6. Maclaurin Series

1 SAJC/2015Promo/4

Given that the first three non-zero terms of the series expansion of $(1-x)^{\frac{1}{3}}$ are equal to the first three non-zero terms of the series expansion of $\dfrac{3+ax}{3+bx}$, where a and b are constants, find the values of a and b. [4]

2 RI/2009Prelim/I/2

Given that $y^2+\mathrm{e}^x+3y=5,\ \ y>0$,

(i) show that $2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\left(2y+3\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^x=0$, [2]

(ii) find the first three terms of the Maclaurin's series for y. [3]

3 RI/2014Promo/7

It is given that

$$y=\frac{\ln(1+2x)}{1+x}.$$

(i) Show that $(1+x)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\dfrac{\mathrm{d}y}{\mathrm{d}x}+\dfrac{4}{(1+2x)^2}=0$. **[3]**

(ii) Find the Maclaurin's series for y, up to and including the term in x^3. **[3]**

(iii) Verify that the same result is obtained if the standard series expansions for $\ln(1+2x)$ and $(1+x)^{-1}$ are used. **[3]**

4 YJC/2015Promo/6

It is given that $\mathrm{f}\left(x\right)=\dfrac{1}{\sqrt{9-x^2}}$.

(i) Find the expansion of f (x) in ascending powers of x, up to and including the term in x^4. Find the range of values of x for which the expansion is valid. [4]

(ii) By letting $x=\dfrac{1}{2}$, find an approximation for $\sqrt{35}$, leaving your answer as a fraction in its lowest terms. [2]

5 DHS/2015Promo/11

(i) Express $\mathrm{f}(x)=\dfrac{5x}{(1+2x)(1+x^2)}$ as $\dfrac{A}{1+2x}+\dfrac{Bx+C}{1+x^2}$. [2]

(ii) Hence obtain the series expansion of $\mathrm{f}(x)$ up to and including the term in x^3. [3]

(iii) Using your result in part **(ii)**, find an approximate value for $\displaystyle\int_0^1 \frac{x}{(1+2x)(1+x^2)}\,\mathrm{d}x$. Give your answer in its lowest fraction. [2]

(iv) Use your calculator to evaluate $\displaystyle\int_0^1 \frac{x}{(1+2x)(1+x^2)}\,\mathrm{d}x$. Justify whether your answer in part **(iii)** is an appropriate approximation. [2]

6 RI/2015Promo/8

(a) (i) Expand $\mathrm{f}(x)=\dfrac{2}{2-x}-\dfrac{1}{(1+x)^2}$ as a series in ascending powers of x up to and including the term in x^2. **[3]**

(ii) State the equation of the tangent to the curve $y=\mathrm{f}(x)$ at the origin. **[1]**

(b) Using the standard series given in the List of Formulae (MF26) or otherwise, show that the first three non-zero terms in the Maclaurin series for $\mathrm{e}^{\sqrt{1+x}}$ can be expressed as $\mathrm{e}(1+px+qx^3+\ldots)$ where p and q are constants to be determined. **[6]**

7 TPJC/2009Prelim/I/7

It is given that $\tan^{-1} y=\ln(1+x)$.

(i) Prove that $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+y^2$. [1]

(ii) By successively differentiating this result three times, find the Maclaurin's series for $\tan(\ln(1+x))$, up to and including the term in x^4. [6]

(iii) Show on a sketch the shape of the graph of $y=\tan(\ln(1+x))$ for small x, indicating clearly the relationship of the graph to that of $y=x$. [2]

8 NYJC/2015Promo/9 (Part)

(a) Expand $\dfrac{8}{(2-x)^2}$ in ascending powers of x up to and including the term in x^2, stating the range of values of x for which the expansion is valid. **[4]**

(b) Given that $y = \cot\left(2x + \dfrac{\pi}{4}\right)$, show that $\dfrac{dy}{dx} = -2\left(1 + y^2\right)$. **[2]**

By further differentiation, find the Maclaurin series for y, up to and including the term in x^3. **[5]**

Hence show that $\tan\left(2x + \dfrac{\pi}{4}\right) \approx a + bx + cx^2$ where a, b and c are constants to be determined. **[4]**

9 NJC/2015Promo/3

Given that x and y are related by $\left(4 - x^2\right)\dfrac{dy}{dx} = x^2 - 3,$ where $-2 < x < 2$,

and that $y = 2$ when $x = 0$,

(i) find y in terms of x, [3]

(ii) by differentiating further, or otherwise, find the first three non-zero terms of the Maclaurin's series for y. [3]

10 SAJC/2009Prelim/I/3

Given that $y = e^{3\tan^{-1}x}$, show that

$$\left(1 + x^2\right)\frac{dy}{dx} = 3y.$$

By repeated differentiation of this result, find the Maclaurin's expansion for y, up to and including the term in x^3. [5]

Hence, deduce the Maclaurin's expansion for $e^{2x+3\tan^{-1}x}$, up to and including the term in x^2. [2]

11 RVHS/2015Prelim/I/8

Given that $y = \sin^{-1}\left[\ln(x+1)\right]$, show that

$$\cos y\frac{dy}{dx} = \frac{1}{x+1} \quad \text{and} \quad \cos y\frac{d^2y}{dx^2} - \sin y\left(\frac{dy}{dx}\right)^2 = -\frac{1}{(x+1)^2}.$$ [2]

(i) By further differentiation, find the Maclaurin series for y, up to and including the term in x^3. [3]

(ii) Find the set of values of x for which the value of y is within ± 0.1 of the value found by its Maclaurin series. [3]

(iii) Deduce the series expansion for $\dfrac{1}{(x+1)\sqrt{1-(\ln(x+1))^2}}$ up to and including the term in x^2. [2]

12 DHS/2009Prelim/II/3(a)

It is given that $\dfrac{dy}{dx} = \dfrac{e^{\tan^{-1} x}}{1+x^2}$, where $\tan^{-1} x$ denotes the principal value.

(i) Find an expression for y in terms of x given that $y = 1$ when $x = 0$. [2]

(ii) Show that $\left(1+x^2\right)\dfrac{d^2y}{dx^2} + (2x-1)\dfrac{dy}{dx} = 0$. [2]

(iii) By further differentiation of the result in **(ii)**, find the Maclaurin series for y up to and including the term in x^3. [4]

(iv) State the series expansion for $\dfrac{e^{\tan^{-1} x}}{1+x^2}$ up to and including the term in x^2. [1]

13 RVHS/2015Promo/4

Given that $y = e^x \sin^2 x + 1$, show that $\dfrac{dy}{dx} = e^x \sin 2x + y + a$ and

$\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2e^x \cos 2x + b$, where a and b are constants to be determined. [3]

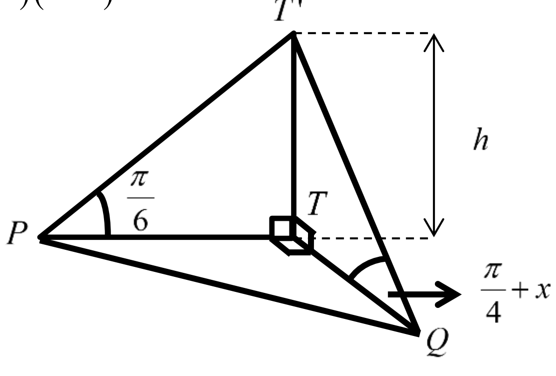
(i) Hence find the Maclaurin series for y, up to and including the term in x^2. [2]

(ii) Write down the first three terms in the series expansion for $(1-x)^{-1}$. Deduce the Maclaurin series of $\dfrac{e^{-2x}\sin^2 2x + 1}{1-x}$ up to and including the term in x^2. [3]

14 JJC/2015Prelim/I/4

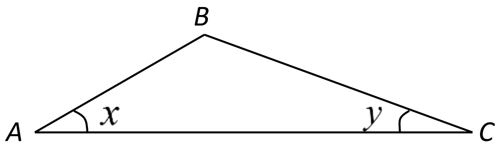
Two ground spotlights P and Q are shining at the top of a tower TT' of height h m. P is due west and Q is due south of the tower. The angle TPT' is $\dfrac{\pi}{6}$ radians and the angle TQT' is $\left(\dfrac{\pi}{4} + x\right)$ radians, where x is small.

Show that $QT = h(1-x)(1+x)^{-1}$. [3]



Hence, by using the standard results in MF26, show that $PQ^2 \approx 4h^2\left(1 - x + 2x^2\right)$. [3]

15 ACJC/2015Promo/3



The diagram above shows triangle *ABC* which has fixed points *A* and *C* and a variable point *B* such that $AB = kBC$, where *k* is a constant such that $0 < k < 1$. The angles *BAC* and *BCA*, measured in radians, are x and y respectively.

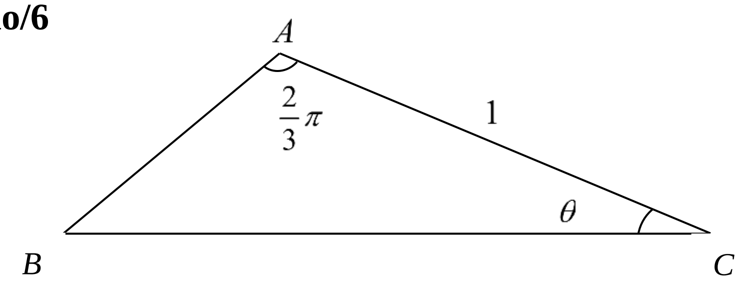
(i) Show that $\sin y = k \sin x$. [1]

(ii) By successively differentiating the equation in part **(i)**, show that

$$\cos y \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \sin y \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -k \sin x\,. \quad [1]$$

Hence find the Maclaurin series for y, up to and including the term x^3. [3]

16 IJC/2015Promo/6



In the triangle *ABC*, $AC = 1$, angle $BAC = \frac{2}{3}\pi$ radians and angle $ACB = \theta$ radians (see diagram).

(i) Show that $BC = \dfrac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that $BC \approx 1 + \dfrac{\theta}{\sqrt{3}} + p\theta^2$, where *p* is a constant to be determined. [4]

17 IJC/2015Promo/10

(i) Given that $y = \mathrm{e}^{\sin^{-1} 3x}$, show that $\left(1 - 9x^2\right)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 9x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 9y$. [3]

(ii) By further differentiation of the result in part **(i)**, find the Maclaurin series for *y* in ascending powers of *x*, up to and including the term in x^3. [4]

(iii) Hence find an approximate value of $\mathrm{e}^{-\frac{\pi}{2}}$, giving your answer as a fraction in its simplest form. [2]

18 MI/I/2015Promo/10

(a) Given that $\mathrm{f}(x)=\sin\left(2x+\frac{\pi}{4}\right)$, find the exact values of $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$.

Hence, write down the first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [5]

(b) In the triangle *ABC*, *AB* = 2, *BC* = 4 and angle *ABC* = θ radians. Given that θ is a sufficiently small angle, show that $AC \approx \left(4+8\theta^2\right)^{\frac{1}{2}} \approx a+b\theta^2$, where a and b are constants to be determined. [5]

19 SRJC/2009Prelim/I/11

(i) Given that $y=\ln\left(\frac{\cos x}{\mathrm{e}-x}\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\tan x+\frac{1}{(\mathrm{e}-x)}$. By further differentiation of this result, or otherwise, find the Maclaurin's series of y up to and including the term in x^3. [6]

(ii) Deduce the Maclaurin's series for the equation $y=\ln\left(\frac{\cos(\mathrm{e}x)}{1+x}\right)$ up to and including the term in x^3. [2]

20 TJC/2014Promo/8

Given that $y=(1-\sin x)^{\frac{1}{2}}$, show that $2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+y^2-1=0$. [2]

By further differentiation, find the Maclaurin's series of y in ascending powers of x up to and including the term in x^3. [4]

Deduce the Maclaurin's series of cos x up to and including the term in x^2. [3]

21 PJC/2015Promo/10

(iii) Given that $y=\tan(3\mathrm{e}^x-3)$, show that

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\mathrm{e}^x\left(1+y^2\right).$$ [2]

Hence find the Maclaurin's series for y, up to and including the term in x^2. [4]

(ii) Use the standard results given in the List of Formulae (MF26) to find the first two non-zero terms of $(1+ax)^{10}\sin bx$, where a and b are constants. [2]

The first two non-zero terms in the Maclaurin series for y are equal to the first two non-zero terms in the series expansion of $(1+ax)^{10}\sin bx$, find the values of a and b. [2]

22 VJC/2015Promo/8 (modified)

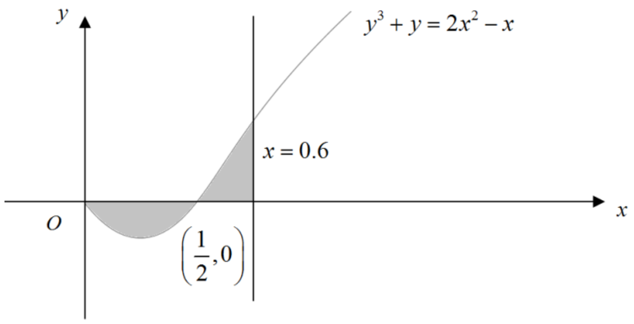
(a) **(i)** Find the derivative of $\sqrt{1-x^2}$. [2]

(ii) Hence, find $\int \sin^{-1} x \, dx$. [3]

(b) **(i)** Given that $y^3 + y = 2x^2 - x$, show that $6y\left(\frac{dy}{dx}\right)^2 + \frac{d^2y}{dx^2}\left(3y^2+1\right) = 4$. [2]

(ii) Find the series expansion of y, up to and including the term in x^2. [3]

The diagram below shows the graph of C, given by $y^3 + y = 2x^2 - x$ for $x \geq 0$ and the line $x = 0.6$.



(iii) Find an approximation for the area of the shaded region. How can the approximation be made better? [2]

23 MI/2020Promo/PU2/P1/Q6

(a) Using standard series from the List of Formulae (MF26), expand $e^{3x} \ln(1+ax)$ as far as the term in x^3, where a is a non-zero constant. Given that there is no term in x^2, determine the coefficient of x^3. [5]

(b) Find the expansion of $\frac{1+3x}{\sqrt{9-x^2}}$ in ascending powers of x, up to and including the term in x^3. State the set of values of x for which the expansion is valid. [4]

Answers

1. $b = -1$, $a = -2$
2. (ii) $y = 1 - \frac{1}{5}x - \frac{27}{250}x^2 + ...$
3. (ii) $y = 2x - 4x^2 + \frac{20}{3}x^3 + ...$
4. (i) $\frac{1}{3}\left(1 + \frac{x^2}{18} + \frac{x^4}{216} + ...\right)$, $-3 < x < 3$ (ii) $\frac{20736}{3505}$

5. (i) $\mathrm{f}(x)=-\frac{2}{1+2x}+\frac{x+2}{1+x^2}$ (ii) $\mathrm{f}(x)=5x-10x^2+15x^3+\ldots$ (iii) $\frac{7}{12}$ (iv) 0.164

6. (a)(i) $\frac{5x}{2}-\frac{11}{4}x^2+\ldots$ (ii) $y=\frac{5x}{2}$ (b) $\mathrm{e}\left(1+\frac{1}{2}x+\frac{1}{48}x^3+\ldots\right)$

7. (ii) $x-\frac{1}{2}x^2+\frac{2}{3}x^3-\frac{3}{4}x^4+\cdots$.

8. (a) $2+2x+\frac{3}{2}x^2+\ldots$, $-2<x<2$ (b) $y=1-4x+8x^2+\ldots$

9. (i) $y=-x+\frac{1}{4}\ln\left(\frac{2+x}{2-x}\right)+2$ (ii) $2-\frac{3}{4}x+\frac{1}{48}x^3+\ldots$

10. $1+3x+\frac{9}{2}x^2+\frac{7}{2}x^3+\ldots$, $1+5x+\frac{25}{2}x^2+\ldots$.

11. (i) $y=x-\frac{1}{2}x^2+\frac{1}{2}x^3+\ldots$ (ii) $\{x\in\mathbb{R}:-0.521<x<0.783\}$ (iii) $1-x+\frac{3}{2}x^2+\ldots$

12. (i) $y=\mathrm{e}^{\tan^{-1}x}$, (iii) $1+x+\frac{1}{2}x^2-\frac{1}{6}x^3+\cdots$, (iv) $1+x-\frac{1}{2}x^2+\cdots$.

13. $a=-1, b=1$ (i) $y=1+x^2+\ldots$ (ii) $1+x+x^2+\ldots$, $1+x+5x^2+\ldots$

15. (ii) $y=kx+\frac{k\left(k^2-1\right)}{6}x^3+\ldots$

16. (ii) $1+\frac{\theta}{\sqrt{3}}+\frac{5}{6}\theta^2$

17. (ii) $y=1+3x+\frac{9}{2}x^2+9x^3+\ldots$ (iii) $\frac{1}{6}$

18. (a) $\mathrm{f}\left(0\right)=\frac{\sqrt{2}}{2}$, $\mathrm{f}'\left(0\right)=\sqrt{2}$, $\mathrm{f}''\left(0\right)=-2\sqrt{2}$, $\mathrm{f}\left(x\right)=\frac{\sqrt{2}}{2}+\sqrt{2}x-\sqrt{2}x^2+\ldots$ (b) $2+2\theta^2$

19. (i) $-1+\frac{1}{\mathrm{e}}x+\frac{1}{2}\left(\frac{1}{\mathrm{e}^2}-1\right)x^2+\frac{1}{3\mathrm{e}^3}x^3+\cdots$, (ii) $-x+\frac{1}{2}\left(1-\mathrm{e}^2\right)x^2-\frac{1}{3}x^3+\cdots$.

20. $\left(1-\sin x\right)^{\frac{1}{2}}=1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{48}x^3+\ldots$; $\cos x=1-\frac{1}{2}x^2+\ldots$

21. (i) $y=3x+\frac{3}{2}x^2+\ldots$ (ii) $bx+10abx^2+\ldots$, $a=\frac{1}{20}$ and $b=3$

22. (a)(i) $-\frac{x}{\sqrt{1-x^2}}$ (a)(ii) $x\sin^{-1}x+\sqrt{1-x^2}+C$ (b)(ii) $-x+2x^2+\ldots$ (b)(iii) 0.0473

23. (a) $ax+\left(3a-\frac{a^2}{2}\right)x^2+\left(\frac{a^3}{3}-\frac{3a^2}{2}+\frac{9a}{2}\right)x^3+\ldots$; 45,

(b) $\frac{1}{3}+x+\frac{1}{54}x^2+\frac{1}{18}x^3+\ldots$; $-3<x<3$