\left(\sqrt{u_{N+1}} - \cancel{\sqrt{u_N}}\right)\Big\}$ $= \frac{1}{4}\left(\sqrt{u_{N+1}} - \sqrt{u_1}\right)$ $= \frac{1}{4}\left(\sqrt{4N+9} - 3\right)$

(iii)

$$\frac{1}{\sqrt{53}+\sqrt{49}}+\frac{1}{\sqrt{57}+\sqrt{53}}+\cdots+\frac{1}{\sqrt{361}+\sqrt{357}}$$

$$=\frac{1}{\sqrt{4(11)+9}+\sqrt{4(11)+5}}+\frac{1}{\sqrt{4(12)+9}+\sqrt{4(12)+5}}+\cdots+\frac{1}{\sqrt{4(88)+9}+\sqrt{4(88)+5}}$$

$$=\sum_{n=1}^{88}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}-\sum_{n=1}^{10}\frac{1}{\sqrt{u_{n+1}}+\sqrt{u_n}}$$

$$=\frac{1}{4}\left(\sqrt{4(88)+9}-3\right)-\frac{1}{4}\left(\sqrt{4(10)+9}-3\right)$$

$$=4-1$$

$$=3$$

43. CJC/2022/1/11

A new coating is applied to TX tennis balls to improve its elasticity. To examine its effect, a TX tennis ball is dropped vertically from a fixed height H cm onto the floor. The height reached after each bounce decreased by 5 cm from the height reached by the previous bounce.

Find, in terms of H and n,

(i) the height reached by the tennis ball after the $(n-1)$th bounce, [1]

(ii) the total distance travelled by the tennis ball just before it rebounds off the floor for the nth bounce. [5]

(iii) State an assumption used in your calculation for part **(ii).** [1]

The time interval between any successive bounces is also measured. Given that the time interval between the first and second bounce is 1.2 seconds, and the time interval between subsequent consecutive bounces is 0.75 times of the time interval of the previous bounce, find

(iv) the least integer m such that it takes less than 0.02 seconds between the mth and $(m + 1)$th bounce, [3]

(v) total time taken before the tennis ball comes to a stop given that it takes 2 seconds for the tennis ball to first hit the ground. [2]

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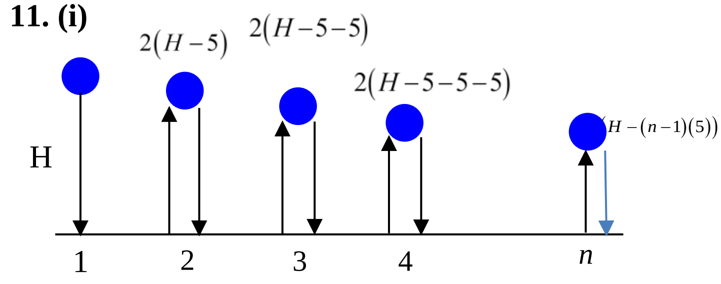
11. (i)



Height reached after 1st bounce = $H-5$

Height reached after 2nd bounce = $H-(2)5$

Height reached after 3rd bounce = $H-(3)5$

…

Height reached after the $(n-1)$th bounce = $H-(n-1)(5)$

(ii)
Total distance travelled by the ball just before it rebounds off the floor for the n^{th} time

$$=H+2(H-5)+2(H-2(5))+2(H-3(5))+\ldots+2(H-(n-1)(5))$$

$$=H+\underbrace{2H+2H+\ldots+2H}_{(n-1)terms}-2[5+(5+5)+(5+5+5)+\ldots+(n-1)(5)]$$

$$=H+2(n-1)H-2\left[\frac{n-1}{2}(5+(n-1)5)\right]$$

$$=H+2(n-1)H-5n(n-1)$$

OR

Total distance travelled by the ball just before it rebounds off the floor for the n^{th} time

$$=H+2(H-5)+2(H-2(5))+2(H-3(5))+\ldots+2(H-(n-1)(5))$$

$$=H+\underbrace{2H+2H+\ldots+2H}_{(n-1)terms}-2[5+(5+5)+(5+5+5)+\ldots+(n-1)(5)]$$

$$=H+2(n-1)H-2\left[\frac{n-1}{2}(2[5]+([n-1]-1)5)\right]$$

$$=H+2(n-1)H-(n-1)(10+5n-10)$$

$$=H+2(n-1)H-(n-1)(5n)$$

$$=H+2(n-1)H-5n(n-1)$$

(iii)
State an assumption made in the calculation above.
- The ball only travels vertically
- Upon hitting the floor, the tennis ball rebounds instantly

(iv)
Time taken between 1^{st} and 2^{nd} bounce = 1.2
Time taken between 2^{nd} and 3^{rd} bounce = $1.2(0.75)$
Time taken between 3^{rd} and 4^{th} bounce = $1.2(0.75)^2$
…
Time taken between m^{th} and $(m+1)$th bounce = $1.2(0.75)^{m-1}$

$$1.2(0.75)^{m-1}<0.02$$

$$(0.75)^{m-1}<\frac{0.02}{1.2}$$

$$m-1>\frac{\ln\left(\frac{0.02}{1.2}\right)}{\ln(0.75)}$$

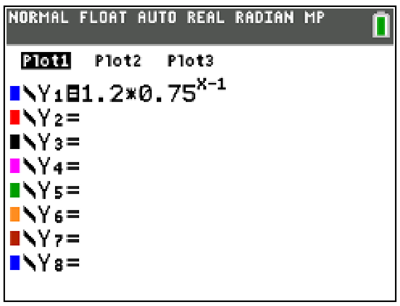
$m-1>14.232$

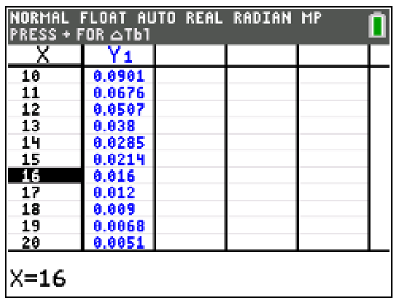
$m>15.232$

Least m = 16

Alternative Method (using GC table)

$1.2(0.75)^{m-1}<0.02$





Least m = 16

(v)

total time taken before the tennis balls comes to a stop $=\frac{1.2}{1-0.75}+2$

= 6.8s

44. EJC/2022/2/3

The sum S_n of the first n terms of a sequence u_1, u_2, u_3, … is given by

$$S_n=\ln\left(\frac{e^n}{3^{n^2}}\right).$$

(a) Show that $u_n=1+(1-2n)\ln 3$. [2]

(b) Hence, show that the sequence is an arithmetic progression. [2]

(c) Find the sum of the first ten odd-numbered terms. [2]

(d) A geometric sequence e^{u_1}, e^{u_2}, e^{u_3}, ... has a common ratio, r. Find the value of r. [2]

(e) Find the least value of n for which the sum of the first n terms of this geometric sequence is within 10^{-8} of its sum to infinity. [3]

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3(a) $S_n = \ln\left(\frac{e^n}{3^{n^2}}\right)$

$$u_n = S_n - S_{n-1}$$
$$= \ln\left(\frac{e^n}{3^{n^2}}\right) - \ln\left(\frac{e^{n-1}}{3^{(n-1)^2}}\right)$$
$$= \ln\left(\frac{e^n}{3^{n^2}} \times \frac{3^{(n-1)^2}}{e^{n-1}}\right)$$
$$= \ln\left(\frac{3^{-2n+1}}{e^{-1}}\right)$$
$$= 1 + (1-2n)\ln 3 \quad \text{(Shown)}$$

(b) $u_n - u_{n-1} = [1+(1-2n)\ln 3] - [1+(1-2(n-1))\ln 3]$

$= -2\ln 3$ which is a constant independent of n

Hence, the sequence is an arithmetic progression.

(c) First term $= 1 - \ln 3$

Common difference $= -4\ln 3$

Sum of the first ten odd-numbered terms

$$= \frac{10}{2}[2(1-\ln 3) + 9(-4\ln 3)]$$
$$= 10 - 190\ln 3$$

(d) <u>Method 1 – use</u> $r = \frac{T_n}{T_{n-1}}$

$$r = \frac{e^{u_n}}{e^{u_{n-1}}}$$
$$= e^{u_n - u_{n-1}}$$
$$= e^{-2\ln 3} \quad \text{(from (b))}$$
$$= 3^{-2}$$
$$= \frac{1}{9}$$

<u>Method 2</u>

$$r = \frac{e^{1-3\ln 3}}{e^{1-\ln 3}} = e^{-2\ln 3} = \frac{1}{9}$$

(e) **We want** $|S_n - S_\infty| \le 10^{-8}$**.**

$$\left|\frac{a(1-r^n)}{1-r}-\frac{a}{1-r}\right|\le 10^{-8}$$

$$\Rightarrow \left|\frac{-ar^n}{1-r}\right|\le 10^{-8}$$

$$\Rightarrow \frac{e^{1-\ln 3}\left(\frac{1}{9}\right)^n}{1-\frac{1}{9}}\le 10^{-8}$$

$$\Rightarrow \frac{3e}{8}\left(\frac{1}{9}\right)^n \le 10^{-8}$$

Method 1 (algebraic)

$$n \ge \frac{\ln \frac{8(10^{-8})}{3e}}{\ln \frac{1}{9}} = 8.39233808$$

Least $n=9$**.**

Method 2 (use GC)

n	$\frac{3e}{8}\left(\frac{1}{9}\right)^n$
8	2.368×10^{-8} $(>10^{-8})$
9	2.631×10^{-9} $(<10^{-8})$
10	2.923×10^{-10} $(<10^{-8})$

Least $n=9$**.**