Chapter 5: Graphing Techniques II

Content outline :

- Effect of transformations on the graph of $y = \mathrm{f}(x)$ as represented by $y = a\mathrm{f}(x)$, $y = \mathrm{f}(x) \pm a$, $y=\mathrm{f}\left(x \pm a\right)$ and $y = \mathrm{f}(ax)$, and combinations of these transformations
- Relating the graphs of $y=\left|\mathrm{f}\left(x\right)\right|$, $y=\mathrm{f}\left(\left|x\right|\right)$, and $y=\dfrac{1}{\mathrm{f}\left(x\right)}$ to the graph of $y=\mathrm{f}\left(x\right)$

References

- http://www.h2maths.site
  [This website offers Applets which allow self-exploration of numerous types of graphs covered in this topic. Refer to "Transformation of Graphs" under the link "Graphs and Functions" in this website]
- H2 Mathematics For 'A' Level, Federick Ho, David Khor, Yui-P'ng Lam, B.S. Ong Volume 1, Call No 510.76 HO
- Mathematics The Core Course for A-level, L Bostock and Chandler, Call No 510 BOS
- MEI Structured Mathematics (2nd Edition), Pure Mathematics 4, Terry Heard and David Martin, Call No 510 HEA
- Pure Mathematics 4, Hugh Neill and Douglas Quadling, Call No 510 NEI

Prerequisites

- Coordinate geometry from Add Maths
- Using GC to draw simple curves from Chapter 2 Graphing Techniques I
- Functions learnt in Chapter 4
- O Level Trigonometry

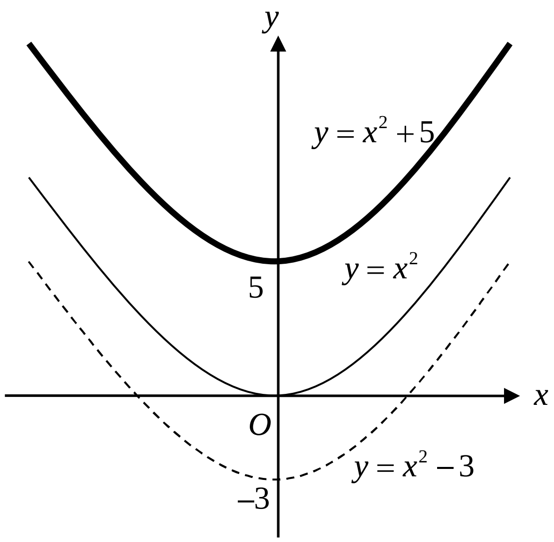
1.1 Translations

(I) Graphs of the form $y = \mathrm{f}(x) + a$

Example 1

The diagram below shows the graphs of $y = x^2$, $y = x^2 + 5$ and $y = x^2 - 3$ on the same axes. How are the graphs of $y = x^2 + 5$ and $y = x^2 - 3$ related to the graph of $y = x^2$?

Solution:



The graph of $y = x^2 + 5$ is obtained by translating the graph of $y = x^2$ ____________
____

The graph of $y = x^2 - 3$ is obtained by translating the graph of $y = x^2$ ____________
____

This can be generalized as follows:

For $a > 0$,
the graph of $y = \mathrm{f}(x) + a$ is a **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units** in the **positive *y* direction**,

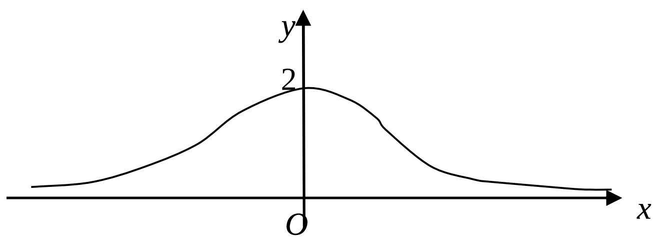
the graph of $y = \mathrm{f}(x) - a$ is a **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units** in the **negative *y* direction**.

Note: This transformation affects the *y*-coordinates only. The *x*-coordinates remain unchanged.

Example 2:

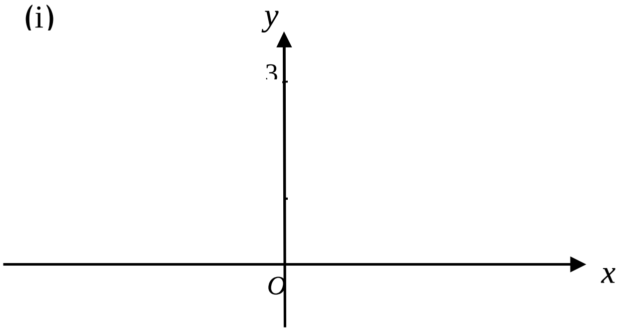
Given the graph of $y = \mathrm{f}(x)$ below, where $\mathrm{f}(x) = \dfrac{4}{x^2 + 2}$, sketch, on separate diagrams, the graphs of (i) $y = \mathrm{f}(x) + 1$, (ii) $y = \mathrm{f}(x) - 2$.

Students may use the GC to check the accuracy of their answers. However, this may take a longer time.

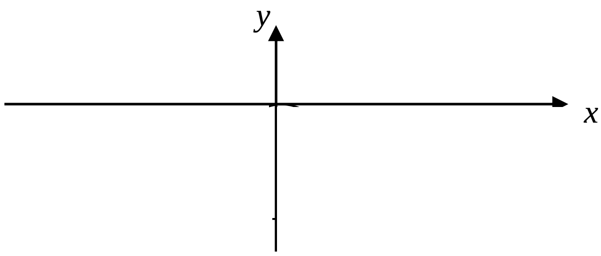


Solution:

(i)



(ii)

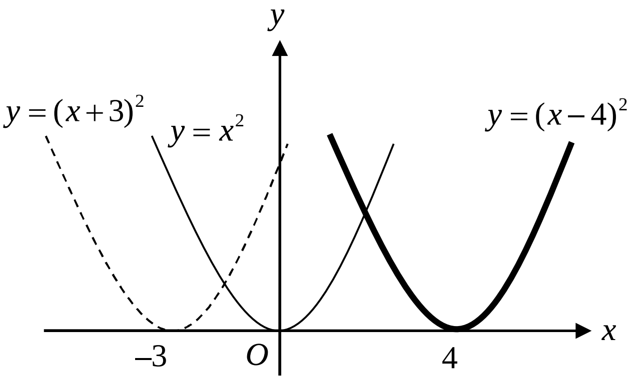


(II) Graphs of the form $y = \mathrm{f}(x - a)$

Example 3

The diagram below shows the graphs of $y = x^2$, $y = (x+3)^2$ and $y = (x-4)^2$ on the same axes. How are the graphs of $y = (x+3)^2$ and $y = (x-4)^2$ related to the graph of $y = x^2$?

Solution:



The graph of $y = (x+3)^2$ is obtained by translating the curve of $y = x^2$ by

____

The graph of $y = (x-4)^2$ is obtained by translating the curve of $y = x^2$ by

____

This can be generalized as follows:

For $a > 0$,
the graph of $y = \mathrm{f}(x - a)$ is obtained by a **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units** in the **positive *x* direction**.

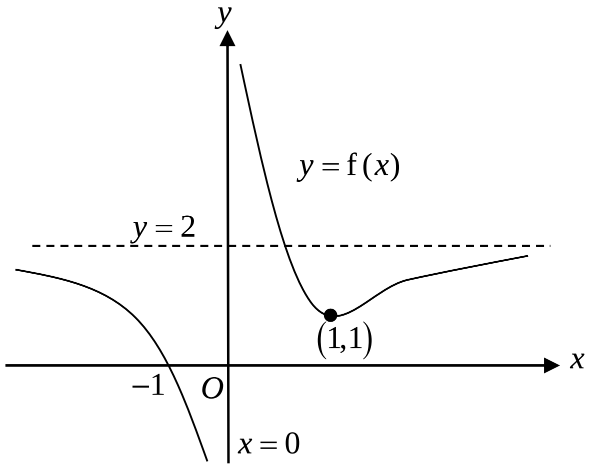
the graph of $y = \mathrm{f}(x + a)$ is obtained by a **translation** of the graph of $y = \mathrm{f}(x)$ **by *a* units** in the **negative *x* direction**.

Note: (I) The formal descriptive word to use is "**translation**".
Avoid using other vague descriptive words such as "shift" or "move".
(II) This transformation affects the *x*-coordinates only.
The *y*-coordinates remain unchanged.

Example 4:
The diagram shows the graph of $y=\mathrm{f}(x)$. On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}(x-2)$,

(ii) $y=\mathrm{f}(x+1)$.



Solution:

(i)

(ii)

Check: Have you ensured that key features such as the given points and the asymptotes in the $y=\mathrm{f}(x)$ graph are indicated in your answers?

1.2 Scaling

(I) Graphs of the form $y = a\,\mathrm{f}(x)$, $a > 0$

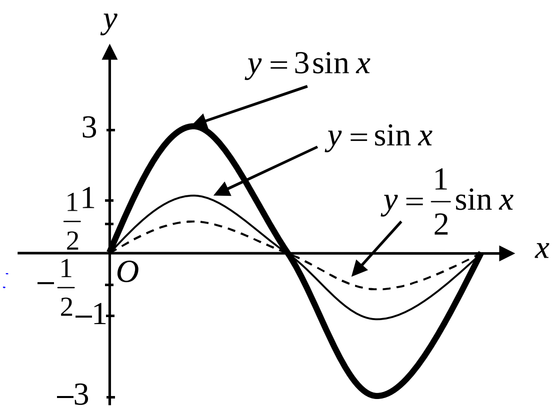
(Scaling parallel to the y-axis)

If f(x) is multiplied by a positive constant a where $a > 1$, the geometric effect is to **“**stretch" the graph of $y = \mathrm{f}(x)$ parallel to the y-axis.

If f(x) is multiplied by a positive constant a where $0 < a < 1$, the geometric effect is to “compress” the graph of $y = \mathrm{f}(x)$ parallel to the y-axis.

Example 5:
The diagram below shows the graphs of $y = \sin x$, $y = 3\sin x$ and $y = \frac{1}{2}\sin x$, for $0 \le x \le 2\pi$, on the same axes. How are the graphs of $y = 3\sin x$ and $y = \frac{1}{2}\sin x$ related to the graph of $y = \sin x$?

Solution:



The graph of $y = 3\sin x$ is obtained by **scaling** the curve $y = \sin x$ ______________

____

The graph of $y = \frac{1}{2}\sin x$ is obtained by **scaling** the curve $y = \sin x$ ______________

____

In general,

For $a > 0$, the graph of $y = a\,\mathrm{f}(x)$ is obtained from the graph of $y = \mathrm{f}(x)$ by a **scaling parallel to the y-axis by a scale factor of a**.

i.e. “stretch” the original curve ($y = \mathrm{f}(x)$) away from the x-axis in both directions ($a > 1$)
or “compress” the original curve ($y = \mathrm{f}(x)$) towards the x-axis in both directions. ($0 < a < 1$)

Note: (1) Scale factor has no units. **Do not** write “scale factor of k **units**”.
(2) The ***x*-coordinates** of all the points remain **unchanged.**

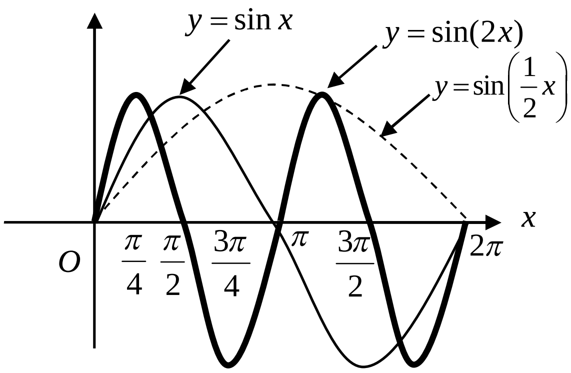
(II) Graphs of the form $y = \mathrm{f}(ax)$, $a > 0$

(Scaling parallel to the x-axis)

Example 6:

The diagram below shows the graphs of $y=\sin x$, $y=\sin(2x)$ and $y=\sin\left(\frac{1}{2}x\right)$, on the same axes. How are the graphs of $y=\sin(2x)$ and $y=\sin\left(\frac{1}{2}x\right)$ related to the graph of $y=\sin x$?

Solution:



The graph of $y=\sin(2x)$ is obtained by **scaling** the curve $y=\sin x$ ______________________

____

The graph of $y=\sin\left(\frac{1}{2}x\right)$ is obtained by **scaling** the curve $y=\sin x$ ______________________

____

In general,

For $a > 0$, the graph of $y = \mathrm{f}(ax)$ is obtained from the graph of $y = \mathrm{f}(x)$ by a **scaling parallel to the x-axis by a scale factor of $\frac{1}{a}$**.

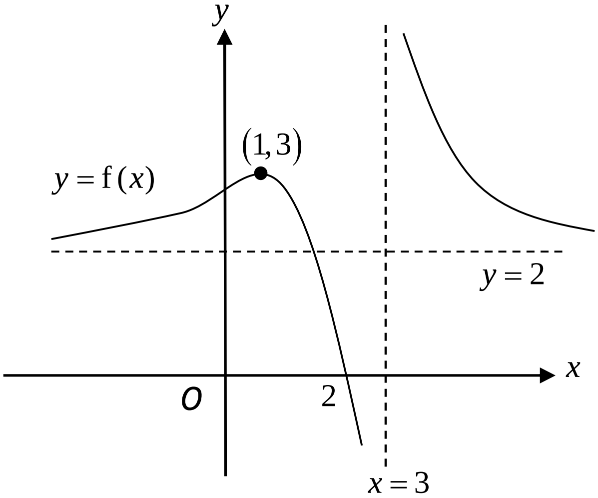
Note: (1) Scale factor has no units. **Do not** write "scale factor of k **units**".
(2) The ***y*-coordinates** of all points remain **unchanged.**

Example 7:

The diagram shows the graph of $y = \mathrm{f}(x)$. On separate diagrams, sketch the graphs of

(i) $y = 2\mathrm{f}(x)$,

(ii) $y = \mathrm{f}(3x)$.



Solution:

(i)

(ii)

Check: Have you ensured that key features such as the given points and the asymptotes in the graph are indicated in your answers?

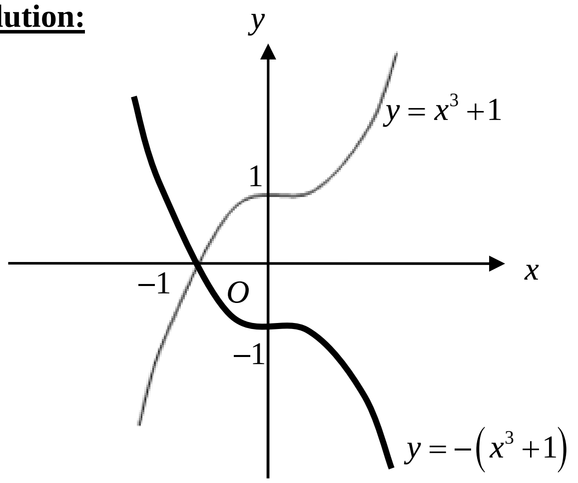
1.3 Reflections

(I) Graphs of the form $y = -f(x)$

Example 8:

The diagram shows the graphs of $y = x^3 + 1$ and $y = -(x^3 + 1)$. Are these two graphs related in any way?

Solution:



The graph of $y = -(x^3 + 1)$ is obtained by ______________

____

In general,

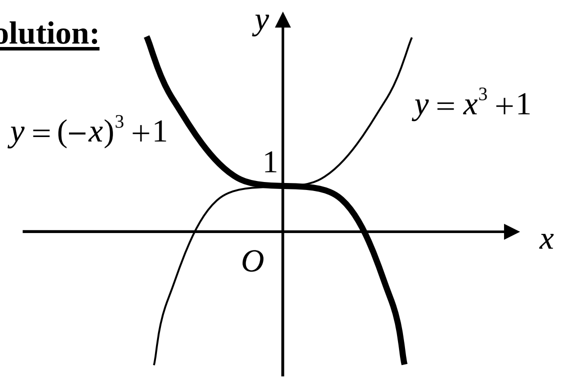
The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

(II) Graphs of the form $y = f(-x)$

Example 9:

The diagram shows the graphs of $y = x^3 + 1$ and $y = (-x)^3 + 1$. How are these two graphs related to each other?

Solution:



The graph of $y = (-x)^3 + 1$ is obtained ____________ $y = x^3 + 1$ __________

____

In general,

The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the y-axis.

1.4 Combinations of transformations

A foolproof guide to obtaining transformations which involve various combinations of basic transformations, for example, $\boldsymbol{y = c\,\mathrm{f}(bx + a) + d}$ is the

AM MA Procedure :

Do the transformations inside the bracket followed by those outside the bracket.
Within the bracket, **addition** is done before **multiplication** (i.e. horizontal translation followed by scaling parallel to x-axis).
Outside the bracket, **multiplication** is done before **addition** (i.e. scaling parallel to y-axis followed by vertical translation).

Note:

The AM-MA procedure is just a guide in performing Combinations of Transformations, it is NOT a THEOREM to be quoted explicitly in examinations.

THE AM-MA PROCEDURE

To obtain $y = c\,\mathrm{f}(bx + a) + d$ from $y = \mathrm{f}(x)$, remember:

A M ***M A***

$\longrightarrow \boxed{x} \longrightarrow$ $\longrightarrow \boxed{y} \longrightarrow$

Do <u>inside</u> bracket first
i.e. $\mathrm{f}(\boldsymbol{b}x + \boldsymbol{a})$ which transforms parallel to x-axis

↓

Within bracket, do translation first i.e. **<u>A</u>**dd**:** $\mathrm{f}(x)$ to $\mathrm{f}(x + \boldsymbol{a})$

then do scale/reflect i.e.
<u>M</u>ultiply**:** $\mathrm{f}(x + a)$ to $\mathrm{f}(\boldsymbol{b}x + a)$

Finally, <u>outside</u> bracket i.e.
$\boldsymbol{c}\mathrm{f}(bx+a) + \boldsymbol{d}$ which transforms parallel to y -axis

↓

Outside bracket, do scale/reflect first i.e. **<u>M</u>**ultiply: $\boldsymbol{c}\mathrm{f}(bx+a)$

then do translation i.e.
<u>A</u>dd: $\boldsymbol{c}\mathrm{f}bx+a) + \boldsymbol{d}$

<u>Note:</u>

Combinations of Transformations can be executed in varied (other) ways but for the following examples we shall use the AM-MA procedure.

Example 10:

Describe the following sequences of transformations:

i) $y = \mathrm{f}(2x + 1)$

→ represents a translation by 1 unit in the negative x-direction **followed by** a scaling parallel to the x-axis with a scale factor of $\frac{1}{2}$.

ii) $y = 3\,\mathrm{f}(-x)$

→

iii) $y = \mathrm{f}\left(\frac{x}{3}\right) - 1$

→

iv) $y = 2\mathrm{f}\left(3x - 1\right) + 1$

For (iv), the graph of $y = 2\mathrm{f}\left(3x - 1\right) + 1$ is obtained from the graph of $y = \mathrm{f}(x)$ through a sequence of transformations.

A – Addition –

M – Multiplication –

M – Multiplication –

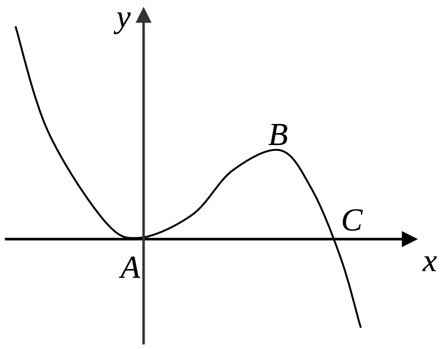
A – Addition –

Example 11:

The graph of $y = \mathrm{f}(x)$ is shown below, where points A, B, and C have coordinates (0, 0), (2,1) and (3,0) respectively. Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2x + 1)$ (ii) $y = \mathrm{f}(1 - x)$ (iii) $y = 2\,\mathrm{f}(x) - 1$

giving the coordinates of the points corresponding to A, B and C.



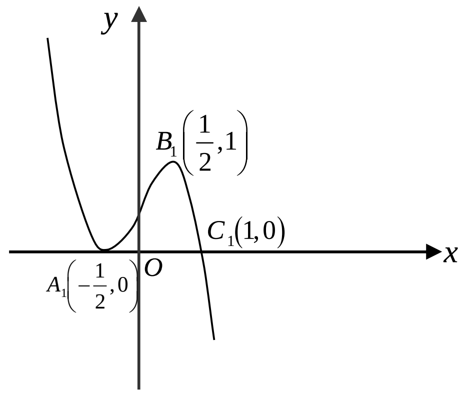
Solution:

(i) $y=\mathrm{f}(x) \to y=\mathrm{f}(x+1) \to y=\mathrm{f}(2x+1)$

A: $(0,0) \to (-1,0) \to A_1(-\frac{1}{2},0)$

B: $(2,1) \to (1,1) \to B_1(\frac{1}{2},1)$

C: $(3,0) \to (2,0) \to C_1(1,0)$

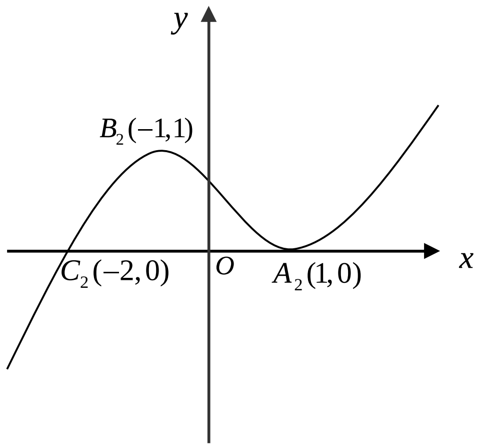


(ii) $y=\mathrm{f}(x) \to y=\mathrm{f}(1+x) \to y=\mathrm{f}(1-x)$

A: $(0,0) \to (-1,0) \to A_2(1,0)$

B: $(2,1) \to \quad \to B_2(-1,1)$

C: $(3,0) \to \quad \to C_2(-2,0)$

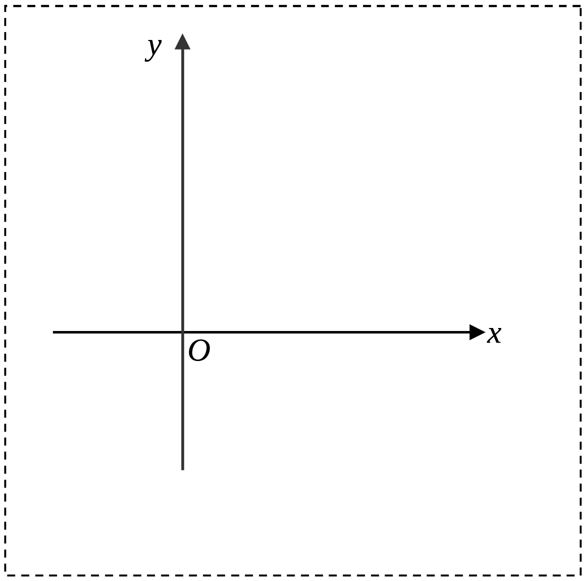


(iii) $y=\mathrm{f}(x) \to y=2\mathrm{f}(x) \to y=2\mathrm{f}(x)-1$

A: $(0,0) \to \quad \to A_3(0,-1)$

B: $(2,1) \to \quad \to B_3(2,1)$

C: $(3,0) \to \quad \to C_3(3,-1)$



Example 12:

The curve $y=x^2+2x-3$ undergoes the following transformations, in succession:

A: a translation by 2 units in the positive *x*-direction
B: a reflection in the *x*-axis
C: scale parallel to *y*-axis by a scale factor of 2

Give the equation of the resulting curve.

Solution:

$y=\text{f}(x)=x^2+2x-3$ (after *A*)

$y=\text{f}(x-2)=$

(after *B*)

$y=-\text{f}(x-2)=$

(after *C*)

$y=2\left[-\text{f}(x-2)\right]=$

Example 13:

The curve $y=\text{f}\left(x\right)$ undergoes the following transformations, in succession:

A: a translation by 2 units in the negative *x*-direction
B: a reflection in the *y*-axis
C: a scaling parallel to the *y*-axis by a factor of 5

The equation of the resulting curve is $y=5\text{e}^{-x+2}$.

Determine the equation of the curve $y=\text{f}\left(x\right)$.

Solution:

We work backwards:

Before C :

and we get

Before B :

and we get

Before A :

and we get

Therefore $y=\text{f}\left(x\right)=$

Alternatively, we work forward:

From $y=\text{f}(x)$
After *A*: $\text{f}(x+2)$
After *B*: $\text{f}(-x+2)$
After *C*: $5\,\text{f}(-x+2)$

$5\,\text{f}(-x+2)=5\,\text{e}^{-x+2}$

$\therefore\ \text{f}(-x+2)=\text{e}^{-x+2}$

Let $u=-x+2\Rightarrow x=2-u$

$\text{f}(u)=\text{e}^{-(2-u)+2}=\text{e}^{u}$

Replace u by x

$\Rightarrow \text{f}(x)=\text{e}^{x}$

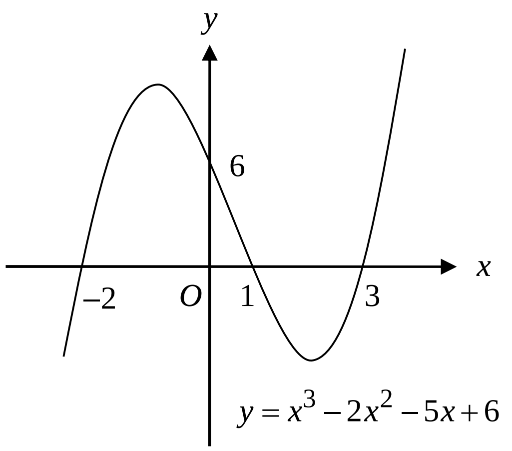
2 Other Transformations

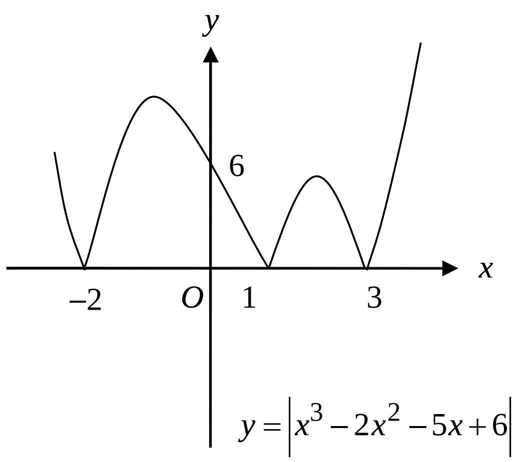
2.1 Graphs of $y=|\mathrm{f}(x)|$ and $y=\mathrm{f}(|x|)$:

(I) Graphs of the form $y=|\mathrm{f}(x)|$

Example 14:

The diagrams show the graphs of $y=x^3-2x^2-5x+6$ and $y=\left|x^3-2x^2-5x+6\right|$. How are these two graphs related to each other?





The graph of $y=\left|x^3-2x^2-5x+6\right|$ is obtained by

(a) retaining $y=\mathrm{f}(x)$, where $\mathrm{f}(x)\geq 0$ (the part of the original curve above and on the x-axis)

followed by

(b) reflecting $y=\mathrm{f}(x)$, where $\mathrm{f}(x)<0$ (the part of the original curve below the x-axis) in the x-axis.

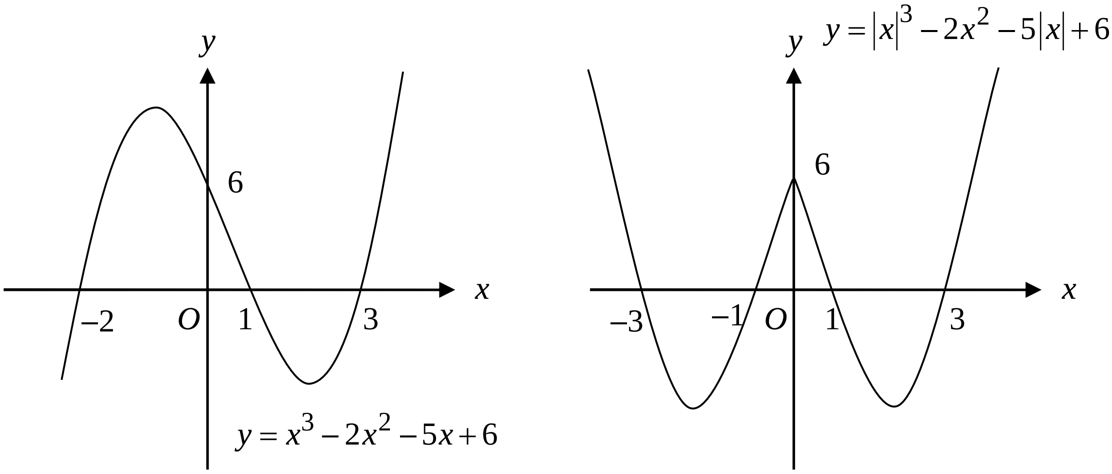
In general,

the graph of $y=\lvert\mathrm{f}(x)\rvert$ is obtained by (a) retaining $y=\mathrm{f}(x)$, where $\mathrm{f}(x)\geq 0$ (the part of the $y=\mathrm{f}(x)$ graph above and on the x – axis) (b) reflecting $y=\mathrm{f}(x)$, where $\mathrm{f}(x)<0$ (the part of the $y=\mathrm{f}(x)$ graph below the x – axis) in the x – axis.

(II) Graphs of the form $y = \mathrm{f}\left(|x|\right)$

Example 15:

The diagrams show the graphs of $y = x^3 - 2x^2 - 5x + 6$ and $y = |x|^3 - 2x^2 - 5|x| + 6$. How are these two graphs related to each other?



The graph of $y = |x|^3 - 2x^2 - 5|x| + 6$ is obtained by
(a) discarding $y = \mathrm{f}(x)$, where $x < 0$ (the part of the original curve on the left of the *y*-axis)
(b) retaining $y = \mathrm{f}(x)$, where $x \geq 0$ (the part of the original curve on the right and on the *y*-axis)
(c) reflecting $y = \mathrm{f}(x)$, where $x \geq 0$ (the part of the original curve on the right and on the *y*-axis) in the *y*-axis.

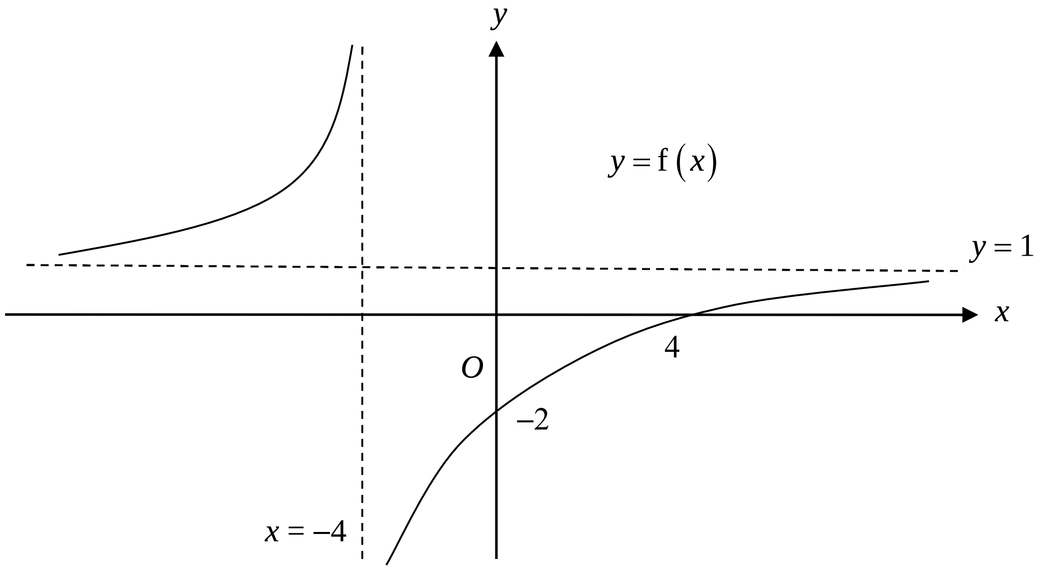
In general,

the graph of $y = \mathrm{f}\left(|x|\right)$ is obtained by
(a) discarding the part of the graph $y = \mathrm{f}(x)$ where $x < 0$ (the part of the $y = \mathrm{f}(x)$ curve on the left of the *y*-axis)
(b) retaining $y = \mathrm{f}(x)$, where $x \geq 0$ (the part of the $y = \mathrm{f}(x)$ curve on the right and on the *y*-axis)
(c) reflecting $y = \mathrm{f}(x)$, where $x \geq 0$ (the part of the $y = \mathrm{f}(x)$ curve on the right of the *y*-axis) in the *y*-axis.

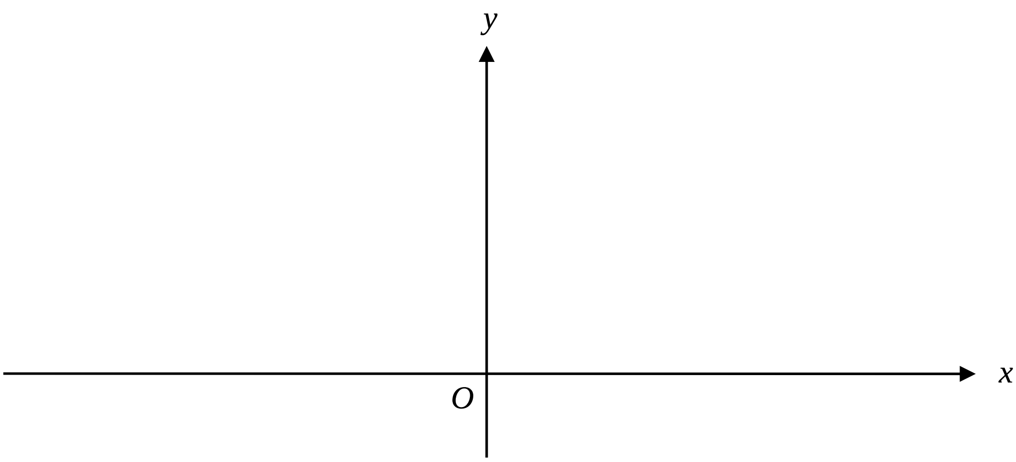
Example 16:

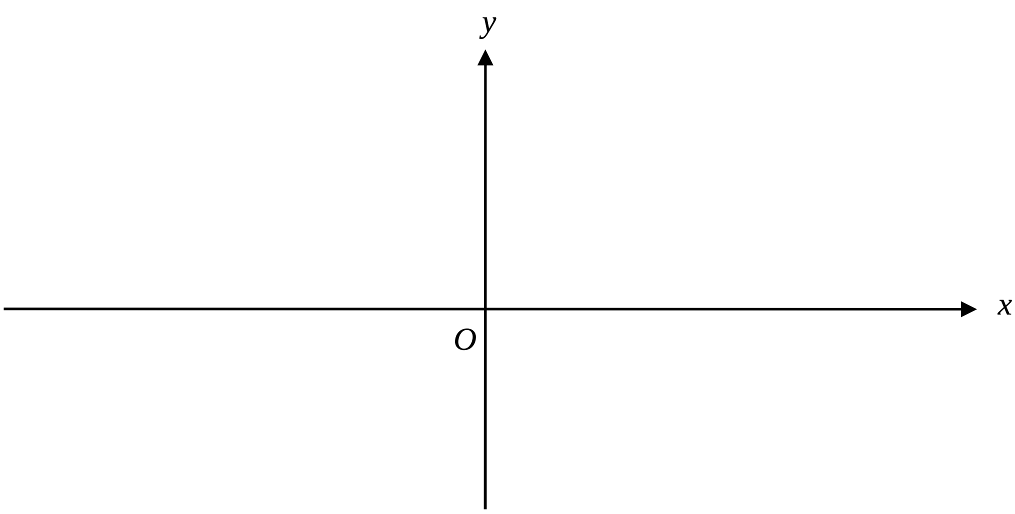
The diagram below shows the graph of $y = \mathrm{f}(x)$.

Sketch, on separate diagrams, the curves $y = |\mathrm{f}(x)|$ and $y = \mathrm{f}(|x|)$.



Solution:

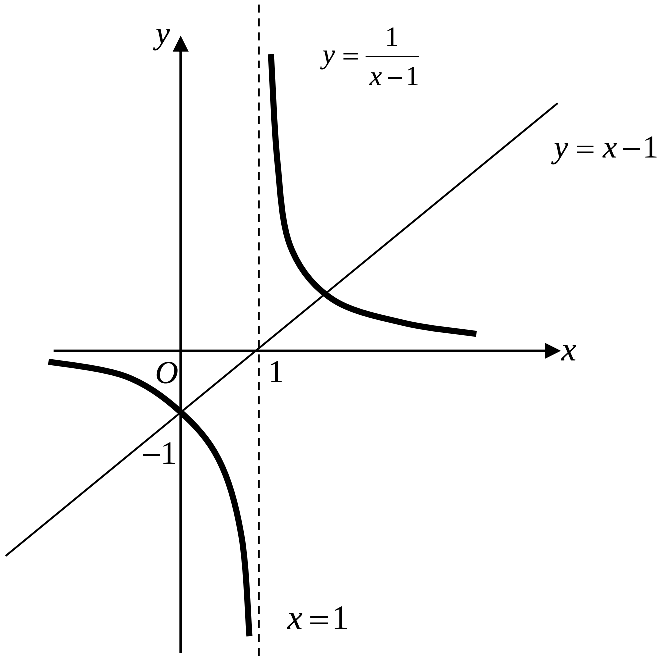




2.2 Graphs of $y=\dfrac{1}{\mathbf{f}(x)}$

Example 17:

The diagram below shows the graph of $y=x-1$ and $y=\dfrac{1}{x-1}$ on the same axes.



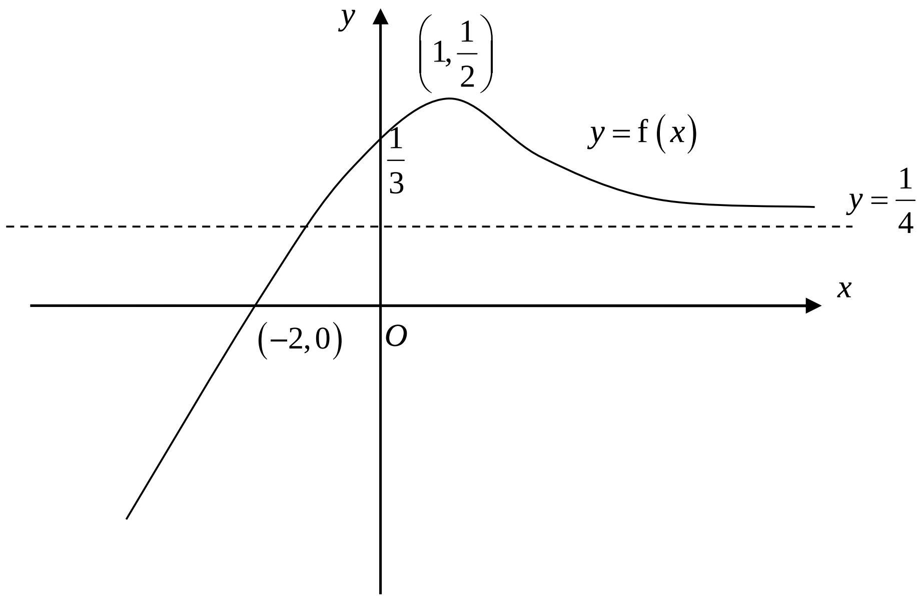
In general, the graph of $y=\dfrac{1}{\mathrm{f}(x)}$ can be deduced from the graph of $y=\mathrm{f}(x)$ using the following guidelines:

	Points/Lines on $y=\mathbf{f}(x)$	**Corresponding Points/Lines on $y=\dfrac{1}{\mathbf{f}(x)}$**
1	(i) x-intercept at $(h,0)$ (ii) Vertical asymptote $x=h$	(i) Vertical asymptote $x=h$ (ii) x-intercept at $(h,0)$
2	(i) Horizontal asymptote $y=k$, $k\neq 0$ (ii) Horizontal asymptote $y=0$ (iii) $\mathrm{f}(x)\to$ $\mathrm{f}(x)\to$	(i) Horizontal asymptote $y=\dfrac{1}{k}$ (ii) $\dfrac{1}{\mathrm{f}(x)}\to\pm\ \infty$, no horizontal asymptote (iii) $\dfrac{1}{\mathrm{f}(x)}\to$ $\dfrac{1}{\mathrm{f}(x)}\to$
3	Oblique asymptote $y=\mathrm{g}(x)$	Horizontal asymptote $y=0$
4	(i) (h,k) , $k\neq 0$ (ii) Maximum point (h,k) , $k\neq 0$ (iii) Minimum point (h,k) , $k\neq 0$	(i) $\left(h,\dfrac{1}{k}\right)$ (ii) Minimum point $\left(h,\dfrac{1}{k}\right)$ (iii) Maximum point $\left(h,\dfrac{1}{k}\right)$
5	(i) $\mathrm{f}(x)>0$ (curve above x-axis) (ii) $\mathrm{f}(x)<0$ (curve below x-axis)	(i) $\dfrac{1}{\mathrm{f}(x)}>0$ (curve above x-axis) (ii) $\dfrac{1}{\mathrm{f}(x)}<0$ (curve below x-axis)
6	(i) $\mathrm{f}(x)$ is **<u>increasing</u>** as x increases (ii) $\mathrm{f}(x)$ is **<u>decreasing</u>** as x increases	(i) $\dfrac{1}{\mathrm{f}(x)}$ is **<u>decreasing</u>** as x increases (ii) $\dfrac{1}{\mathrm{f}(x)}$ is **<u>increasing</u>** as x increases

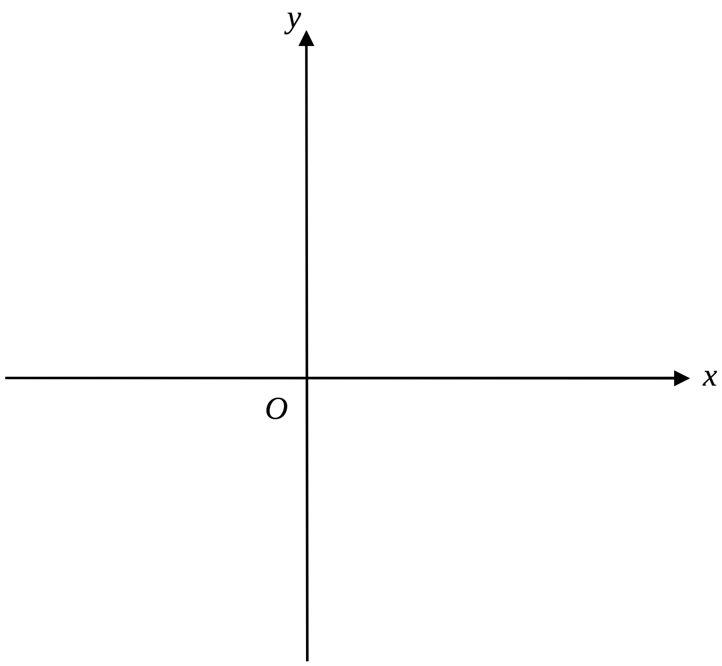
Example 18:

The diagram below shows the graph of $y = \mathrm{f}(x)$.

Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$.



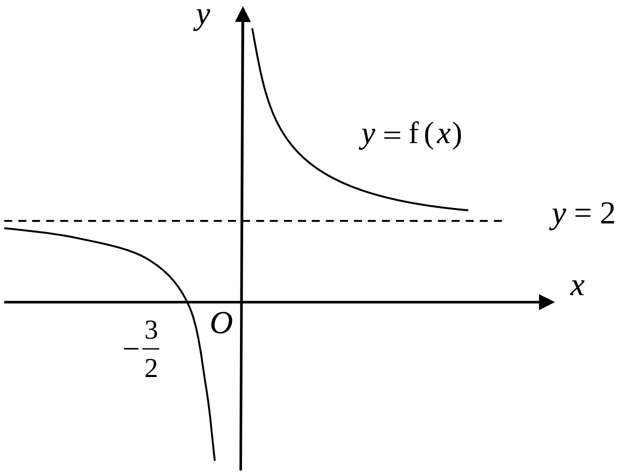
Solution:



Example 19:

The diagram below shows the graph of $y = \mathrm{f}(x)$.

Sketch the graph of $y = \dfrac{1}{\mathrm{f}(x)}$.



Solution: