H2 Mathematics (9758)
Chapter 6A 3D Vector Geometry (Lines)
Discussion Questions Solutions

Level 1

1 Find a vector equation, a cartesian equation and a set of parametric equations of the following lines:

(a) passing through the point with position vector $7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ and parallel to $\mathbf{i}-3\mathbf{j}+\mathbf{k}$,

(b) passing through the points $(1,-2,1)$ and $(0,4,9)$,

(c) passing through the point $(3,0,2)$ and parallel to the line $x=\frac{y+4}{3}, z=1$.

Q1	Vector Equation	Cartesian Equation	Parametric Equation
(a)	$l_1:\mathbf{r}=\begin{pmatrix}7\\2\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\1\end{pmatrix}, \lambda\in\mathbb{R}$	$x-7=\frac{y-2}{-3}=z+4$ **OR** $x-7=\frac{2-y}{3}=z+4$	$x=7+\lambda$ $y=2-3\lambda$ $z=-4+\lambda$, $\lambda\in\mathbb{R}$
(b)	$l_2:\mathbf{r}=\begin{pmatrix}1\\-2\\1\end{pmatrix}+\mu\begin{pmatrix}1\\-6\\-8\end{pmatrix}, \mu\in\mathbb{R}$	$x-1=\frac{y+2}{-6}=\frac{z-1}{-8}$ **OR** $x-1=-\frac{y+2}{6}=\frac{1-z}{8}$	$x=1+\mu$ $y=-2-6\mu$ $z=1-8\mu$, $\mu\in\mathbb{R}$
(c)	$l_5:\mathbf{r}=\begin{pmatrix}3\\0\\2\end{pmatrix}+\gamma\begin{pmatrix}1\\3\\0\end{pmatrix}, \gamma\in\mathbb{R}$	$x-3=\frac{y}{3}$, $z=2$	$x=3+\gamma$ $y=3\gamma$ $z=2$, $\gamma\in\mathbb{R}$

2 For the following pairs of lines, determine whether they are parallel lines, intersecting lines or skew lines. Find the coordinates of the point of intersection for intersecting lines.

(a) $\frac{x-1}{3}=\frac{y-1}{-2}=z-1$, $\mathbf{r}=-2\mathbf{i}+3\mathbf{j}+\alpha(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$ where α is a real parameter.

(b) $\mathbf{r}=-2\mathbf{i}+3\mathbf{j}+\lambda(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$, $\mathbf{r}=(-1-6\mu)\mathbf{i}+(3-9\mu)\mathbf{j}+(3\mu)\mathbf{k}$, where λ and μ are real parameters**.**

Q2	**Solution**
(a)	$l_1:\frac{x-1}{3}=\frac{y-1}{-2}=z-1\Rightarrow \mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+\beta\begin{pmatrix}3\\-2\\1\end{pmatrix},\ \beta\in\mathbb{R}$ $l_2:\ \mathbf{r}=\begin{pmatrix}-2\\3\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\3\\-1\end{pmatrix},\ \alpha\in\mathbb{R}$ $\begin{pmatrix}3\\-2\\1\end{pmatrix}$ is not parallel to $\begin{pmatrix}2\\3\\-1\end{pmatrix}$, therefore l_1 and l_2 are not parallel lines. Let $\begin{pmatrix}1\\1\\1\end{pmatrix}+\beta\begin{pmatrix}3\\-2\\1\end{pmatrix}=\begin{pmatrix}-2\\3\\0\end{pmatrix}+\alpha\begin{pmatrix}2\\3\\-1\end{pmatrix}\Rightarrow \begin{cases}1+3\beta=-2+2\alpha & -(1)\\1-2\beta=3+3\alpha & -(2)\\1+\beta=-\alpha & -(3)\end{cases}$ $(3)\times2:\quad 2+2\beta=-2\alpha \quad -(4)$ $(2)+(4):\quad 3=3+\alpha \quad\Rightarrow\quad \alpha=0$ From (3), $\beta=-1$ Put $\beta=-1,\ \alpha=0$ in equation (1): $\text{LHS}=1-3=-2$ $\text{RHS}=-2=\text{LHS}$ Thus (1) holds. $\therefore l_1$ and l_2 are intersecting lines and the coordinates of their point of intersection are $(-2,3,0)$. **<u>Alternative</u>** Re-arranging, we get $\begin{cases}-2\alpha+3\beta=-3 & -(5)\\-3\alpha-2\beta=2 & -(6)\\\alpha+\beta=-1 & -(7)\end{cases}$ Using GC, to solve equations (5), (6) and (7), $\alpha=0$, $\beta=-1$. $\therefore l_1$ and l_2 are intersecting lines and the coordinates of their point of intersection are $(-2,3,0)$.

(b)	$l_3 : \mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + \lambda(2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$ $\quad l_4 : \mathbf{r} = (-1-6\mu)\mathbf{i} + (3-9\mu)\mathbf{j} + (3\mu)\mathbf{k}$ $l_3 :\ \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}$ $\quad l_4 :\ \mathbf{r} = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix},\ \mu \in \mathbb{R}$ Since $\mathbf{d}_4 = \begin{pmatrix} -6 \\ -9 \\ 3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = -3\mathbf{d_3}$, $\therefore \mathbf{d}_3 // \mathbf{d}_4$, thus l_3 and l_4 are parallel lines. If l_3 and l_4 are the same line, then there exist $\lambda \in \mathbb{R}$ such that $\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} 1 = 2\lambda \\ 0 = \lambda \\ 0 = \lambda \end{cases}$ Since the values of λ are inconsistent for all 3 equations, $(-1,3,0)$ does not lie on l_3. Thus l_3 and l_4 are distinct parallel lines.

3 The lines l_1 and l_2 have equations $\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}$ respectively, where s and t are real parameters.

(i) Show that l_1 passes through the point $A(2,-1,-4)$, but that l_2 does not.

(ii) Find the acute angle between l_2 and the line joining $A(2,-1,-4)$ and $B(1,-1,1)$.

Q3	Solution
(i)	$l_1:\ \mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\2\\4\end{pmatrix},\ s\in\mathbb{R}$ $\quad l_2:\ \mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix},\ t\in\mathbb{R}$ When $s=-1$, $\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+(-1)\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}2\\-1\\-4\end{pmatrix}$. Therefore, l_1 passes through $(2,-1,-4)$. **Alternatively,** Let $\begin{pmatrix}3\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}2\\-1\\-4\end{pmatrix}$ for some $s\in\mathbb{R}$ $\Rightarrow$ $\begin{cases}3+s=2 & \Rightarrow s=-1\\1+2s=-1 & \Rightarrow s=-1\\4s=-4 & \Rightarrow s=-1\end{cases}$ The value of s are consistent for all 3 equations, so l_1 passes through $(2,-1,-4)$. Let $\begin{pmatrix}1\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}=\begin{pmatrix}2\\-1\\-4\end{pmatrix}$ for some $t\in\mathbb{R}$ $\Rightarrow$ $\begin{cases}1+2t=2 & \Rightarrow t=\frac{1}{2}\\-1+t=-1 & \Rightarrow t=0\\1-t=-4 & \Rightarrow t=5\end{cases}$ The values of t are inconsistent for all 3 equations, so l_2 does not pass through $(2,-1,-4)$.
(ii)	$\overrightarrow{AB}=\begin{pmatrix}1\\-1\\1\end{pmatrix}-\begin{pmatrix}2\\-1\\-4\end{pmatrix}=\begin{pmatrix}-1\\0\\5\end{pmatrix}$ The line joining $A(2,-1,-4)$ and $B(1,-1,1)$ has direction vector $\begin{pmatrix}-1\\0\\5\end{pmatrix}$. Let θ be the required acute angle. $\cos\theta=\dfrac{\left\|\begin{pmatrix}2\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\5\end{pmatrix}\right\|}{\sqrt{6}\sqrt{26}}=\dfrac{7}{\sqrt{6}\sqrt{26}} \quad\Rightarrow\quad \theta=55.9^\circ\ (1\text{ d.p.})$

4 Given that point A has position vector $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and point B has position vector $\begin{pmatrix}1\\0\\2\end{pmatrix}$, find the

(i) length of the projection of $\overrightarrow{AB}$ onto the z-axis,

(ii) projection vector of $\overrightarrow{AB}$ onto the z-axis.

Q4	Solution
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\0\\2\end{pmatrix}-\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$ $\therefore$ Length of projection of $\overrightarrow{AB}$ onto z-axis $=\left\lvert\dfrac{\begin{pmatrix}-1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{(0+0+1)}}\right\rvert=\dfrac{\lvert 0+0+1\rvert}{\sqrt{1}}=1$ unit
(ii)	Projection vector of $\overrightarrow{AB}$ onto z-axis $=\dfrac{\begin{pmatrix}-1\\0\\1\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}}{\sqrt{(0+0+1)}}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

5 Find the coordinates of the foot of perpendicular from the point $P(7, -2, 4)$ to the line $\mathbf{r} = (2+\lambda)\mathbf{i} + (3-2\lambda)\mathbf{j} + \mathbf{k}$, where λ is a real parameter.

Q5	**Solution**
	$\mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$ Let F be the foot of perpendicular from P to the line. Since F lies on the line, $\overrightarrow{OF} = \begin{pmatrix} 2+\lambda \\ 3-2\lambda \\ 1 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. $\overrightarrow{PF} = \begin{pmatrix} 2+\lambda \\ 3-2\lambda \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -5+\lambda \\ 5-2\lambda \\ -3 \end{pmatrix}$ Since PF is perpendicular to the line, $\overrightarrow{PF} \bullet \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 0$.$\Rightarrow (-5+\lambda) - 2(5-2\lambda) = 0 \Rightarrow$ $\lambda = 3$ Substitute $\lambda = 3$ into $\overrightarrow{OF}$: The position vector of F is $\overrightarrow{OF} = \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}$ and the coordinates of F is $(5, -3, 1)$

Level 2

6 The equation of a straight line l is $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}$, where t is a parameter.

The point A on l is given by $t=0$, and the origin of the position vectors is O.

(a) Calculate the acute angle between OA and l, giving your answer correct to the nearest degree.

(b) Find the position vector of the point P on l such that OP is perpendicular to l.

(c) A point Q on l is such that the length of OQ is 5 units. Find the two possible position vectors of Q.

(d) The points R and S on l are given by $t=\lambda$ and $t=2\lambda$ respectively. Show that there is no value of λ for which OR and OS are perpendicular.

Q6	Solution
(a)	$l:\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}, t\in\mathbb{R}$ At $t=0$, $\overrightarrow{OA}=\begin{pmatrix}1\\2\\3\end{pmatrix}$ Recall: $\cos\theta=\dfrac{\lvert\mathbf{d}_1\cdot\mathbf{d}_2\rvert}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$ where θ is the acute angle between the two lines l_1 and l_2 Let θ be the required (acute) angle $\cos\theta=\dfrac{\left\lvert\begin{pmatrix}1\\2\\3\end{pmatrix}.\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rvert}{\sqrt{14}\sqrt{3}}=\dfrac{4}{\sqrt{14}\sqrt{3}}$ $\Rightarrow$ $\theta=52^\circ$ (nearest degree) Learning Point: Remember to include modulus when finding acute angle
(b)	Since P is on l, $\overrightarrow{OP}=\begin{pmatrix}1-t\\2+t\\3+t\end{pmatrix}$ for some $t\in\mathbb{R}$. Learning Point: If a point lies on the line, then the position vector of that point satisfies the line equation $\overrightarrow{OP}\perp l \Rightarrow \overrightarrow{OP}\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=0 \Rightarrow \begin{pmatrix}1-t\\2+t\\3+t\end{pmatrix}\bullet\begin{pmatrix}-1\\1\\1\end{pmatrix}=0 \Rightarrow t=-\dfrac{4}{3}$ $\overrightarrow{OP}=\begin{pmatrix}1+\frac{4}{3}\\2-\frac{4}{3}\\3-\frac{4}{3}\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}7\\2\\5\end{pmatrix}$ Learning Point: $\overrightarrow{OP}$ is perpendicular to $l \Leftrightarrow \overrightarrow{OP}\cdot(\text{direction vector of } l)=0$

(c)

Since Q is on l, $\overrightarrow{OQ}=\begin{pmatrix}1-t\\2+t\\3+t\end{pmatrix}$ for some $t\in\mathbb{R}$.

$\left|\overrightarrow{OQ}\right|=\sqrt{(1-t)^2+(2+t)^2+(3+t)^2}=5$

$\Rightarrow 3t^2+8t-11=0$

$\Rightarrow (3t+11)(t-1)=0$

$\therefore t=-\dfrac{11}{3}$ or $t=1$

$\overrightarrow{OQ}=\begin{pmatrix}1\\2\\3\end{pmatrix}-\dfrac{11}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}=\dfrac{1}{3}\begin{pmatrix}14\\-5\\-2\end{pmatrix}$ or $\overrightarrow{OQ}=\begin{pmatrix}1\\2\\3\end{pmatrix}+\begin{pmatrix}-1\\1\\1\end{pmatrix}=\begin{pmatrix}0\\3\\4\end{pmatrix}$

> Recall:
> $\left|\begin{pmatrix}x\\y\\z\end{pmatrix}\right|=\sqrt{x^2+y^2+z^2}$

(d)

$\overrightarrow{OR}=\begin{pmatrix}1-\lambda\\2+\lambda\\3+\lambda\end{pmatrix}$, $\overrightarrow{OS}=\begin{pmatrix}1-2\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix}$

$$\begin{aligned}\overrightarrow{OR}\bullet\overrightarrow{OS}&=\begin{pmatrix}1-\lambda\\2+\lambda\\3+\lambda\end{pmatrix}\bullet\begin{pmatrix}1-2\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix}\\&=(1-\lambda)(1-2\lambda)+(2+\lambda)(2+2\lambda)+(3+\lambda)(3+2\lambda)\\&=1-2\lambda-\lambda+2\lambda^2+4+4\lambda+2\lambda+2\lambda^2+9+6\lambda+3\lambda+2\lambda^2\\&=6\lambda^2+12\lambda+14\\&=6\left(\lambda^2+2\lambda\right)+14\\&=6\left[(\lambda+1)^2-1^2\right]+14\\&=6(\lambda+1)^2+8>0 \quad \text{for all } \lambda\in\mathbb{R}\end{aligned}$$

> Learning Point:
> Complete the square to show that the expression is always positive or always negative so that it cannot be zero

Since there is no value of λ such that $\overrightarrow{OR}\bullet\overrightarrow{OS}=0$, there is no value of λ for which OR and OS are perpendicular.

Alternative Method:

If OR and OS are perpendicular, then $\overrightarrow{OR}\bullet\overrightarrow{OS}=0$

$$\begin{pmatrix}1-\lambda\\2+\lambda\\3+\lambda\end{pmatrix}\bullet\begin{pmatrix}1-2\lambda\\2+2\lambda\\3+2\lambda\end{pmatrix}=0$$

$$(1-\lambda)(1-2\lambda)+(2+\lambda)(2+2\lambda)+(3+\lambda)(3+2\lambda)=0$$

$$1-2\lambda-\lambda+2\lambda^2+4+4\lambda+2\lambda+2\lambda^2+9+6\lambda+3\lambda+2\lambda^2=0$$

$$6\lambda^2+12\lambda+14=0$$

Since discriminant $=12^2-4(6)(14)=-192<0$, there is no real roots for $6\lambda^2+12\lambda+14$. Hence, there is no value of λ for which OR and OS are perpendicular.

7 N2015/I/7

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC:CA = 3:2$. Point D lies on OB, between O and B, such that $OD:DB = 5:6$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

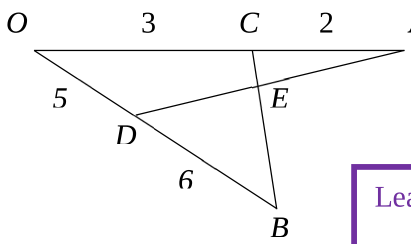
(ii) Show that the vector equation of the line BC can be written as $\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of a parameter μ. [3]

(iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE:ED$. [5]

Learning Point:
Draw diagram to show the ratio clearly before applying ratio theorem.

Q7	Solution

(i)



From the diagram,

$\overrightarrow{OC} = \frac{3}{5}\overrightarrow{OA} = \frac{3}{5}\mathbf{a}$, $\overrightarrow{OD} = \frac{5}{11}\overrightarrow{OB} = \frac{5}{11}\mathbf{b}$

Learning Point:
Points O, A, and C are collinear
$\Leftrightarrow \overrightarrow{OA} = \lambda\overrightarrow{OC}$, for some $\lambda \in \mathbb{R}, \lambda \neq 0$.

(ii)

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$
$$= \frac{3}{5}\mathbf{a} - \mathbf{b}$$

Equation of line BC:

$$\mathbf{r} = \overrightarrow{OB} + \lambda\overrightarrow{BC}$$
$$\mathbf{r} = \mathbf{b} + \lambda\left(\frac{3}{5}\mathbf{a} - \mathbf{b}\right)$$
$$\mathbf{r} = \frac{3}{5}\lambda\mathbf{a} + (1-\lambda)\mathbf{b}, \quad \lambda \in \mathbb{R} \quad \text{(shown)}$$

Learning point:
Position vector of fixed point on line BC is $\overrightarrow{OB}$ or $\overrightarrow{OC}$

Direction vector of line BC is $\overrightarrow{BC}$ or $\overrightarrow{CB}$.

But since this is a showing question, we can only use $\mathbf{r} = \overrightarrow{OB} + \lambda\overrightarrow{BC}$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$$
$$= \frac{5}{11}\mathbf{b} - \mathbf{a}$$

Equation of line AD:

$$\mathbf{r} = \overrightarrow{OA} + \mu\overrightarrow{AD}$$
$$\mathbf{r} = \mathbf{a} + \mu\left(\frac{5}{11}\mathbf{b} - \mathbf{a}\right)$$
$$\mathbf{r} = \frac{5}{11}\mu\mathbf{b} + (1-\mu)\mathbf{a}, \quad \mu \in \mathbb{R}$$

Learning point:
Position vector of fixed point on line BC is $\overrightarrow{OA}$ or $\overrightarrow{OD}$

Direction vector of line BC is $\overrightarrow{AD}$ or $\overrightarrow{DA}$.

(iii) **Position vector of E, intersection of AD and BC**

$=\frac{3}{5}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}=\frac{5}{11}\mu\mathbf{b}+(1-\mu)\mathbf{a}$

Since a and b are non-zero and non-parallel vectors,

$\frac{3}{5}\lambda=1-\mu$ or $1-\lambda=\frac{5}{11}\mu$ --- (2)

$\mu=1-\frac{3}{5}\lambda$ --- (1)

Solving (1) and (2),

$1-\lambda=\frac{5}{11}\left(1-\frac{3}{5}\lambda\right)$

$1-\lambda=\frac{5}{11}-\frac{3}{11}\lambda$

$\lambda=\frac{3}{4}$

$\therefore \overrightarrow{OE}=\frac{3}{5}\left(\frac{3}{4}\right)\mathbf{a}+\left(1-\frac{3}{4}\right)\mathbf{b}$

$=\frac{9}{20}\mathbf{a}+\frac{1}{4}\mathbf{b}$

$\overrightarrow{AE}=\overrightarrow{OE}-\overrightarrow{OA}$

$=\frac{9}{20}\mathbf{a}+\frac{1}{4}\mathbf{b}-\mathbf{a}$

$=\frac{1}{4}\mathbf{b}-\frac{11}{20}\mathbf{a}$

$=\frac{11}{20}\left(\frac{5}{11}\mathbf{b}-\mathbf{a}\right)$

$=\frac{11}{20}\overrightarrow{AD}$

$\therefore AE:ED \Rightarrow 11:9$

Useful technique: Solving 2 unknowns in a given equation involving vectors.

Recall:
For **a** and **b** are non-zero and non parallel vectors,

$\lambda\mathbf{a}=\mu\mathbf{b} \;\Rightarrow\; \lambda=\mu=0$

Alternative (Shorter)

$\mu=1-\frac{3}{5}\lambda=1-\frac{3}{5}\left(\frac{3}{4}\right)=\frac{11}{20}$

$\mathbf{r}=\overrightarrow{OA}+\mu\overrightarrow{AD}$

$\overrightarrow{OE}=\overrightarrow{OA}+\frac{11}{20}\overrightarrow{AD}$

$=\overrightarrow{OA}+\overrightarrow{AE}$

So, $\overrightarrow{AE}=\frac{11}{20}\overrightarrow{AD}$

$\therefore AE:ED \Rightarrow 11:9$

Learning point:

Express $\overrightarrow{AE}=\frac{11}{20}\overrightarrow{AD}$ as *A*, *D* and *E* are collinear.

To find the ratio, can consider its fraction :

$\frac{AE}{AD}=\frac{11}{20}$ (using **length** *AE* and *AD*)

$\therefore AE:ED \Rightarrow 11:9$ (using **length** *AE* and *ED*)

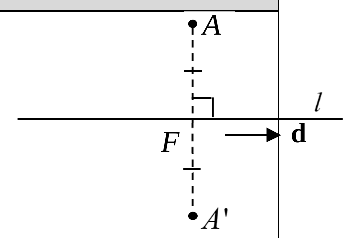
Level 3

8 Relative to the origin O, the point A has coordinates (4, 4, 7) and the line l has equation $\mathbf{r} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(6\mathbf{i} + \mathbf{j} + \mathbf{k})$. Find the position vector of

(i) the foot of perpendicular from A to l,

(ii) the point A', the reflection of A in the line l.

Hence or otherwise, find the shortest distance from A to line l.

Q8	Solution
(i)	Let F be the foot of the perpendicular from A to l.  $l\text{:}\ \mathbf{r} = \begin{pmatrix}-1\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}6\\1\\1\end{pmatrix},\ \lambda \in \mathbb{R}$ Since F lies on l then $\overrightarrow{OF} = \begin{pmatrix}-1+6\lambda\\1+\lambda\\2+\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$. $\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix}-1+6\lambda\\1+\lambda\\2+\lambda\end{pmatrix} - \begin{pmatrix}4\\4\\7\end{pmatrix} = \begin{pmatrix}6\lambda-5\\\lambda-3\\\lambda-5\end{pmatrix}$ $\overrightarrow{AF} \perp l \Rightarrow \begin{pmatrix}6\lambda-5\\\lambda-3\\\lambda-5\end{pmatrix}.\begin{pmatrix}6\\1\\1\end{pmatrix} = 0 \Rightarrow 36\lambda - 30 + \lambda - 3 + \lambda - 5 = 0 \Rightarrow \lambda = 1$ Therefore, the position vector of the foot of the perpendicular from A to l is $\overrightarrow{OF} = \begin{pmatrix}-1+6(1)\\1+1\\2+1\end{pmatrix} = \begin{pmatrix}5\\2\\3\end{pmatrix}$.
(ii)	Let A' be the reflection of A in the line l. By Ratio Theorem, $\overrightarrow{OF} = \dfrac{\overrightarrow{OA} + \overrightarrow{OA'}}{2} \Rightarrow \overrightarrow{OA'} = 2\overrightarrow{OF} - \overrightarrow{OA} = 2\begin{pmatrix}5\\2\\3\end{pmatrix} - \begin{pmatrix}4\\4\\7\end{pmatrix} = \begin{pmatrix}6\\0\\-1\end{pmatrix}$
	Hence shortest distance from A to line l is $\left\lvert\overrightarrow{AF}\right\rvert = \left\lvert\begin{pmatrix}6(1)-5\\1-3\\1-5\end{pmatrix}\right\rvert = \left\lvert\begin{pmatrix}1\\-2\\-4\end{pmatrix}\right\rvert = \sqrt{21}$.

9 2012/I/9 (modified)

(i) Find a vector equation of the line through the points A and B with position vectors $7\mathbf{i}+8\mathbf{j}+9\mathbf{k}$ and $-\mathbf{i}-8\mathbf{j}+\mathbf{k}$ respectively. [3]

(ii) The perpendicular to this line from the point C with position vector $\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ meets the line at the point N. Find the position vector of N and the ratio $AN : NB$. [5]

(iii) Find a Cartesian equation of the line which is a reflection of the line AC in the line AB. [4]

(iv) The point D has position vector $\mathbf{i}+8\mathbf{j}-2\mathbf{k}$. Find the length of projection of $\overrightarrow{CD}$ onto line AB. [4]

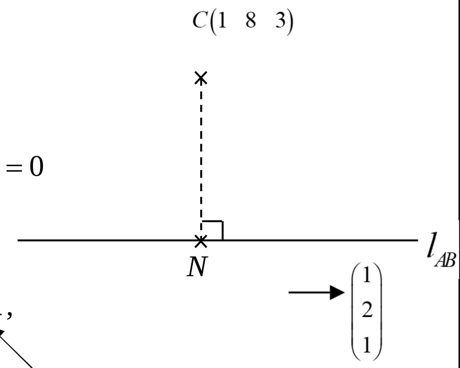
Q9	Solution
(i)	$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-1\\-8\\1\end{pmatrix}-\begin{pmatrix}7\\8\\9\end{pmatrix}=\begin{pmatrix}-8\\-16\\-8\end{pmatrix}=-8\begin{pmatrix}1\\2\\1\end{pmatrix}$ ← Always good to simplify the direction vector Vector equation of the line AB: $\mathbf{r}=\begin{pmatrix}7\\8\\9\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$ **Alternatively,** can write $\mathbf{r}=\begin{pmatrix}-1\\-8\\1\end{pmatrix}+\mu\begin{pmatrix}1\\2\\1\end{pmatrix},\ \mu\in\mathbb{R}$

(ii)

What is the aim of the question?

To find $\overrightarrow{ON}$

What can be observed from the diagram?



$$\overrightarrow{CN} \perp l_{AB} \Rightarrow \overrightarrow{CN} \bullet \begin{pmatrix}1\\2\\1\end{pmatrix} = 0 \Rightarrow \left(\overrightarrow{ON} - \overrightarrow{OC}\right) \bullet \begin{pmatrix}1\\2\\1\end{pmatrix} = 0$$

Since N is on l_{AB}, $\overrightarrow{ON} = \begin{pmatrix}7+\lambda\\8+2\lambda\\9+\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$,

N must be on the line AB

$$\left[\begin{pmatrix}7+\lambda\\8+2\lambda\\9+\lambda\end{pmatrix} - \begin{pmatrix}1\\8\\3\end{pmatrix}\right] \bullet \begin{pmatrix}1\\2\\1\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}6+\lambda\\2\lambda\\6+\lambda\end{pmatrix} \bullet \begin{pmatrix}1\\2\\1\end{pmatrix} = 0$$

$$\Rightarrow (6+\lambda) + 4\lambda + (6+\lambda) = 0 \Rightarrow \lambda = -2$$

Thus $\overrightarrow{ON} = \begin{pmatrix}7-2\\8+2(-2)\\9-2\end{pmatrix} = \begin{pmatrix}5\\4\\7\end{pmatrix}$

Learning Point:

1. If point N is on the line l_1, then

$$\overrightarrow{ON} = \begin{pmatrix}7+\lambda\\8+2\lambda\\9+\lambda\end{pmatrix} \text{for some } \lambda \in \mathbb{R}$$

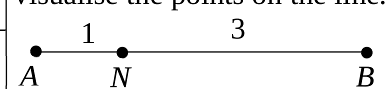
2. $\overrightarrow{CN} \perp \mathbf{d} \Rightarrow \overrightarrow{CN} \bullet \mathbf{d} = 0$
3. Aim is to solve for unknown λ
4. Substitute λ into $\overrightarrow{ON}$

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA}$$

$$= \begin{pmatrix}5\\4\\7\end{pmatrix} - \begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}-2\\-4\\-2\end{pmatrix} = \frac{1}{4}\begin{pmatrix}-8\\-16\\-8\end{pmatrix} = \frac{1}{4}\overrightarrow{AB}$$

$\therefore AN : NB = 1:3$

Sketch a diagram if it helps to visualise the points on the line.



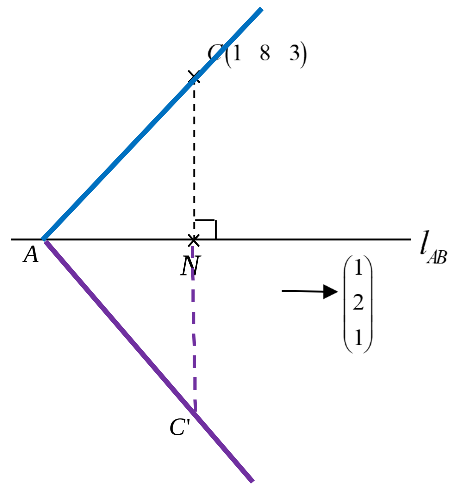
(iii) Let *C'* be the point of reflection of *C* in line *AB*

Using Ratio Theorem,

$$\overrightarrow{ON} = \frac{\overrightarrow{OC} + \overrightarrow{OC'}}{2}$$

$$\overrightarrow{OC'} = 2\overrightarrow{ON} - \overrightarrow{OC}$$

$$= 2\begin{pmatrix}5\\4\\7\end{pmatrix} - \begin{pmatrix}1\\8\\3\end{pmatrix} = \begin{pmatrix}9\\0\\11\end{pmatrix}$$



Direction vector

$$\overrightarrow{AC'} = \overrightarrow{OC'} - \overrightarrow{OA}$$

$$= \begin{pmatrix}9\\0\\11\end{pmatrix} - \begin{pmatrix}7\\8\\9\end{pmatrix}$$

$$= \begin{pmatrix}2\\-8\\2\end{pmatrix} = 2\begin{pmatrix}1\\-4\\1\end{pmatrix}$$

Alternative: Ratio Theorem

$$\overrightarrow{AN} = \frac{\overrightarrow{AC} + \overrightarrow{AC'}}{2}$$

$$\overrightarrow{AC'} = 2\overrightarrow{AN} - \overrightarrow{AC}$$

$$= 2\begin{pmatrix}-2\\-4\\-2\end{pmatrix} - \left[\begin{pmatrix}1\\8\\3\end{pmatrix} - \begin{pmatrix}7\\8\\9\end{pmatrix}\right]$$

$$= \begin{pmatrix}2\\-8\\2\end{pmatrix} = 2\begin{pmatrix}1\\-4\\1\end{pmatrix}$$

Vector equation of line AC': $\mathbf{r} = \begin{pmatrix}7\\8\\9\end{pmatrix} + \mu\begin{pmatrix}1\\-4\\1\end{pmatrix}$, $\mu \in \mathbb{R}$

$$x = 7 + \mu \Rightarrow \mu = x - 7$$

$$y = 8 - 4\mu \Rightarrow \mu = \frac{y-8}{-4}$$

$$z = 9 + \mu \Rightarrow \mu = z - 9$$

Cartesian equation of line of reflection is

$$x - 7 = \frac{y-8}{-4} = z - 9$$

(iv)

$$\overrightarrow{CD} = \begin{pmatrix}1\\8\\-2\end{pmatrix} - \begin{pmatrix}1\\8\\3\end{pmatrix} = \begin{pmatrix}0\\0\\-5\end{pmatrix}$$

Length of projection of $\overrightarrow{CD}$ on onto line *AB* is $\left|\dfrac{\overrightarrow{CD}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}{\sqrt{1^2+2^2+1^2}}\right| = \left|\dfrac{\begin{pmatrix}0\\0\\-5\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix}}{\sqrt{6}}\right|$

$$= \left|\frac{-5}{\sqrt{6}}\right| = \frac{5\sqrt{6}}{6}$$

Interpret as length of projection of $\overrightarrow{CD}$ onto direction vector of line *AB*, $\mathbf{d} = \begin{pmatrix}1\\2\\1\end{pmatrix}$.

10 2017(9758)/I/10

Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0,0,0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix}3\\1\\-2\end{pmatrix}$.

A new cable is installed which passes through points $P(1,2,-1)$ and $Q(5,7,a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

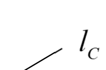
To ensure that the cables do not meet, the engineers use $a=-3$. The engineers wish to connect each of the points P and Q to a point R on C.

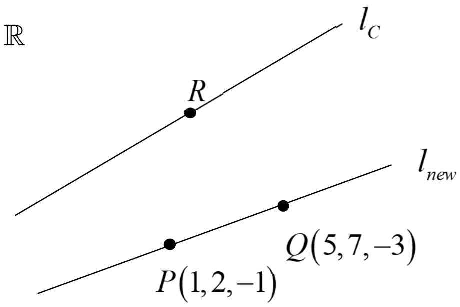
(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be 90°. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

Extend: How can we find the exact minimum length without first finding the coordinates of R?

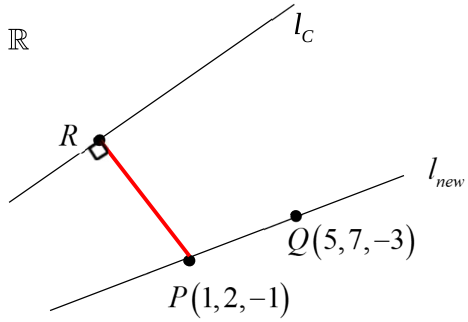
Q10	Solution
(i)	$l_C : \mathbf{r} = \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}, \lambda \in \mathbb{R}$ ← Cable C starts at **main switching** site at $(0,0,0)$, hence it must pass through $(0,0,0)$ $\overrightarrow{PQ} = \begin{pmatrix}5\\7\\a\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}4\\5\\a+1\end{pmatrix}$ $l_{new} : \mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}4\\5\\a+1\end{pmatrix}, \mu \in \mathbb{R}$ ← Vector equation of new cable that passes through point P and Q. For the cables to meet, $\begin{pmatrix}3\lambda\\\lambda\\-2\lambda\end{pmatrix} = \begin{pmatrix}1+4\mu\\2+5\mu\\-1+\mu(a+1)\end{pmatrix}$, for some $\lambda, \mu \in \mathbb{R}$ $3\lambda = 1+4\mu$ ----- (1) $\lambda = 2+5\mu$ ------ (2) $-2\lambda = -1+\mu(a+1)$ --- (3) Using GC to solve (1) and (2), $\lambda = -\frac{3}{11}$ and $\mu = -\frac{5}{11}$ Substitute $\lambda = -\frac{3}{11}$ and $\mu = -\frac{5}{11}$ into (3)



	$\therefore a=-\frac{22}{5}$
(ii)	Since R lies on l_C, $\overrightarrow{OR}=\begin{pmatrix}3\lambda\\ \lambda\\ -2\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$  $\overrightarrow{RP}=\begin{pmatrix}1\\2\\-1\end{pmatrix}-\begin{pmatrix}3\lambda\\ \lambda\\ -2\lambda\end{pmatrix}=\begin{pmatrix}1-3\lambda\\2-\lambda\\-1+2\lambda\end{pmatrix}$ $\overrightarrow{RQ}=\begin{pmatrix}5\\7\\-3\end{pmatrix}-\begin{pmatrix}3\lambda\\ \lambda\\ -2\lambda\end{pmatrix}=\begin{pmatrix}5-3\lambda\\7-\lambda\\-3+2\lambda\end{pmatrix}$ $\overrightarrow{RP}\bullet\overrightarrow{RQ}=\begin{pmatrix}1-3\lambda\\2-\lambda\\-1+2\lambda\end{pmatrix}\bullet\begin{pmatrix}5-3\lambda\\7-\lambda\\-3+2\lambda\end{pmatrix}$ ← In order to show that angle PRQ is not 90°, we need to show that the $\overrightarrow{RP}\bullet\overrightarrow{RQ}\neq 0$ using algebraic method. $=\lambda^2(9+1+4)+\lambda(-18-9-8)+(5+14+3)$ $=14\lambda^2-35\lambda+22$ $=14\left(\lambda^2-\frac{35}{14}\lambda+\left(\frac{35}{28}\right)^2-\left(\frac{35}{28}\right)^2\right)+22$ $=14\left(\lambda-\frac{35}{28}\right)^2+\frac{1}{8}$ Since $\left(\lambda-\frac{35}{28}\right)^2\geq 0$ for all $\lambda\in\mathbb{R}$, $14\left(\lambda-\frac{35}{28}\right)^2+\frac{1}{8}>0$. Since $\overrightarrow{RP}\bullet\overrightarrow{RQ}\neq 0$, it is not possible that $\angle PRQ=90^\circ$. **Alternative Method:** If angle PRQ is 90°., then $\overrightarrow{RP}\bullet\overrightarrow{RQ}=0$ $\begin{pmatrix}1-3\lambda\\2-\lambda\\-1+2\lambda\end{pmatrix}\bullet\begin{pmatrix}5-3\lambda\\7-\lambda\\-3+2\lambda\end{pmatrix}=0$ $\lambda^2(9+1+4)+\lambda(-18-9-8)+(5+14+3)=0$ $14\lambda^2-35\lambda+22=0$ Since discriminant $=(-35)^2-4(14)(22)=-7<0$, there is no real roots for $14\lambda^2-35\lambda+22=0$ and thus $\overrightarrow{RP}\bullet\overrightarrow{RQ}\neq 0$. Hence, it is not possible that $\angle PRQ=90^\circ$.

(iii)

Since R lies on l_C, $\overrightarrow{OR}=\begin{pmatrix}3\lambda\\ \lambda\\ -2\lambda\end{pmatrix}$ for some $\lambda\in\mathbb{R}$



$$\overrightarrow{PR}=\begin{pmatrix}3\lambda\\ \lambda\\ -2\lambda\end{pmatrix}-\begin{pmatrix}1\\ 2\\ -1\end{pmatrix}=\begin{pmatrix}3\lambda-1\\ \lambda-2\\ 1-2\lambda\end{pmatrix}$$

In order for the length of *PR* to be as small as possible, $\overrightarrow{PR}$ has to be perpendicular to l_C.

$$\overrightarrow{PR}\bullet\begin{pmatrix}3\\ 1\\ -2\end{pmatrix}=0$$

Therefore, we are trying to find *R*, which is the foot of the perpendicular from *P* to l_C.

$$\begin{pmatrix}3\lambda-1\\ \lambda-2\\ 1-2\lambda\end{pmatrix}\bullet\begin{pmatrix}3\\ 1\\ -2\end{pmatrix}=0$$

$$9\lambda-3+\lambda-2-2+4\lambda=0$$

$$14\lambda=7$$

$$\lambda=\frac{1}{2}$$

Reminder to answer the question and give the coordinates of *R*.

$\therefore\ \overrightarrow{OR}=\frac{1}{2}\begin{pmatrix}3\\ 1\\ -2\end{pmatrix}$, coordinates of R is $\left(\frac{3}{2},\frac{1}{2},-1\right)$

$$\overrightarrow{PR}=\begin{pmatrix}\frac{3}{2}-1\\ \frac{1}{2}-2\\ 1-1\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\\ -\frac{3}{2}\\ 0\end{pmatrix}$$

Exact Minimum length $=\sqrt{\left(\frac{1}{2}\right)^2+\left(-\frac{3}{2}\right)^2}$

$$=\frac{\sqrt{10}}{2}$$

11 9758 Specimen Paper/I/6

(a) The non-zero vectors **a**, **b** and **c** are such that $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$. Given that $\mathbf{b}\neq-\mathbf{c}$, find a linear relationship between **a**, **b** and **c**. [3]

(b) The variable vector **v** satisfies the equation $\mathbf{v}\times(\mathbf{i}-3\mathbf{k})=2\mathbf{j}$. Find the set of vectors **v** and fully describe this set geometrically. [5]

Q11	Solution
(a)	$\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$ **Useful technique**: Shift all terms to one side so that a zero vector is "created" on the other side. $\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{a}=\mathbf{0}$ $(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})=\mathbf{0}$ $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{0}$ Properties of vector product: 1. $\mathbf{a}\times(\mathbf{c}+\mathbf{b})=\mathbf{a}\times\mathbf{c}+\mathbf{a}\times\mathbf{b}$ 2. $\mathbf{c}\times\mathbf{c}=\mathbf{0}$ (zero vector) 3. $\mathbf{a}\times\mathbf{c}=-\mathbf{c}\times\mathbf{a}$ Learning point: $\mathbf{a}\times\mathbf{b}=\mathbf{0}$ $\Rightarrow$ **a** is parallel to **b** **OR** $\mathbf{a}=\mathbf{0}$ **OR** $\mathbf{b}=\mathbf{0}$ Since $\mathbf{b}\neq-\mathbf{c}$, $\mathbf{b}+\mathbf{c}\neq\mathbf{0}$ and given $\mathbf{a}\neq\mathbf{0}$, hence **a** is parallel to $\mathbf{b}+\mathbf{c}$. Therefore, a linear relationship between **a**, **b** and **c** is $\mathbf{a}=\lambda(\mathbf{b}+\mathbf{c})$ for some $\lambda\in\mathbb{R},\lambda\neq0$.

(b)

Given $\mathbf{v}\times\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$, let $\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

> Since column vector is involved, we can start by expressing $\mathbf{v}=\begin{pmatrix}x\\y\\z\end{pmatrix}$

$$\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\0\\-3\end{pmatrix}=\begin{pmatrix}-3y\\-(-3x-z)\\-y\end{pmatrix}=\begin{pmatrix}0\\2\\0\end{pmatrix}$$

$\Rightarrow y=0$ and $3x+z=2\Rightarrow x=\frac{2}{3}-\frac{1}{3}z$ (Replacing z with λ)

Arranging,

$x=\frac{2}{3}-\frac{1}{3}\lambda$

$y=0 \qquad ,\lambda\in\mathbb{R}$

$z=\qquad \lambda$

Therefore,

the set of vectors $\mathbf{v}$ is $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{2}{3}-\frac{1}{3}\lambda\\0\\\lambda\end{pmatrix}\right\}$ or

$$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{2}{3}\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}-\frac{1}{3}\\0\\1\end{pmatrix}\right\}$$

> **<u>Alternatively,</u>**
>
> $y=0,\ 3x+z=2\Rightarrow 3x=2-z$
>
> $3x=\lambda\Rightarrow x=\frac{\lambda}{3}$
>
> $2-z=\lambda\Rightarrow z=2-\lambda$
>
> $x=\qquad \frac{1}{3}\lambda$
>
> $y=0 \qquad ,\lambda\in\mathbb{R}$
>
> $z=2\ -\ \lambda$
>
> So, set of vectors $\mathbf{v}$ is
>
> $$\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}\frac{1}{3}\\0\\-1\end{pmatrix}\right\}$$

This set is the set of position vectors of all points lying on the line passing through point $\left(\frac{2}{3},0,0\right)$ with direction $\begin{pmatrix}-\frac{1}{3}\\0\\1\end{pmatrix}$.