H2 Mathematics (9758)
Chapter 8 Applications of Differentiation
Assignment Suggested Solutions

1 **2019/SAJC Promo/Q8**

The curve C has parametric equations

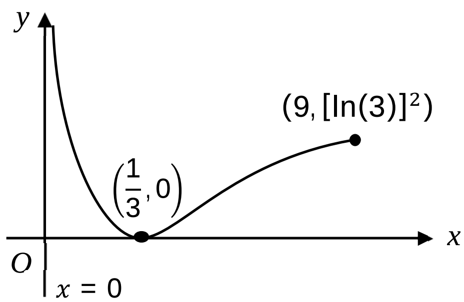
$$x = \frac{t^3}{3}, \quad y = \left[\ln(t)\right]^2, \qquad \text{for } 0 < t \le 3.$$

(i) Sketch the graph of C, giving the coordinates of its endpoint(s) and the point(s) where C meets the axes. State also the equation of the vertical asymptote. [3]

(ii) Find the equation of the tangent to the curve C at the point $\left(\frac{p^3}{3}, \left[\ln(p)\right]^2\right)$, simplifying your answer. [5]

(iii) Hence find the exact coordinates of the points Q and R where the tangent to the curve C when $t = \mathrm{e}$ meets the x-axis and y-axis respectively. [3]

(iv) Find the area of triangle OQR in exact form. [2]

Q1	**Solution**
(i)	 Remember to press p and set Tmin = 0 Tmax = 3 For sketch, need to label the endpoint as a closed circle, since endpoint is at t = 3 which is included. As $t \to 0$, $y \to +\infty$, $x \to 0$. Hence vertical asymptote is $x = 0$. Question asks to state equation of vertical asymptote. Keep this in mind and make sure you have a vertical asymptote in your graph. As $t \to 0$, $x = \frac{t^3}{3} \to 0$ $\ln t \to -\infty$, $y = (\ln t)^2 \to \infty$ Asymptote: $x = 0$ Intercept: $y = 0, t = 1 \Rightarrow x = \frac{1}{3}$

(ii)

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{3t^2}{3}=t^2,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{2\ln t}{t}$$

Note: t is the parameter, p is a constant that gives a specific point

So, you have to differentiate with respect to t, then substitute $t = p$ to find the value of the gradient at that point

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$=\frac{2\ln t}{t}\times\frac{1}{t^2}$$

$$=\frac{2\ln t}{t^3}$$

At $t = p$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\ln p}{p^3}$.

You need to find the value of the gradient at point P by substituting $t = p$

Equation of the tangent of C at point p :

$$y-[\ln p]^2=\frac{2\ln p}{p^3}\left(x-\frac{p^3}{3}\right)$$

$$y-[\ln p]^2=\left(\frac{2}{p^3}\ln p\right)x-\frac{2}{3}\ln p$$

$$y=\left(\frac{2}{p^3}\ln p\right)x+[\ln p]^2-\frac{2}{3}\ln p$$

Recall:
Equation of line:
$$y-y_0=m(x-x_0)$$
where m is the gradient and (x_0, y_0) is a point on the line

ALERT: $[\ln p]^2 \neq 2\ln p$

(iii)

Equation of the tangent of C at point $t = p$:

$$y=\left(\frac{2}{p^3}\ln p\right)x+[\ln p]^2-\frac{2}{3}\ln p$$

Keyword: 'Hence'
Means you need to use the previous result (equation of tangent at point $t = p$) to solve

At $p = \mathrm{e}$,

$$y=\left(\frac{2}{\mathrm{e}^3}\ln \mathrm{e}\right)x+[\ln \mathrm{e}]^2-\frac{2}{3}\ln \mathrm{e}$$

$$y=\frac{2}{\mathrm{e}^3}x+\frac{1}{3}$$

Note: e is a constant
Use equation of tangent found in (ii), substituting $p = \mathrm{e}$.
Simplify using $\ln \mathrm{e} = 1$

When the tangent cuts the x – axis,

$$y=0 \Rightarrow x=-\frac{\mathrm{e}^3}{6}.$$

When the tangent cuts the axis at y – axis,

$$x=0 \Rightarrow y=\frac{1}{3}.$$

The coordinates of Q are $\left(-\frac{\mathrm{e}^3}{6}, 0\right)$.

Reminder:
Question wants **exact coordinates** so remember to answer the question

The coordinates of R are $\left(0, \frac{1}{3}\right)$.

(iv)	Area of triangle $OQR = \frac{1}{2}\left(\frac{e^3}{6}\right)\left(\frac{1}{3}\right)$ $= \frac{e^3}{36}$ units^2

Reminder:
Area is always positive

Triangle *OQR* is a right angled triangle as *Q* and *R* are on the *x*- and *y*-axes respectively

2 2019/JPJC Promo/Q4

Steel sheets of negligible thickness are used to make cans in the shape of a right-circular cylinder of radius *r* cm and height *h* cm. Given that a can is to hold 300 cm^3 of liquid,

(i) show that the external surface area, *A* cm^2, of a closed can is

$$A = 2\pi r^2 + \frac{600}{r} .$$ [2]

(ii) Find, using differentiation, the exact value of *r* which produces a can with a minimum value of *A*. Hence, find the minimum value of *A*. [4]

(iii) Deduce that $\frac{h}{r} = 2$ when *A* is a minimum. [2]

(iv) Liquid is dispensed into an empty can at a rate of 100 cm^3/s. Find the rate of increase of the depth of the liquid, *H*, given that *A* is a minimum. [4]

2	**2019/JPJC Promo/Q4**
(i)	Given that $V=\pi r^2 h=300$, $h=\dfrac{300}{\pi r^2}$ $A=2\pi r^2+2\pi rh$ $=2\pi r^2+2\pi r\left(\dfrac{300}{\pi r^2}\right)$ $=2\pi r^2+\dfrac{600}{r}$
(ii)	see below
(iii)	A is minimum when $r=\sqrt[3]{\dfrac{150}{\pi}}$.

(ii)

$$\frac{\mathrm{d}A}{\mathrm{d}r}=4\pi r-\frac{600}{r^2}$$

When $\dfrac{\mathrm{d}A}{\mathrm{d}r}=0$, $4\pi r-\dfrac{600}{r^2}=0$

$$r=\sqrt[3]{\frac{150}{\pi}}$$

r	$\left(\sqrt[3]{\dfrac{150}{\pi}}\right)^-$	$\sqrt[3]{\dfrac{150}{\pi}}$	$\left(\sqrt[3]{\dfrac{150}{\pi}}\right)^+$
$\dfrac{\mathrm{d}A}{\mathrm{d}r}$	\	—	/

$\therefore$ A is minimum when $r=\sqrt[3]{\dfrac{150}{\pi}}$.

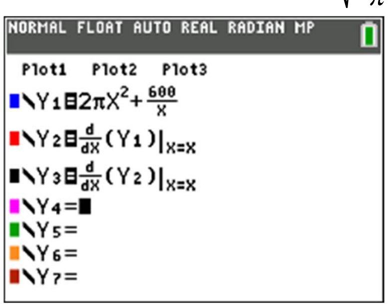
> Remember to prove that value of *r* gives minimum *A*.

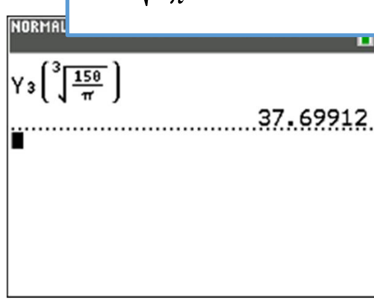
Alternatively:

Using GC, $\left.\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}\right|_{r=\sqrt[3]{\frac{150}{\pi}}}=37.7>0$

> For second derivative test:
>
> Write down the value of $\dfrac{\mathrm{d}^2A}{\mathrm{d}r^2}=37.7$ at $r=\sqrt[3]{\dfrac{150}{\pi}}$ before comparing with 0.

$\therefore$ A is minimum when $r=\sqrt[3]{\dfrac{150}{\pi}}$.





Minimum A

$$=2\pi\left(\sqrt[3]{\frac{150}{\pi}}\right)^2+\frac{600}{\sqrt[3]{\frac{150}{\pi}}}$$

$=248$ (to 3 s.f.)

> Question does not require *A* to be exact.

	From $h=\dfrac{300}{\pi r^2}$, $\dfrac{h}{r}=\dfrac{300}{\pi r^3}$ $=\dfrac{300}{\pi\left(\dfrac{150}{\pi}\right)}$ $=2$ Deduce that $\dfrac{h}{r}=2$ when A is a minimum. ➔ Use value of r found in (ii)
(iv)	Let W be volume of liquid. $\dfrac{dH}{dt}=\dfrac{dW}{dt}\times\dfrac{dH}{dW}$ When A is a minimum, $r=\sqrt[3]{\dfrac{150}{\pi}}$. When water is added to the can, depth increases but radius remains constant. $W=\pi r^2 H=\pi\left(\sqrt[3]{\dfrac{150}{\pi}}\right)^2 H=\pi\left(\dfrac{150}{\pi}\right)^{\frac{2}{3}} H$ $W=\pi r^2 H=\pi\left(\dfrac{H}{2}\right)^2 H=\dfrac{1}{4}\pi H^3$ refers to the volume of the can when A is minimum. When liquid is added to the empty can, the depth of the liquid increases, but the radius of the liquid surface remains as a constant, thus in $W=\pi r^2 H$, r is constant and $\dfrac{dW}{dH}=\pi r^2$ and the value of r is the value of r when A is minimum (i.e $r=\sqrt[3]{\dfrac{150}{\pi}}$). $\dfrac{dW}{dH}=\pi\left(\dfrac{150}{\pi}\right)^{\frac{2}{3}}$ $\dfrac{dH}{dt}=100\times\dfrac{1}{\pi\left(\dfrac{150}{\pi}\right)^{\frac{2}{3}}}$ $=2.4186$ $=2.42$ cm/s The rate of increase of depth of liquid is 2.42 cm/s.

3 2015/MJC/Prelim/P2/Q2

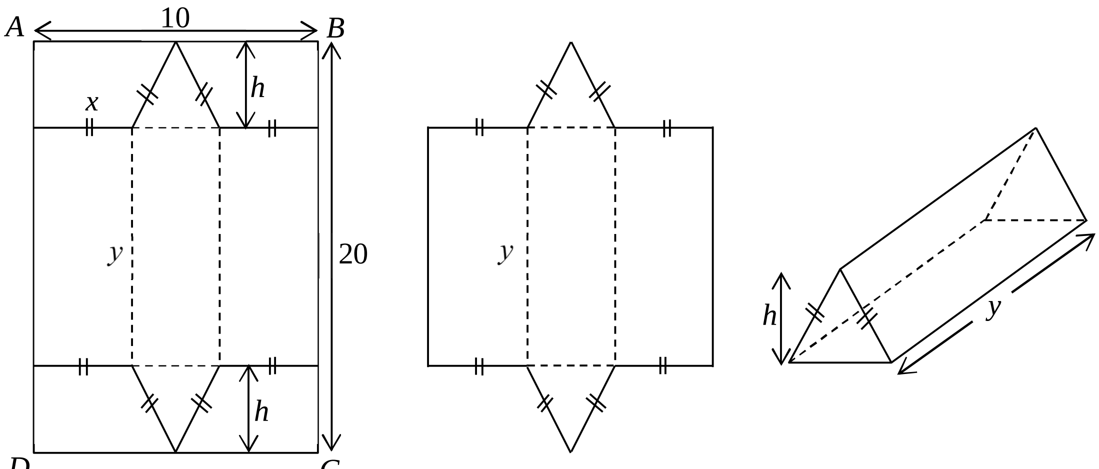


Fig. 1 **Fig. 2** **Fig. 3**

Fig. 1 shows a rectangular piece of cardboard *ABCD* of sides 10 cm and 20 cm. A trapezium shape is cut out from each corner, to give the shape shown in Fig. 2. This shape consists of 2 isosceles triangles and 3 rectangles of different sizes. The remaining cardboard shown in Fig. 2 is folded along the dotted lines, to form a closed triangular prism shown in Fig. 3.

(i) Show that the volume V cm^3 of the closed triangular prism is given by $V = \frac{1}{5}\left(h^4 - 10h^3 - 25h^2 + 250h\right)$. [4]

(ii) Use differentiation to find the maximum value of V, proving that it is a maximum. [5]

Q3	Solution
(i)	$y + 2h = 20 \quad \Rightarrow \quad y = 20 - 2h$ By Pythagoras Theorem, $h^2 + (5-x)^2 = x^2$ $h^2 + 25 - 10x + x^2 = x^2$ $x = \dfrac{25 + h^2}{10}$ Note that if you used other variables in your solution, you are supposed to define your variables clearly. Base of isosceles triangle $= 10 - 2x$

$$V=\frac{1}{2}(10-2x)hy$$
$$=\frac{1}{2}\left[10-2\left(\frac{25+h^2}{10}\right)\right]h(20-2h)$$
$$=\frac{1}{5}h(10-h)(25-h^2)$$
$$=\frac{1}{5}h(h^3-10h^2-25h+250)$$
$$=\frac{1}{5}(h^4-10h^3-25h^2+250h) \quad \text{(shown)}$$

(ii)

Differentiate wrt x,

$$\frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{5}(4h^3-30h^2-50h+250)$$

Obtain $\frac{\mathrm{d}V}{\mathrm{d}h}$ first before applying $\frac{\mathrm{d}V}{\mathrm{d}h}=0$ for maximum value of V.

For maximum V, $\frac{\mathrm{d}V}{\mathrm{d}h}=0$.

$$\therefore \frac{\mathrm{d}V}{\mathrm{d}h}=\frac{1}{5}(4h^3-30h^2-50h+250)=0$$
$$4h^3-30h^2-50h+250=0$$

Use GC to solve if possible

Express intermediate solutions in 5 significant figures.

Using GC, $h=8.0902$ cm (rejected $\because 2x=18.090>10$),
$h=-3.0902$ cm (rejected $\because h>0$)
or $h=2.5$ cm.

Give reasons for rejection.

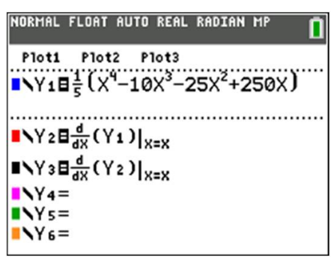
h	2.5^-	2.5	2.5^+
$\frac{\mathrm{d}V}{\mathrm{d}h}$	/	—	\

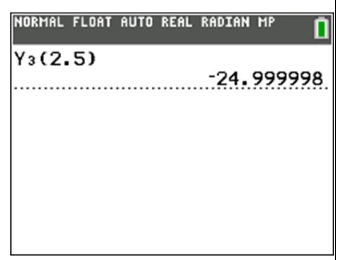
$\therefore$ V is maximum at $h=2.5$.

<u>Alternatively</u>

Using GC, $\left.\frac{\mathrm{d}^2V}{\mathrm{d}h^2}\right|_{h=2.5}=-25.0<0$

$\therefore$ V is maximum at $h=2.5$.





$$\therefore \text{ maximum } V=\frac{1}{5}\left((2.5)^4-10(2.5)^3-25(2.5)^2+250(2.5)\right)$$
$$=70.3125 \text{ cm}^3$$

70.3125 is an exact value. No rounding off is allowed.