7. Integration and its Applications (solutions)

1	**CJC PROMO 2010/QN10**
(a)	$\int \frac{x}{3+x}\,dx$ $= \int \frac{3+x-3}{3+x}\,dx$ $= \int 1 - \frac{3}{3+x}\,dx$ $= x - 3\ln\lvert 3+x\rvert + c$
(b)	$\int x^2 \ln x\,dx$ $= \quad [\frac{x^3}{3}\ln x] - \int \frac{x^3}{3}\frac{1}{x}\,dx$ $= \quad [\frac{x^3}{3}\ln x] - \frac{1}{3}\int x^2\,dx$ $= \quad [\frac{x^3}{3}\ln x] - \frac{x^3}{9} + C$
(c)	$\int \frac{x+3}{x^2+4x+7}\,dx$ $= \frac{1}{2}\int \frac{2x+6}{x^2+4x+7}\,dx$ $= \frac{1}{2}\int \frac{2x+4+2}{x^2+4x+7}\,dx$ $= \frac{1}{2}[\int \frac{2x+4}{x^2+4x+7}\,dx + \int \frac{2}{x^2+4x+7}\,dx]$ $= \frac{1}{2}[\ln(x^2+4x+7) + \int \frac{2}{(x+2)^2+3}\,dx]$ $= \frac{1}{2}[\ln(x^2+4x+7) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}] + C$
(d)	Let $x = 2\sin\theta$ $dx = 2\cos\theta\,d\theta$ $\int \frac{2x-1}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin\theta - 1}{2\cos\theta}\,2\cos\theta\,d\theta$ $= \quad \int 4\sin\theta - 1\,d\theta$ $= \quad -4\cos\theta - \theta + c$ $= \quad -4\frac{\sqrt{4-x^2}}{2} - \sin^{-1}\frac{x}{2} + c$ $= \quad -2\sqrt{4-x^2} - \sin^{-1}\frac{x}{2} + c$

2 **DHS PROMO 2009/QN9**

(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x) = 2\cos 2x$$

(ii)

$$\int \frac{\sin x + \cos x}{\left(\cos x - \sin x\right)^2}\,\mathrm{d}x = \int (\sin x + \cos x)\left(\cos x - \sin x\right)^{-2}\,\mathrm{d}x$$

$$= -\int (-\sin x - \cos x)\left(\cos x - \sin x\right)^{-2}\,\mathrm{d}x$$

$$= -\frac{\left(\cos x - \sin x\right)^{-1}}{-1} + C$$

$$= \frac{1}{\left(\cos x - \sin x\right)} + C$$

(iii)

$$\int_0^{\frac{\pi}{6}} \frac{\sin x \sin 2x + \cos x \sin 2x}{\left(\cos x - \sin x\right)^2}\,\mathrm{d}x$$

$$= \left[\sin 2x . \frac{1}{\cos x - \sin x}\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \frac{1}{\cos x - \sin x} . 2\cos 2x\,\mathrm{d}x$$

$$= \sin\frac{\pi}{3} . \frac{1}{\cos\frac{\pi}{6} - \sin\frac{\pi}{6}} - 2\int_0^{\frac{\pi}{6}} \frac{\cos^2 x - \sin^2 x}{\cos x - \sin x}\,\mathrm{d}x$$

$$= \frac{\sqrt{3}}{\sqrt{3}-1} - 2\int_0^{\frac{\pi}{6}} \left(\cos x + \sin x\right)\mathrm{d}x$$

$$= \frac{3+\sqrt{3}}{2} - 2\left[\sin x - \cos x\right]_0^{\frac{\pi}{6}}$$

$$= \frac{3\sqrt{3}}{2} - \frac{3}{2}$$

3 **ACJC PROMO 2010/QN9**

(a)

$$\int \frac{\cos 3x - \operatorname{cosec}^2 3x}{\sin 3x + \cot 3x}dx = \frac{1}{3}\ln\left|\sin 3x + \cot 3x\right| + c$$

(b)

$$\int \frac{1-x}{\sqrt{1-16x^2}}dx = \int \frac{1}{4\sqrt{\frac{1}{16} - x^2}}dx - \int \frac{x}{\sqrt{1-16x^2}}dx$$

$$= \frac{1}{4}\int \frac{1}{\sqrt{\left(\frac{1}{4}\right)^2 - x^2}}dx + \frac{1}{32}\int \left(-32x\right)\left(1-16x^2\right)^{-\frac{1}{2}}dx$$

$$= \frac{1}{4}\sin^{-1}\left(\frac{x}{1/4}\right) + \frac{2}{32}\sqrt{1-16x^2} + C$$

$$= \frac{1}{4}\sin^{-1}\left(4x\right) + \frac{1}{16}\sqrt{1-16x^2} + C$$

(c)	$\int (1-x)^{-2}\ln x\ dx = (\ln x)\left(\frac{1}{1-x}\right) - \int \frac{1}{x(1-x)}dx$ $= \frac{\ln x}{1-x} - \int\left(\frac{1}{1-x}+\frac{1}{x}\right)dx$ $= \frac{\ln x}{1-x} + \ln\lvert 1-x\rvert - \ln x + C$
4	**TJC PROMO 2009/QN2** (a) $\int \frac{1}{x\ln x^2}dx = \int \frac{\frac{1}{x}}{2\ln x}dx = \frac{1}{2}\int \frac{\frac{1}{x}}{\ln x}dx = \frac{1}{2}\ln\lvert\ln x\rvert + C$ (b) $\int \frac{e^{\sqrt{2x-1}}}{\sqrt{2x-1}}dx = \int \frac{1}{\sqrt{2x-1}}e^{\sqrt{2x-1}}dx = e^{\sqrt{2x-1}} + C$
5	$\int \sec^4 d\theta = \int \sec^2\theta(1+\tan^2\theta)d\theta = \tan\theta + \frac{1}{3}\tan^3\theta + C$
6 **(a)**	**JJC PROMO 2010/QN12** $A(2x+6)+b = 2Ax+6A+b$ Comparing coefficients of x, $2A=1 \Rightarrow A=\frac{1}{2}$ Comparing coefficients of constant term, $3+B=4 \Rightarrow B=1$ $\therefore x+4 = \frac{1}{2}(2x+6)+1$ $\int \frac{x+4}{x^2+6x+13}\ \mathrm{d}x$ $= \int \frac{\frac{1}{2}(2x+6)+1}{x^2+6x+13}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{2x+6}{x^2+6x+13}\ \mathrm{d}x + \int \frac{1}{x^2+6x+13}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{2x+6}{x^2+6x+13}\ \mathrm{d}x + \int \frac{1}{(x+3)^2+2^2}\ \mathrm{d}x$ $= \frac{1}{2}\ln\left(x^2+6x+13\right) + \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c$

(b)

Using $x = \frac{1}{u}$,

$\mathrm{d}x = -\frac{1}{u^2}\,\mathrm{d}u$

When $x = 2$, $u = \frac{1}{2}$.

When $x = 4$, $u = \frac{1}{4}$.

$$\int_2^4 \frac{1}{x^3} e^{\frac{1}{x}}\ \mathrm{d}x$$

$$= \int_{\frac{1}{2}}^{\frac{1}{4}} u^3 e^u \left(-\frac{1}{u^2}\right)\ \mathrm{d}u$$

$$= \int_{\frac{1}{2}}^{\frac{1}{4}} -ue^u\ \mathrm{d}u$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} ue^u\ \mathrm{d}u \ \text{ (shown)}$$

$$\int_2^4 \frac{1}{x^3} e^{\frac{1}{x}}\ \mathrm{d}x$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} ue^u\ \mathrm{d}u$$

$$= \left[ue^u\right]_{1/4}^{1/2} - \int_{\frac{1}{4}}^{\frac{1}{2}} e^u\ \mathrm{d}u$$

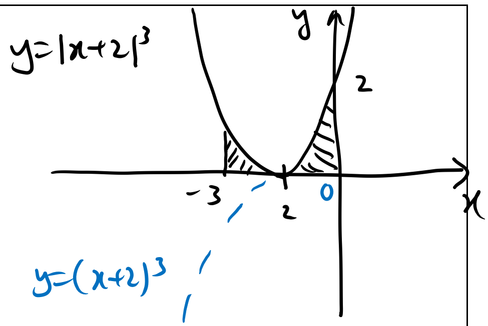
$$= \left(\frac{1}{2}e^{\frac{1}{2}} - \frac{1}{4}e^{\frac{1}{4}}\right) - \left[e^u\right]_{1/4}^{1/2}$$

$$= \left(\frac{1}{2}e^{\frac{1}{2}} - \frac{1}{4}e^{\frac{1}{4}}\right) - \left(e^{\frac{1}{2}} - e^{\frac{1}{4}}\right)$$

$$= \frac{3}{4}e^{\frac{1}{4}} - \frac{1}{2}e^{\frac{1}{2}}$$

(c)

$$|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -(x+2) & \text{if } x < -2 \end{cases}$$



$$\int_{-3}^{0} |x+2|^3 \, \mathrm{d}x$$

$$= \int_{-3}^{-2} [-(x+2)]^3 \, \mathrm{d}x + \int_{-2}^{0} (x+2)^3 \, \mathrm{d}x$$

$$= -\left[\frac{(x+2)^4}{4}\right]_{-3}^{-2} + \left[\frac{(x+2)^4}{4}\right]_{-2}^{0}$$

$$= -\frac{1}{4}(0-1) + \frac{1}{4}(2^4 - 0)$$

$$= \frac{17}{4}$$

7

RVHS PROMO 2010/QN11

(a)

$$u = e^x \Rightarrow \frac{du}{dx} = e^x = u$$

Then

$$\int \frac{1}{e^x + 2e^{-x}} \, dx = \int \frac{1}{u + \frac{2}{u}} \left(\frac{du}{u}\right)$$

$$= \int \frac{1}{u^2 + 2} \, du$$

$$= \int \frac{1}{\left(\sqrt{2}\right)^2 + u^2} \, du$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c \quad = \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{e^x}{\sqrt{2}}\right) + c$$

(b)

$$\int_0^1 \frac{4x-5}{\sqrt{3+2x-x^2}} \, dx$$

$$= \int_0^1 \frac{-2(2-2x)-1}{\sqrt{3+2x-x^2}} \, dx$$

$$= -2\int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}} \, dx - \int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \, dx$$

$$= -2\int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}} \, dx - \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}} \, dx$$

$$= -2\left[2\sqrt{3+2x-x^2}\right]_0^1 - \left[\sin^{-1}\left(\frac{x-1}{2}\right)\right]_0^1$$

$$= -2\left[2\sqrt{4} - 2\sqrt{3}\right]_0^1 - \left[\sin^{-1}\left(\frac{1-1}{2}\right) - \sin^{-1}\left(\frac{0-1}{2}\right)\right]$$

	$= 4\sqrt{3} - 8 - 0 - \frac{\pi}{6}$ $= \frac{24\sqrt{3}-48-\pi}{6}$
8	**VJCPROMO2013/QN5** (a)(i) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x^2+1}\right) = \frac{x^2+1-x(2x)}{\left(x^2+1\right)^2}$ $= \frac{1-x^2}{\left(x^2+1\right)^2}$ $= \frac{2-1-x^2}{\left(x^2+1\right)^2}$ $= \frac{2}{\left(x^2+1\right)^2} - \frac{1+x^2}{\left(x^2+1\right)^2}$ $= \frac{2}{\left(x^2+1\right)^2} - \frac{1}{x^2+1}$ (ii) $\int_0^1 \left[\frac{2}{\left(x^2+1\right)^2} - \frac{1}{x^2+1}\right] \mathrm{d}x = \left[\frac{x}{x^2+1}\right]_0^1$ $2\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x - \left[\tan^{-1} x\right]_0^1 = \frac{1}{2}$ $2\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x = \frac{1}{2} + \frac{\pi}{4}$ $\int_0^1 \frac{1}{\left(x^2+1\right)^2}\,\mathrm{d}x = \frac{1}{4} + \frac{\pi}{8}$ (b) RHS $= A + \frac{\mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}$ $= \frac{A - A\mathrm{e}^{2x} + \mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}$ Comparing the numerator to that of the LHS, $A - A\mathrm{e}^{2x} + \mathrm{e}^{2x} = 1$ $\Rightarrow A = 1$ $\int \frac{1}{1-\mathrm{e}^{2x}}\,\mathrm{d}x = \int \left(1 + \frac{\mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}\right)\mathrm{d}x$ $= x - \frac{1}{2}\ln\left\|1-\mathrm{e}^{2x}\right\| + C$ If $x \geq 2$, then $\|x-2\| = (x-2)$ If $x > 2$, then $\|x-2\| = -(x-2)$

9	**HCI/2020Prelim/I/6**

a.

$$\int \frac{3e^x}{5-0.3e^x}\,dx$$
$$=-10\int \frac{-0.3e^x}{5-0.3e^x}\,dx$$
$$=-10\ln\left|5-0.3e^x\right|+C$$

b.

Let $I=\int \cos(\ln x)\,dx$

$u=\cos(\ln x)$, $v'=1$

$u'=-\frac{1}{x}\sin(\ln x)$, $v=x$

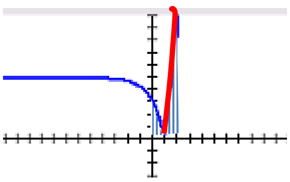
$$I=x\cos(\ln x)-\int -\frac{1}{x}\left[x\sin(\ln x)\right]dx$$
$$=x\cos(\ln x)+\int \sin(\ln x)\,dx$$

$u=\sin(\ln x)$, $v'=1$

$u'=\frac{1}{x}\cos(\ln x)$, $v=x$

$$I=x\cos(\ln x)+x\sin(\ln x)-\int \frac{1}{x}\left[x\cos(\ln x)\right]dx$$
$$=x\cos(\ln x)+x\sin(\ln x)-\int \cos(\ln x)\,dx$$
$$2I=x\cos(\ln x)+x\sin(\ln x)$$
$$I=\frac{x}{2}\left[\cos(\ln x)+\sin(\ln x)\right]+C$$

c.



When $2e^x-5=0$, $x=\ln 2.5$

$$\left|2e^x-5\right| \begin{cases} 2e^x-5 & ,x\ge \ln 2.5 \\ -(2e^x-5) & ,x<\ln 2.5 \end{cases}$$

	$\int_0^3 \left\|2e^x - 5\right\| \, dx$ $= -\int_0^{\ln 2.5} 2e^x - 5 \, dx + \int_{\ln 2.5}^3 2e^x - 5 \, dx$ $= \left[5x - 2e^x\right]_0^{\ln 2.5} + \left[2e^x - 5x\right]_{\ln 2.5}^3$ $= \left[\left(5\ln 2.5 - 2e^{\ln 2.5}\right) + 2e^0\right] + \left[\left(2e^3 - 15\right) - \left(2e^{\ln 2.5} - 5\ln 2.5\right)\right]$ $= 10\ln 2.5 - 4e^{\ln 2.5} + 2e^3 - 13$ $= 10\ln 2.5 - 4(2.5) + 2e^3 - 13$ $= 10\ln 2.5 + 2e^3 - 23$
10	**NJC/2020Promo/6** (i) $\int \frac{x}{\sqrt{1-k^2x^2}} \, dx$ $= \int x\left(1-k^2x^2\right)^{-\frac{1}{2}} \, dx$ $= \frac{-1}{2k^2}\int -2k^2x\left(1-k^2x^2\right)^{-\frac{1}{2}} \, dx$ $= \frac{-1}{2k^2}\frac{\left(1-k^2x^2\right)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$ $= \frac{-1}{k^2}\left(1-k^2x^2\right)^{\frac{1}{2}} + C$ (ii) $\int \left(\sin^{-1} kx\right)\frac{x}{\sqrt{1-k^2x^2}} \, dx$ $u = \left(\sin^{-1} kx\right), \quad \frac{dv}{dx} = \frac{x}{\sqrt{1-k^2x^2}}$ $\frac{du}{dx} = \frac{k}{\sqrt{1-k^2x^2}}, \quad v = \frac{-1}{k^2}\left(1-k^2x^2\right)^{\frac{1}{2}}$

$$\int \left(\sin^{-1} kx\right)\frac{x}{\sqrt{1-k^2x^2}}\ \mathrm{d}x$$

$$=\left(\sin^{-1} kx\right)\frac{-1}{k^2}\left(1-k^2x^2\right)^{\frac{1}{2}}-\int \frac{-1}{k^2}\left(1-k^2x^2\right)^{\frac{1}{2}}\frac{k}{\sqrt{1-k^2x^2}}\ \mathrm{d}x$$

$$=\frac{-\left(\sin^{-1} kx\right)\left(1-k^2x^2\right)^{\frac{1}{2}}}{k^2}+\int \frac{1}{k}\ \mathrm{d}x$$

$$=\frac{-\left(\sin^{-1} kx\right)\left(1-k^2x^2\right)^{\frac{1}{2}}}{k^2}+\frac{x}{k}+D$$

(iii)

When $k=1$, $\int \left(\sin^{-1} kx\right)\frac{x}{\sqrt{1-k^2x^2}}\ \mathrm{d}x$ becomes

$\int \left(\sin^{-1} x\right)\frac{x}{\sqrt{1-x^2}}\ \mathrm{d}x$ so

$$\int_0^{\frac{1}{\sqrt{2}}} \left(\sin^{-1} x\right)\frac{x}{\sqrt{1-x^2}}\ \mathrm{d}x$$

$$=\left[-\left(\sin^{-1} x\right)\left(1-x^2\right)^{\frac{1}{2}}+x\right]_0^{\frac{1}{\sqrt{2}}}$$

$$=-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\left(1-\frac{1}{2}\right)^{\frac{1}{2}}+\frac{1}{\sqrt{2}}$$

$$=-\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}\right)+\frac{1}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)$$

(iv)

Method 1

$$\int_m^{\frac{1}{\sqrt{2}}+m} \left[\sin^{-1}\left(x-m\right)\right]\frac{x-m}{\sqrt{1-\left(x-m\right)^2}}\ \mathrm{d}x$$

$$=\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)$$

Both integrand and limits of integration underwent a translation of m units in the positive or negative x-direction so the area under the curve is preserved.

Or

Method 2

Let $u = x - m$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$x = m,\ u = m - m = 0$

$x = \frac{1}{\sqrt{2}} + m,\ u = \frac{1}{\sqrt{2}} + m - m = \frac{1}{\sqrt{2}}$

$$\int_{m}^{\frac{1}{\sqrt{2}}+m} \left[\sin^{-1}(x-m)\right] \frac{x-m}{\sqrt{1-(x-m)^2}}\,\mathrm{d}x$$

$$= \int_{0}^{\frac{1}{\sqrt{2}}} \left(\sin^{-1} u\right) \frac{u}{\sqrt{1-u^2}}\,\mathrm{d}u$$

$$= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$$

11(a)

TJC/2014Promo/6

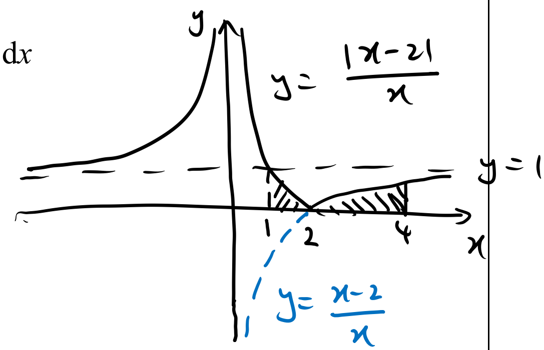
$$\int_1^4 \frac{|x-2|}{x}\,\mathrm{d}x = \int_1^2 \frac{-(x-2)}{x}\,\mathrm{d}x + \int_2^4 \frac{(x-2)}{x}\,\mathrm{d}x$$

$$= -\int_1^2 \left(1 - \frac{2}{x}\right)\mathrm{d}x + \int_2^4 \left(1 - \frac{2}{x}\right)\mathrm{d}x$$

$$= -\left[x - 2\ln x\right]_1^2 + \left[x - 2\ln x\right]_2^4$$

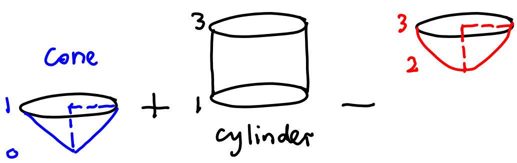
$$= -\left(2 - 2\ln 2 - 1\right) + \left[4 - 2\ln 4 - (2 - 2\ln 2)\right]$$

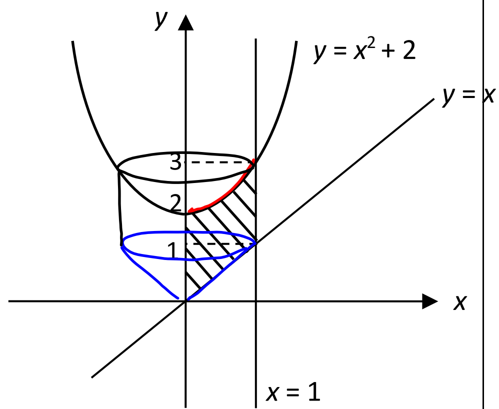
$$= 1$$



(b)

Volume of solid generated





$$= \frac{1}{3}\pi(1)^2(1) + \left[\pi(1)^2(2) - \pi\int_2^3 (y-2)\,\mathrm{d}y\right]$$

$$= \frac{7}{3}\pi - \left[\frac{y^2}{2} - 2y\right]_2^3$$

	$=\frac{7}{3}\pi-\frac{1}{2}\pi$ $=\frac{11}{6}\pi\ \text{unit}^3$
12(i)	**<u>ACJC PROMO 2012/QN15</u>** $x=\cos\theta-1\ \Rightarrow\frac{dx}{d\theta}=-\sin\theta$ $\int_{-2}^{-1}\sqrt{-x^2-2x}\ \mathrm{d}x$ $=\int_{\pi}^{\frac{\pi}{2}}\sqrt{-(\cos\theta-1)^2-2(\cos\theta-1)}\ \ (-\sin\theta)d\theta$ $=\int_{\pi}^{\frac{\pi}{2}}\sqrt{-\cos^2\theta-1+2}\ \ (-\sin\theta)d\theta$ $=\int_{\pi}^{\frac{\pi}{2}}\sqrt{1-\cos^2\theta}\ \ (-\sin\theta)d\theta$ $=\int_{\pi}^{\frac{\pi}{2}}\sqrt{\sin^2\theta}\ \ (-\sin\theta)d\theta$ $=\int_{\frac{\pi}{2}}^{\pi}\sin^2\theta\ d\theta$ $=\int_{\frac{\pi}{2}}^{\pi}\frac{1-\cos2\theta}{2}\ d\theta$ $=\frac{1}{2}\left[\theta-\frac{\sin2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$ $=\frac{1}{2}\left[\left(\pi-\frac{\sin2\pi}{2}\right)-\left(\frac{\pi}{2}-\frac{\sin\pi}{2}\right)\right]=\frac{\pi}{4}$
(ii)	Note: Only two formulas for area, use $\int\ \mathrm{f}(x)\ \mathrm{d}x$ for area between curve and ***x*-axis**, and $\int\ \mathrm{f}^{-1}(y)\ \mathrm{d}y$ for area between curve and ***y*-axis**. $4(x+1)^2+(y-2)^2=4$ $\Rightarrow(y-2)^2=4\left[1-(x+1)^2\right]$ $\Rightarrow(y-2)=\pm2\sqrt{1-(x+1)^2}$ $\Rightarrow y=2\pm2\sqrt{1-(x+1)^2}$

	For region R, $y < 2$, so choose $y = 2-2\sqrt{1-(x+1)^2}$ Area of R = (area between line and x-axis) − (area between curve and x-axis) Area of R $= \int_{-2}^{-1}\left[-x-\left(2-2\sqrt{1-(x+1)^2}\right)\right]dx$ $=\left[-\frac{x^2}{2}-2x\right]_{-2}^{-1}+2\int_{-2}^{-1}\sqrt{-x^2-2x}\ dx$ $=\left(-\frac{1}{2}+2\right)-\left(-2+4\right)+2\left(\frac{\pi}{4}\right)$ $=\frac{\pi}{2}-\frac{1}{2}$
(iii)	Equation of curve after translation of one unit in the positive x-direction is $4x^2+(y-2)^2=4$ i.e. $x^2+\frac{(y-2)^2}{4}=1$ Required volume $=\pi\int_0^2 x^2\ \mathrm{d}y-\frac{1}{3}\pi(1)^2 1$ $=\pi\int_0^2\left[1-\frac{(y-2)^2}{4}\right]\mathrm{d}y-\frac{1}{3}\pi(1)^2 1$ = 3.14 Alternatively, considering the original curve (without translation): $4(x+1)^2+(y-2)^2=4 \Rightarrow (x+1)^2=\frac{4-(y-2)^2}{4}$ Required volume $=\pi\int_0^2(x+1)^2\ \mathrm{d}y-\frac{1}{3}\pi(1)^2 1$ $=\pi\int_0^2\left[1-\frac{(y-2)^2}{4}\right]\mathrm{d}y-\frac{1}{3}\pi(1)^2 1$ = 3.14

13 **NJC PROMO 2010/QN10**

(a)

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{x^2+2x} = (2x+2)\mathrm{e}^{x^2+2x} = 2(x+1)\mathrm{e}^{x^2+2x}$

(ii) From part (a)(i), $\int (x+1)\mathrm{e}^{x^2+2x}\ \mathrm{d}x = \dfrac{1}{2}\mathrm{e}^{x^2+2x} + C$

$$\begin{aligned}\int (x+1)^3\mathrm{e}^{x^2+2x}\ \mathrm{d}x &= \int (x+1)^2\cdot(x+1)\mathrm{e}^{x^2+2x}\ \mathrm{d}x \\ &= (x+1)^2\cdot\frac{\mathrm{e}^{x^2+2x}}{2} - \int 2(x+1)\cdot\frac{\mathrm{e}^{x^2+2x}}{2}\ \mathrm{d}x \\ &= \frac{1}{2}(x+1)^2\mathrm{e}^{x^2+2x} - \int (x+1)\mathrm{e}^{x^2+2x}\ \mathrm{d}x \\ &= \frac{1}{2}(x+1)^2\mathrm{e}^{x^2+2x} - \frac{1}{2}\mathrm{e}^{x^2+2x} + C\end{aligned}$$

(b)

$$x+4 = \frac{y}{y-1} \Rightarrow x = \frac{y}{y-1} - 4$$

$$\text{Volume} = \pi\int_{\frac{4}{3}}^{2}\left(\frac{y}{y-1}-4\right)^2 \mathrm{d}y + \frac{\pi}{3}(2)^2(2)$$

$$= \pi(1.40833) + \frac{8\pi}{3} \quad \text{(by GC)}$$

$$= 12.80197$$

$$\approx 12.802 \text{ units}^3 \quad \text{(to 3 dec. pl.)}$$

14 **RI/2009Prelim/I/9**

Equating the two equations, we have

$$x^4 + x^2 - \frac{3}{4} = 0$$

Solving, $x = \dfrac{1}{\sqrt{2}}$ or $x = -\dfrac{1}{\sqrt{2}}$

$y = \dfrac{1}{2}$ or $y = \dfrac{1}{2}$

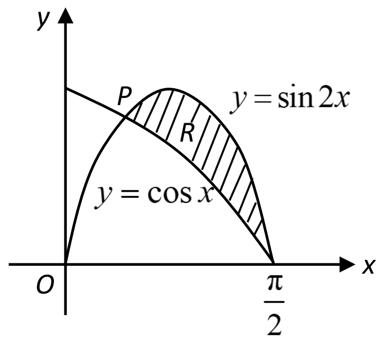
The coordinates of point A and B are $(-\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(\frac{1}{\sqrt{2}}, \frac{1}{2})$ respectively.

(i) Area of $R = 2\left(\int_0^{\frac{1}{\sqrt{2}}} x^2\ \mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \sqrt{\frac{3}{4}-x^2}\ \mathrm{d}x\right)$

= 2 (0.11785 + 0.05403)
= 0.34 (2 d.p.)

(ii) Volume $= \pi\int_0^{\frac{1}{2}}\left(\frac{3}{4} - y^2 - y\right)\mathrm{d}y$

$$= \pi\left[\frac{3}{4}y - \frac{y^3}{3} - \frac{y^2}{2}\right]_0^{\frac{1}{2}} = \frac{5}{24}\pi \text{ unit}^3$$

15(a)	**<u>DHS PROMO 2010/QN11</u>** At *P*, $\sin 2x = \cos x$ $2\sin x\cos x - \cos x = 0$ $\cos x\left(2\sin x - 1\right) = 0$ $\cos x = 0 \quad \text{or} \quad \sin x = \frac{1}{2}$ $x = \frac{\pi}{2} \quad \text{or} \quad x = \frac{\pi}{6}$ Thus *x*-coordinate of *P* is $\frac{\pi}{6}$.  Area required = $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\,\mathrm{d}x$ $= \left[-\frac{\cos 2x}{2} - \sin x \right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ $= \left[\frac{1}{2} - 1 \right] - \left[-\frac{1}{4} - \frac{1}{2} \right]$ $= -\frac{1}{2} + \frac{3}{4}$ $= \frac{1}{4}$ units^2
(b)	Area of region *S* = Area of region *T* $\int_0^2 \frac{x^2}{4}\,\mathrm{d}x = \int_2^b \frac{4}{x^2}\,\mathrm{d}x$ $\frac{1}{4}\left[\frac{x^3}{3} \right]_0^2 = 4\left[-\frac{1}{x} \right]_2^b$ $\frac{8}{12} = 4\left[-\frac{1}{b} + \frac{1}{2} \right]$ $\frac{1}{6} = -\frac{1}{b} + \frac{1}{2}$ $b = 3$

	$V_S+V_T=\pi\int_0^2\left(\frac{x^2}{4}\right)^2\mathrm{d}x+\pi\int_2^b\left(\frac{4}{x^2}\right)^2\mathrm{d}x$ $=0.4\pi+16\pi\left[-\frac{1}{3x^3}\right]_2^b$ $=0.4\pi+16\pi\left[-\frac{1}{3b^3}+\frac{1}{24}\right]$ $=-\frac{16\pi}{3b^3}+\frac{16\pi}{15}$ $W_S=$ Volume of cylinder $-\pi\int_0^1 x^2\mathrm{d}y$ $=\pi(2)^2(1)-\pi\int_0^1 4y\ \mathrm{d}y$ $=4\pi-2\pi$ $=2\pi$ $V_S+V_T=\frac{1}{2}W_S$ $\Rightarrow-\frac{16\pi}{3b^3}+\frac{16\pi}{15}=\pi$ $\Rightarrow\frac{16\pi}{3b^3}=\frac{\pi}{15}$ $\Rightarrow b^3=80$ $\Rightarrow b=4.31$
16 **(i)** **(ii)**	**<u>JJC PROMO 2010/QN10</u>** when $x=3$, $y=\frac{x^2-4}{5}=1$ when $x=3$, $y=\frac{3}{x}=1$ The curves $y=\frac{x^2-4}{5}$ and $y=\frac{3}{x}$ intercept at (3,1). $y=\frac{x^2-4}{5}\Rightarrow x=\sqrt{5y+4}$ $y=\frac{3}{x}\Rightarrow x=\frac{3}{y}$ Area = $\int_0^1\sqrt{5y+4}\ \mathrm{d}y+\int_1^3\left(\frac{3}{y}\right)\mathrm{d}y$ $=5.83$ **Alternative method**

(iii)	Area = $\int_0^1 3\,\mathrm{d}x+\int_1^3\left(\frac{3}{x}\right)\mathrm{d}x-\int_2^3\frac{x^2-4}{5}\,\mathrm{d}x$ $=5.83$ Volume $=\pi\int_0^1 3^2\,\mathrm{d}x+\pi\int_1^3\left(\frac{3}{x}\right)^2\mathrm{d}x-\pi\int_2^3\left(\frac{x^2-4}{5}\right)^2\mathrm{d}x$ $=46.2$
(iv)	Volume $=\pi\int_0^1(5y+4)\,\mathrm{d}y+\pi\int_1^3\left(\frac{3}{y}\right)^2\mathrm{d}y$ $=\pi\left[\frac{5y^2}{2}+4y\right]_0^1+9\pi\left[-\frac{1}{y}\right]_1^3$ $=\pi\left[\frac{5}{2}+4\right]+9\pi\left[-\frac{1}{3}+1\right]$ $=\frac{25}{2}\pi$
17 **(a)**	**RVHS/2020Promo/8** $u=\mathrm{e}^x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x=u.$ When $x=0\Rightarrow u=\mathrm{e}^0=1.$ When $x=\ln\sqrt{3}\Rightarrow u=\mathrm{e}^{\ln\sqrt{3}}=\sqrt{3}.$ $\int_0^{\ln\sqrt{3}}\frac{\mathrm{e}^{3x}}{\mathrm{e}^{2x}+1}\mathrm{d}x=\int_1^{\sqrt{3}}\frac{u^2}{u^2+1}\mathrm{d}u=\int_1^{\sqrt{3}}1-\frac{1}{u^2+1}\mathrm{d}u$ $=\left[u-\tan^{-1}u\right]_1^{\sqrt{3}}=\sqrt{3}-\tan^{-1}\sqrt{3}-\left(1-\tan^{-1}1\right)$ $=\sqrt{3}-1-\frac{\pi}{3}+\frac{\pi}{4}$ $=\sqrt{3}-\frac{\pi}{12}-1.$
(b)(i)	$\int_0^a x\sin x\,\mathrm{d}x$ $=\left[-x\cos x\right]_0^a-\int_0^a(1)(-\cos x)\,\mathrm{d}x$ $=-a\cos a+(0)\cos 0+\int_0^a\cos x\,\mathrm{d}x$ $=\left[\sin x\right]_0^a-a\cos a$ $=\sin a-\sin 0-a\cos a$ $=\sin a-a\cos a.$

b(ii)	Volume generated $= \pi\left(\sqrt{\frac{\pi}{2}}\right)^2\left(\frac{\pi}{2}\right) - \pi\int_0^{\frac{\pi}{2}} x^2 \mathrm{d}y$ $= \frac{\pi^3}{4} - \pi\int_0^{\frac{\pi}{2}} y\sin y \ \mathrm{d}y$ $= \frac{\pi^3}{4} - \pi\left(\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}\right)$ $= \frac{\pi^3}{4} - \pi.$
18(a) Part 1	**<u>NJC PROMO 2010/QN12</u>** **<u>Method 1</u>** $x = \sin t \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = \cos t$ $\int \frac{x\mathrm{e}^{\sin^{-1}x}}{\sqrt{1-x^2}} \ \mathrm{d}x = \int \frac{(\sin t)\mathrm{e}^{\sin^{-1}(\sin t)}}{\sqrt{1-\sin^2 t}}\cos t \ \mathrm{d}t$ $= \int \frac{(\sin t)\mathrm{e}^t}{\cos t}\cos t \ \mathrm{d}t$ $= \int \mathrm{e}^t \sin t \ \mathrm{d}t$ (shown) **<u>Method 2</u>** $x = \sin t \Rightarrow t = \sin^{-1} x \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$ $\int \frac{x\mathrm{e}^{\sin^{-1}x}}{\sqrt{1-x^2}} \ \mathrm{d}x = \int x\mathrm{e}^{\sin^{-1}x} \cdot \frac{1}{\sqrt{1-x^2}} \ \mathrm{d}x$ $= \int (\sin t)\mathrm{e}^{\sin^{-1}(\sin t)} \ \mathrm{d}t$ $= \int \mathrm{e}^t \sin t \ \mathrm{d}t$ (shown)
(a) Part2	$u = \sin t \qquad v = \mathrm{e}^t$ $\frac{\mathrm{d}u}{\mathrm{d}t} = \cos t \qquad \int v \ \mathrm{d}t = \mathrm{e}^t$ $u = \cos t \qquad v = \mathrm{e}^t$ $\frac{\mathrm{d}u}{\mathrm{d}t} = -\sin t \qquad \int v \ \mathrm{d}t = \mathrm{e}^t$ $\int \mathrm{e}^t \sin t \ \mathrm{d}t = \mathrm{e}^t \sin t - \int \mathrm{e}^t \cos t \ \mathrm{d}t$ $= \mathrm{e}^t \sin t - \left[\mathrm{e}^t \cos t - \int \mathrm{e}^t(-\sin t) \ \mathrm{d}t\right]$ $= \mathrm{e}^t \sin t - \mathrm{e}^t \cos t - \int \mathrm{e}^t \sin t \ \mathrm{d}t$ Hence, $\int \mathrm{e}^t \sin t \ \mathrm{d}t = \frac{1}{2}\mathrm{e}^t\left(\sin t - \cos t\right) + c$

	$\int \frac{x\mathrm{e}^{\sin^{-1} x}}{\sqrt{1-x^2}}\ \mathrm{d}x = \int \mathrm{e}^t \sin t\ \mathrm{d}t$ $= \frac{1}{2}\mathrm{e}^t\left(\sin t - \cos t\right) + c$ $= \frac{1}{2}\mathrm{e}^{\sin^{-1} x}\left(x - \sqrt{1-x^2}\right) + c$

(b)

Method 1

$$V = \underbrace{\frac{1}{3}\pi(1)^2(1)}_{\text{Volume of cone generated by } y=x} + \underbrace{\pi\int_1^3 \left(\frac{2}{\sqrt{3+x^2}}\right)^2 \mathrm{d}x}_{\text{Volume generated by } C}$$

$$= \frac{\pi}{3} + \pi\int_1^3 \frac{4}{3+x^2}\,\mathrm{d}x$$

$$= \frac{\pi}{3} + \pi\left[\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$

$$= \frac{\pi}{3} + \frac{4}{\sqrt{3}}\pi\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]_1^3$$

$$= \frac{\pi}{3} + \frac{4}{\sqrt{3}}\pi\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}\left(1 + \frac{2}{\sqrt{3}}\pi\right) \text{ or } 4.85$$

Method 2

$$V = \underbrace{\pi\int_0^1 (x)^2\,\mathrm{d}x}_{\text{Volume of cone generated by } y=x} + \underbrace{\pi\int_1^3 \left(\frac{2}{\sqrt{3+x^2}}\right)^2 \mathrm{d}x}_{\text{Volume generated by } C}$$

$$= \pi\left[\frac{x^3}{3}\right]_0^1 + \pi\int_1^3 \frac{4}{3+x^2}\,\mathrm{d}x$$

$$= \pi\left(\frac{1}{3} - 0\right) + \pi\left[\frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^3$$

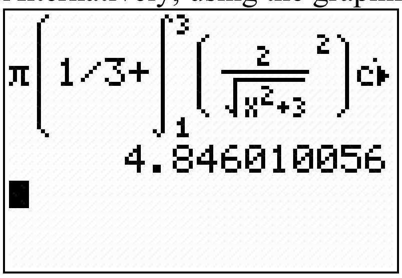
$$= \frac{\pi}{3} + \frac{4}{\sqrt{3}}\pi\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]_1^3$$

$$= \frac{\pi}{3} + \frac{4}{\sqrt{3}}\pi\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}\left(1 + \frac{2}{\sqrt{3}}\pi\right) \text{ or } 4.85$$

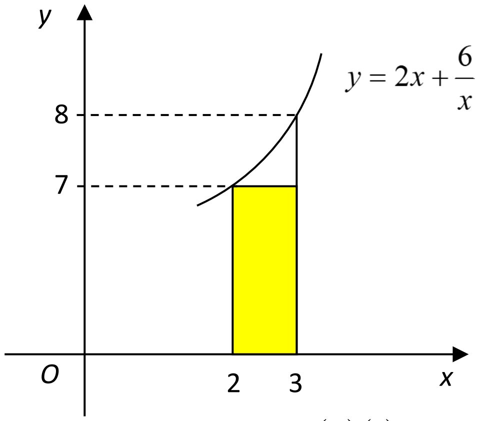
Method 3

Alternatively, using the graphing calculator,



$V = 4.85$

19	**<u>JJC PROMO 2009/QN12</u>**
(i)	Let $u = \ln(x+1)$ $\quad \dfrac{dv}{dx} = 1$ $\dfrac{du}{dx} = \dfrac{1}{x+1} \qquad v = x$ $\int_0^1 \ln(x+1)\,dx = \left[x\ln(x+1)\right]_0^1 - \int_0^1 \dfrac{x}{x+1}\,dx$ $= \ln 2 - \int_0^1 1 - \dfrac{1}{x+1}\,dx$ $= \ln 2 - \left[x - \ln(x+1)\right]_0^1$ $= \ln 2 - [1 - \ln 2]$ $= 2\ln 2 - 1$
(ii)	Area $= \dfrac{1}{n}\ln\left(\dfrac{1}{n}+1\right) + \dfrac{1}{n}\ln\left(\dfrac{2}{n}+1\right) + \dfrac{1}{n}\ln\left(\dfrac{3}{n}+1\right) + ... + \dfrac{1}{n}\ln\left(\dfrac{n-2}{n}+1\right) + \dfrac{1}{n}\ln\left(\dfrac{n-1}{n}+1\right)$ $= \dfrac{1}{n}\left[\ln\left(\dfrac{1+n}{n}\right) + \ln\left(\dfrac{2+n}{n}\right) + ... + \ln\left(\dfrac{n-1+n}{n}\right)\right]$ $= \dfrac{1}{n}\left[\sum_{r=1}^{n-1} \ln\left(\dfrac{r+n}{n}\right)\right]$ (shown)
(iii)	Using (ii) and GC, we have $\dfrac{1}{100}\left[\sum_{r=1}^{99} \ln\left(\dfrac{r+100}{100}\right)\right] = 0.38282$ Using (i), $\dfrac{1}{100}\left[\sum_{r=1}^{99} \ln\left(\dfrac{r+100}{100}\right)\right] \approx 2\ln 2 - 1$ Hence, $\quad 2\ln 2 - 1 \approx 0.38282$ $\ln 2 \approx 0.691$

20	**ACJC PROMO 2008/QN11**  $\int_2^3\left(2x+\frac{6}{x}\right)dx >$ area of shaded rectangle $=(7)(1)=7$ (shown) $\int_2^3\left(2x+\frac{6}{x}\right)dx<(8)(1)=8$ (shown) $\int_2^3\left(2x+\frac{6}{x}\right)dx<\left[x^2+6\ln x\right]_2^3=(9+6\ln 3)-(4+6\ln 2)=5+6\ln\frac{3}{2}$ $\therefore\ 7<5+6\ln\frac{3}{2}<8 \Rightarrow \frac{1}{3}<\ln(1.5)<\frac{1}{2}$ $\therefore\ p=3,\ q=2$