RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Chapter 4B: Complex Numbers II: Polar Form

SYLLABUS INCLUDES

- representation of complex numbers in the Argand diagram
- complex numbers expressed in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ or $re^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta < \pi$.
- calculation of modulus (r) and argument (θ) of a complex number
- multiplication and division of two complex numbers expressed in polar form

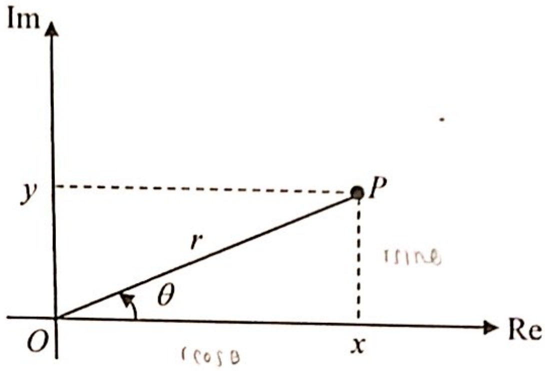
PRE-REQUISITES

- Trigonometry,
- Coordinate Geometry,
- Vectors,
- Indices and algebraic manipulation

CONTENT

1 Complex Numbers in Polar Form

Consider a point P on a Argand diagram representing the complex number, $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$ with $|z| = r$ and $\arg(z) = \theta$, $-\pi < \theta \le \pi$.



It can be easily seen from the diagram that $x = r\cos\theta$ and $y = r\sin\theta$.

So
$$\begin{aligned} z &= x + \mathrm{i}y \\ &= r\cos\theta + \mathrm{i}(r\sin\theta) \\ &= r(\cos\theta + \mathrm{i}\sin\theta) \end{aligned}$$

Furthermore, it can be proven that

$$\boxed{\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta} \text{ (Euler's formula)}}$$

(Refer to Appendix for further details.)

So any complex number can be written as

$$\boxed{\begin{aligned} z &= x + \mathrm{i}y \qquad \text{(cartesian form)} \\ &= \left.\begin{aligned} r(\cos\theta + \mathrm{i}\sin\theta) \\ r\mathrm{e}^{\mathrm{i}\theta} \end{aligned}\right\} \text{(polar form)} \\ &\text{where } |z| = r \text{ and } \arg(z) = \theta,\ -\pi < \theta \le \pi. \end{aligned}}$$

Example 1

(a) Express $2e^{i\frac{\pi}{6}}$ in the form $a+ib$, $a, b \in \mathbb{R}$.

(b) If $|z| = 4$ and $\arg(z) = \frac{2\pi}{3}$, express z in cartesian form.

(c) If $z_1 = 2+2i$ and $z_2 = \sqrt{3} - i$, express z_1 and z_2 in polar form.

Solution

(a)

$2e^{i\frac{\pi}{6}} = 2\cos\frac{\pi}{6} + 2i\sin\frac{\pi}{6}$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= \sqrt{3} + i$

(b)

$z = 4e^{i\frac{2\pi}{3}}$

$z = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$= 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -2 + i(2\sqrt{3})$

(c)

$|z_1| = \sqrt{4+4} = 2\sqrt{2}$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$\arg(z_1) = \frac{\pi}{4}$

$z_1 = 2\sqrt{2}e^{i\frac{\pi}{4}}$

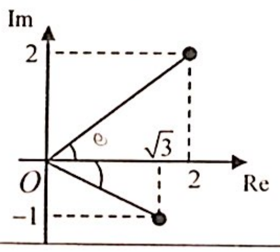
$|z_2| = \sqrt{3+1} = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\arg(z_2) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= -\frac{\pi}{6}$

$z_2 = 2e^{-i\frac{\pi}{6}}$



1.1 Multiplication and Division of Complex Numbers in Polar Form

It is straight forward to add or subtract complex numbers in cartesian form.
However, when it comes to multiplication and division, it is much easier to use the polar form.

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, where $r_1, r_2 > 0$ and $-\pi < \theta_1, \theta_2 \le \pi$.

$|z_1 z_2| = r_1 r_2 = |z_1||z_2|$

$\arg(z_1 z_2) = \theta_1 + \theta_2$

$= \arg(\theta_1) + \arg(\theta_2)$

Then $z_1 z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

$$z_1^n = \underbrace{z_1 \times z_1 \times \ldots \times z_1}_{n \text{ times}} = \underbrace{r_1 e^{i\theta_1} \times r_1 e^{i\theta_1} \times \ldots \times r_1 e^{i\theta_1}}_{n \text{ times}} = \underbrace{r_1 \times \ldots \times r_1}_{n \text{ times}} \times \underbrace{e^{i\theta_1} \times \ldots \times e^{i\theta_1}}_{n \text{ times}} = r_1^n e^{in\theta_1}$$

Hence

(1) $|z_1 z_2| = |z_1||z_2|$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) = \theta_1 + \theta_2$, provided $-\pi < \theta_1 + \theta_2 \le \pi$

(2) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

$\arg\left(\dfrac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2$, provided $-\pi < \theta_1 - \theta_2 \le \pi$

(3) For $n \in \mathbb{Z}^+$, $|z^n| = |z|^n$

$\arg(z_1^{\,n}) = n\arg(z_1) = n\theta_1$, provided $-\pi < n\theta_1 \le \pi$

Note that it is tedious to derive the above results if we use cartesian form of the complex numbers.

Example 2

(a) If $z_1 = 2 + 2\mathrm{i}$ and $z_2 = \sqrt{3} - \mathrm{i}$, express $z_1 z_2$ and $\dfrac{z_1}{z_2}$ in polar form.

(b) If $z = -\sqrt{3} + \mathrm{i}$, find the modulus and argument of z^2.

(c) Given that $|\mathrm{i}z| = 3$ and $\arg(\mathrm{i}z) = \dfrac{\pi}{4}$, express z in cartesian form.

Solution

(a) From Example 1c, $z_1 = 2\sqrt{2}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}$ and $z_2 = 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$

$z_1 z_2 = 2\sqrt{2}e^{i\frac{\pi}{4}} \times 2e^{-i\frac{\pi}{6}}$

$= 4\sqrt{2}e^{i\left(\frac{\pi}{12}\right)}$

$\dfrac{z_1}{z_2} = 2\sqrt{2}e^{i\frac{\pi}{4}} \div 2e^{-i\frac{\pi}{6}}$

$= \sqrt{2}e^{\frac{5i\pi}{12}}$

(b) $|z| = \sqrt{3+1} = 2$, $\arg(z) = \dfrac{5\pi}{6}$ $= \pi - \tan^{-1}\dfrac{1}{\sqrt{3}}$

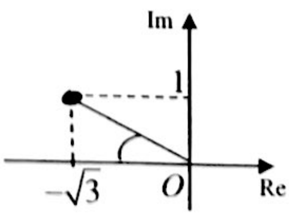
$z = 2e^{i\frac{5\pi}{6}}$ $z^2 = 4e^{i\frac{5\pi}{3}}$

OR

$|z^2| = 4$

$\arg(z^2) = \dfrac{5\pi}{3} - 2\pi$ (need $-\pi <$ $< \pi$)

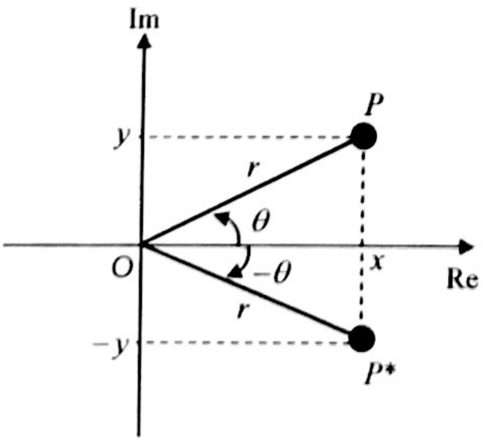
$= -\dfrac{\pi}{3}$ //



(c) $\lvert iz\rvert = \lvert i\rvert\lvert z\rvert = 1\times\lvert z\rvert = 3 \Rightarrow \lvert z\rvert = 3$ $\arg(iz) = \arg(i) + \arg(z)$ $= \frac{\pi}{4}$ $\arg(z) = \frac{\pi}{4} - \frac{\pi}{2}$ $= -\frac{\pi}{4}$ $\therefore \cancel{z = 3e^{-i\frac{\pi}{4}}}$ $z = 3e^{-i\frac{\pi}{4}} = 3\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ $= \frac{3\sqrt{2}}{2} - i\,\frac{3\sqrt{2}}{2}$	OR $\lvert iz\rvert = 3$, $\arg(iz) = \frac{\pi}{4}$ $iz = 3e^{i\frac{\pi}{4}}$ $z = 3e^{i\frac{\pi}{4}} \div e^{i\frac{\pi}{2}}$ $= 3e^{i\left(-\frac{\pi}{4}\right)}$ $= 3\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$ $= \frac{3\sqrt{2}}{2} - i\,\frac{3\sqrt{2}}{2}$

1.2 Relationships between $\arg(z)$ and $\arg(z^*)$, $\lvert z\rvert$ and $\lvert z^*\rvert$

Let the points P and P^* represent the complex numbers $z = x + \mathrm{i}y$ and $z^* = x - \mathrm{i}y$, $x, y \in \mathbb{R}$ respectively on an Argand diagram.



Observe that P and P^* are reflections of each other about the real axis.

From the diagram, it is clear that

$$\lvert z^*\rvert = \lvert z\rvert \text{ and } \arg(z^*) = -\arg(z).$$

Also,

$$zz^* = x^2 + y^2 = \lvert z\rvert^2.$$

In polar form,

if $z = re^{\mathrm{i}\theta}$, then $z^* = re^{-\mathrm{i}\theta}$.

Example 3

A complex number z is such that $|z^*| = \sqrt{2}$ and $\arg(z^*) = \dfrac{3\pi}{4}$.

Find the modulus and argument of the complex number z^2.

Solution

$|z^*| = \sqrt{2} \quad \arg(z^*) = \frac{3\pi}{4}$

$z^* = \sqrt{2}e^{i\frac{3\pi}{4}}$

$z = \sqrt{2}e^{i\,-\frac{3\pi}{4}}$

$z^2 = \left(\sqrt{2}e^{-i\frac{3\pi}{4}}\right)^2 = 2e^{-i\frac{3\pi}{2}} = 2e^{i\left(-\frac{3\pi}{2}+2\pi\right)} = 2e^{i\frac{\pi}{2}}$

$|z^2| = |z||z|$
$= 2$

$\arg(z^2) = 2\left(-\frac{3\pi}{4}\right) = -\frac{3\pi}{2} + 2\pi = \frac{\pi}{2}$ //

1.3 Complex Numbers in Cartesian Form vs Polar Form

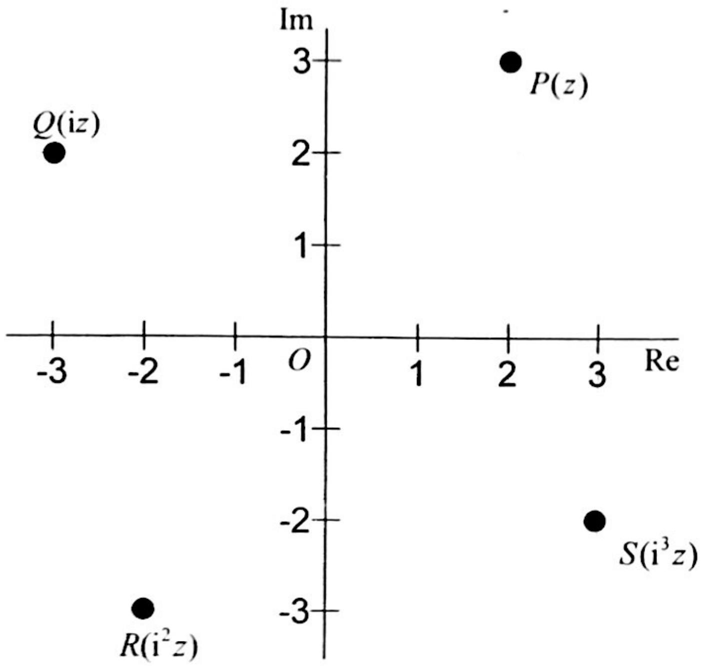
In section 1.1, it seems that multiplication and division of complex using polar form is less tedious. However, when we perform additions and subtractions, it is more direct to use cartesian form. In essence, we have to be flexible when dealing with problems involving complex numbers.

		Cartesian Form: $z = x + \mathrm{i}y$	Polar Form: $z = re^{\mathrm{i}\theta} = r(\cos\theta + \mathrm{i}\sin\theta)$
(a)	$\lvert z\rvert$	$\sqrt{x^2+y^2}$	r
(b)	z^*	$x - \mathrm{i}y$	$re^{-\mathrm{i}\theta}$
(c)	$(z^*)^* = z$	$[(x+\mathrm{i}y)^*]^*$ $= (x-\mathrm{i}y)^*$ $= x + \mathrm{i}y$	$[(re^{\mathrm{i}\theta})^*]^*$ $= (re^{-\mathrm{i}\theta})^*$ $= re^{\mathrm{i}\theta}$
(d)	zz^*	x^2+y^2	r^2
(e)	$z + z^* = 2\,\mathrm{Re}(z)$	$(x+\mathrm{i}y)+(x-\mathrm{i}y)$ $= 2x$	$r(\cos\theta+\mathrm{i}\sin\theta)+r(\cos(-\theta)+\mathrm{i}\sin(-\theta))$ $= 2r\cos\theta$
(f)	$z - z^* = 2\mathrm{i}\,\mathrm{Im}(z)$	$(x+\mathrm{i}y)-(x-\mathrm{i}y)$ $= 2\mathrm{i}y$	$r(\cos\theta+\mathrm{i}\sin\theta)-r(\cos(-\theta)+\mathrm{i}\sin(-\theta))$ $= \mathrm{i}(2r\sin\theta)$

2 Geometrical Effect of Multiplying Two Complex Numbers

2.1 Geometrical Effect of Multiplying a Complex Number by i

Consider the complex number $z = 2 + 3\text{i}$.
Draw the points P, Q, R and S representing z, $\text{i}z$, $\text{i}^2 z$ and $\text{i}^3 z$ respectively on the same Argand diagram.



Now $\text{i}z = -3 + 2\text{i}$, $\text{i}^2 z = -2 - 3\text{i}$, $\text{i}^3 z = 3 - 2\text{i}$

Notice that if P is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain Q.

If Q is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain R.

If R is rotated $\frac{\pi}{2}$ radians anti-clockwise about O, we obtain S.

Why is this so? In general, if $z = re^{\text{i}\theta}$ and we multiply it by i, we have

$$\text{i}z = \left(\text{e}^{\text{i}\frac{\pi}{2}}\right)\left(re^{\text{i}\theta}\right) = re^{\text{i}\left(\theta + \frac{\pi}{2}\right)}$$

Thus, the geometrical effect of multiplying a complex number z by i is an **anti-clockwise rotation** of P through an angle of $\frac{\pi}{2}$ **radians about O**. Note that the modulus is still the same.

Example 4

In an Argand diagram, the points A, B and C represent the complex numbers a, $6+8\mathrm{i}$ and c respectively. $OABC$ is a square described in an anti-clockwise sense, where O is the origin. Give a reason why $c=\mathrm{i}a$ and a reason why $a+c=6+8\mathrm{i}$. Find a and c by calculation (i.e. not by geometry).

Solution

when A is rotated $\frac{\pi}{2}$ radians anticlockwise about O, we get C.

hence $c = ia$

· OC is parallel and equal in length to AB; thus

$c - 0 = 6+8i - a$

thus, $c + a = 6+8i$ – ①

substitute ① into ②:

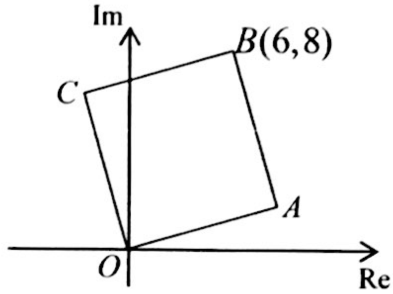
$ia + a = 6+8i$

$a = \frac{6+8i}{i+1} \cdot \frac{i-1}{i-1}$

$= 7+i$

$c = i(7+i)$

$= 7i - 1$

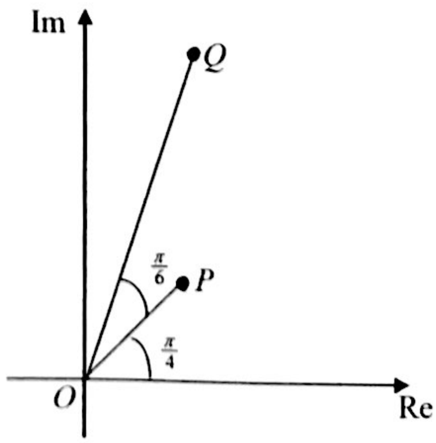


2.2 Geometrical Effect of Multiplying a Complex Number by $re^{\mathrm{i}\theta}$

(learning experience)

Let P and Q represent the complex numbers z_1 and z_1z_2 respectively, where $z_1 = 2e^{\mathrm{i}\frac{\pi}{4}}$ and $z_2 = 3e^{\mathrm{i}\frac{\pi}{6}}$.

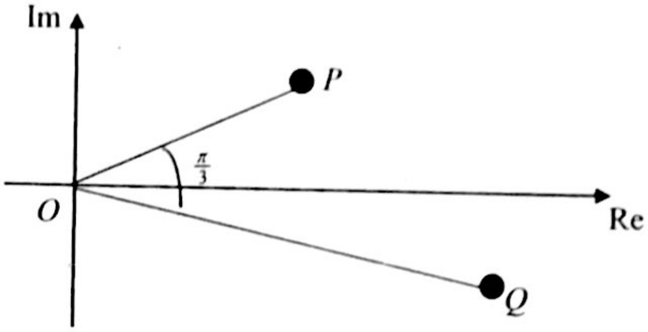
Now $z_1z_2 = \left(2e^{\mathrm{i}\frac{\pi}{4}}\right)\left(3e^{\mathrm{i}\frac{\pi}{6}}\right) = 6e^{\mathrm{i}\frac{5\pi}{12}}$



Thus, Q is obtained by scaling a factor 3 of the length OP, followed by an anti-clockwise rotation through an angle of $\frac{\pi}{6}$ radians about O.

Example 5

P and Q are 2 points on an Argand diagram such that $OQ = 2OP$ and $\angle QOP = \frac{\pi}{3}$ as shown in the diagram below. If P represents the complex number $2+\mathrm{i}$, find the complex number represented by Q in the form $a+\mathrm{i}b$, where the exact real values of a and b are to be found.



Solution

If $OQ = 2OP$ and $\angle QOP = \frac{\pi}{3}$, we can obtain Q by a scaling of factor 2 of the length OP, followed by a clockwise rotation through an angle of $\frac{\pi}{3}$ radians about O

So we have

$|OP| = \sqrt{5}$

$|OQ| = 2\sqrt{5}$

$OQ = 2\sqrt{5}\, e^{-i\frac{\pi}{3}}$

$= 2\sqrt{5}\left[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})\right]$

$= 2\sqrt{5}\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right]$ $\sqrt{5} - \sqrt{15}\,i$

Hence

$\tan\sqrt{3}$

$\arg = \frac{2\pi}{3}$

Example 6

If n is a positive integer, prove that $(-1+\mathrm{i}\sqrt{3})^n+(-1-\mathrm{i}\sqrt{3})^n=2^{n+1}\cos\left(\frac{2n\pi}{3}\right)$.

Solution

$\left|-1+\mathrm{i}\sqrt{3}\right|=2$, $\arg(-1+\mathrm{i}\sqrt{3})=\frac{2\pi}{3}$ $\Rightarrow$ $-1+\mathrm{i}\sqrt{3}=2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}$

Since $-1-\mathrm{i}\sqrt{3}$ and $-1+\mathrm{i}\sqrt{3}$ are conjugates of each other, $-1-\mathrm{i}\sqrt{3}=2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$ $2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}$

$$
\begin{aligned}
\text{Now } (-1+\mathrm{i}\sqrt{3})^n+(-1-\mathrm{i}\sqrt{3})^n &= \left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^n+\left(2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\right)^n \\
&= 2^n\left(\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}\right)+2^n\left(\mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}}\right) \\
&= 2^n\left(\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}+\mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}}\right) \\
&= 2^n\times 2\,\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}\right) \quad \text{since } z+z^*=2\,\mathrm{Re}(z) \\
&= 2^{n+1}\cos\frac{2n\pi}{3}
\end{aligned}
$$

$\left(2\mathrm{e}^{\mathrm{i}\frac{2\pi}{3}}\right)^n+\left(2\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}\right)^n$

$=(2^n)\left(\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}\right)+(2^n)\left(\mathrm{e}^{-\mathrm{i}n\frac{2\pi}{3}}\right)$

$=2^n\left(\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}+\mathrm{e}^{-\mathrm{i}\frac{2n\pi}{3}}\right)$

$=(2^n)\left(2\,\mathrm{Re}\,\mathrm{e}^{\mathrm{i}\frac{2n\pi}{3}}\right)$

$=2^{n+1}\cos\frac{2n\pi}{3}$

Example 7 [HCJC Prelim 9233/2003/02/Q4]

The complex number z is given by $z=\dfrac{(1+\mathrm{i})^3}{\sqrt{2}(a+\mathrm{i})^2}$, where $a>0$.

Given that $|z|=\dfrac{1}{2}$, find the value of a and show that $\arg(z)=\dfrac{5\pi}{12}$.

Solution

$$|z|=\left|\frac{(1+\mathrm{i})^3}{\sqrt{2}(a+\mathrm{i})^2}\right|=\frac{\left|(1+\mathrm{i})^3\right|}{\left|\sqrt{2}(a+\mathrm{i})^2\right|}=\frac{|1+\mathrm{i}|^3}{\left|\sqrt{2}\right||a+\mathrm{i}|^2}==\frac{(\sqrt{2})^3}{\sqrt{2}(\sqrt{a^2+1^2})^2}=\frac{2}{a^2+1}$$

$$\therefore\ \frac{2}{a^2+1}=\frac{1}{2}\quad\Rightarrow a^2=3\ \Rightarrow a=\sqrt{3}\text{ (since } a>0)$$

$$\arg(z)=\arg\left(\frac{(1+\mathrm{i})^3}{\sqrt{2}(\sqrt{3}+\mathrm{i})^2}\right)$$

$$=3\arg(1+\mathrm{i})-\left[\arg(\sqrt{2})+2\arg(\sqrt{3}+\mathrm{i})\right]$$

$$=\ 3\left(\frac{\pi}{4}\right)-\left[0+2\left(\frac{\pi}{6}\right)\right]=\frac{5\pi}{12}\text{ (shown)}$$

CONCLUSION

Basically, we've added another dimension to the real number system so that we can evaluate the square root of negative numbers. Complex numbers are represented geometrically on an Argand diagram. In the Argand diagram, complex numbers behave like vectors, but complex numbers are different from vectors, because vectors cannot be divided or raised to a power, but complex numbers can.

Complex numbers have essential concrete applications in a variety of sciences and related areas such as signal processing (creation of fractals), control theory, electromagnetism, quantum mechanics, cartography, vibration analysis, fluid dynamics, and many others.

If you are interested in the history of imaginary number i, you may want to refer to the book "An Imaginary Tale, the Story of $\sqrt{-1}$", by Paul J.Nahin, Princeton University Press.

APPENDIX
Euler's Formula

$$\cos\theta + \mathrm{i}\sin\theta = \mathrm{e}^{\mathrm{i}\theta}$$

Proof:

From the series expansions of e^{θ}, $\cos\theta$ and $\sin\theta$ (which will be covered in Chapter 7D), we have

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &= 1 + \mathrm{i}\theta + \frac{(\mathrm{i}\theta)^2}{2!} + \frac{(\mathrm{i}\theta)^3}{3!} + \frac{(\mathrm{i}\theta)^4}{4!} + \frac{(\mathrm{i}\theta)^5}{5!} + \ldots \\
&= 1 + \mathrm{i}\theta - \frac{\theta^2}{2!} - \mathrm{i}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{i}\frac{\theta^5}{5!} + \ldots
\end{aligned}$$

(using the result that if $k \in \mathbb{Z}^+$, $\mathrm{i}^{4k} = 1$, $\mathrm{i}^{4k+1} = \mathrm{i}$, $\mathrm{i}^{4k+2} = -1$, $\mathrm{i}^{4k+3} = -\mathrm{i}$)

$$\begin{aligned}
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right) \\
&= \cos\theta + \mathrm{i}\sin\theta
\end{aligned}$$

Summary