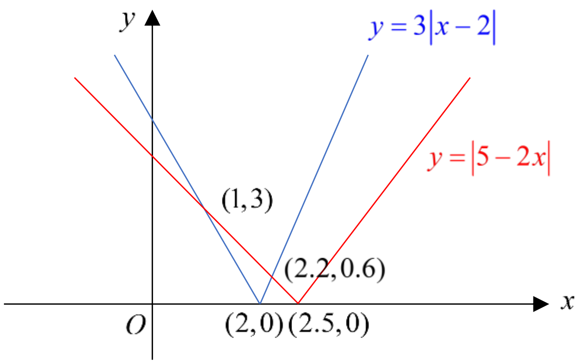
Section A: Pure Mathematics (40 marks)

1 (a) Find the set of values of x for which $3|x-2|<|5-2x|$. [2]

Solutions



$\therefore 1<x<2.2$

(b) Express $\dfrac{x+25}{x^2-4x-5}+3$ as a single fraction. Hence, without using a calculator, solve exactly the inequality $\dfrac{x+25}{x^2-4x-5}>-3$. [4]

Solutions

$$\frac{x+25}{x^2-4x-5}+3=\frac{x+25+3x^2-12x-15}{x^2-4x-5}$$

$$=\frac{3x^2-11x+10}{x^2-4x-5}$$

$$\frac{x+25}{x^2-4x-5}>-3$$

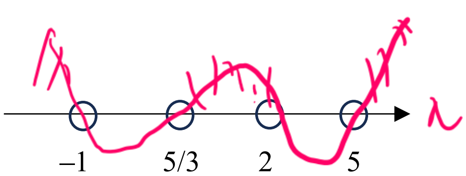
$$\frac{x+25}{x^2-4x-5}+3>0$$

$$\frac{3x^2-11x+10}{x^2-4x-5}>0$$

$$\frac{(3x-5)(x-2)}{(x-5)(x+1)}>0,\ x\neq 5,-1$$

Multiplying by $(x-5)^2(x+1)^2$:

$$(3x-5)(x-2)(x-5)(x+1)>0$$



$\therefore x < -1$ or $\frac{5}{3} < x < 2$ or $x > 5$

2 (a) Given that $y = \ln(\sec x)$, show that $\frac{d^3y}{dx^3} = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)$. [3]

Solutions

$$\frac{dy}{dx} = \frac{1}{\sec x}(\sec x \tan x) = \tan x$$

$$\frac{d^2y}{dx^2} = \sec^2 x$$

$$= 1 + \tan^2 x$$

$$= 1 + \left(\frac{dy}{dx}\right)^2 ---(1)$$

Differentiating with respect to x:

$$\therefore \frac{d^3y}{dx^3} = 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right) ---(2) \text{ (shown)}$$

2 (b) Hence, or otherwise, obtain the Maclaurin expansion of y in terms of x up to and including the term in x^4. [3]

Solutions

Differentiating (2) with respect to x:

$$\frac{d^4y}{dx^4} = 2\left(\frac{d^3y}{dx^3}\right)\left(\frac{dy}{dx}\right) + 2\left(\frac{d^2y}{dx^2}\right)\left(\frac{d^2y}{dx^2}\right)$$

When $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$,

$$\frac{d^2y}{dx^2} = 1, \frac{d^3y}{dx^3} = 0, \frac{d^4y}{dx^4} = 2.$$

$$\therefore y = \frac{x^2}{2} + 2\left(\frac{x^4}{4!}\right) + \ldots$$

$$\approx \frac{x^2}{2} + \frac{x^4}{12}$$

2 (c) By putting $x = \frac{1}{4}\pi$, find an approximation for $\ln 2$ in terms of π. [2]

Solutions
$\ln(\sec x) \approx \frac{x^2}{2} + \frac{x^4}{12}$ Substituting $x = \frac{\pi}{4}$: $\ln\left(\sec\frac{\pi}{4}\right) \approx \frac{\left(\frac{\pi}{4}\right)^2}{2} + \frac{\left(\frac{\pi}{4}\right)^4}{12}$ $\ln\left(\sqrt{2}\right) \approx \frac{\pi^2}{32} + \frac{\pi^4}{3072}$ $\frac{1}{2}\ln(2) \approx \frac{\pi^2}{32} + \frac{\pi^4}{3072}$ $\therefore \ln(2) \approx \frac{\pi^2}{16} + \frac{\pi^4}{1536}$

2 (d) Using your answer to part (b), find an approximation to $\int_0^{\frac{1}{10}\pi} \ln(\sec x)\, dx$. Give your answer correct to 4 significant figures. [1]

Solutions
$\int_0^{\frac{1}{10}\pi} \ln(\sec x)\, dx \approx \int_0^{\frac{1}{10}\pi} \left(\frac{x^2}{2} + \frac{x^4}{12}\right) dx$ $= 0.005219$ (to 4 sig fig)

3 The points A and B have position vectors $\begin{pmatrix}-1\\2\\5\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\8\end{pmatrix}$ respectively.

The point C is such that $\overrightarrow{BC}=2\overrightarrow{AB}$.

(a) Find the position vector of C. [2]

Solutions
$\overrightarrow{OA}=\begin{pmatrix}-1\\2\\5\end{pmatrix},\overrightarrow{OB}=\begin{pmatrix}1\\-2\\8\end{pmatrix}$ $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}1\\-2\\8\end{pmatrix}-\begin{pmatrix}-1\\2\\5\end{pmatrix}=\begin{pmatrix}2\\-4\\3\end{pmatrix}$ Given $\overrightarrow{BC}=2\overrightarrow{AB}$, $\overrightarrow{OC}-\overrightarrow{OB}=2\begin{pmatrix}2\\-4\\3\end{pmatrix}$ $\overrightarrow{OC}=\begin{pmatrix}4\\-8\\6\end{pmatrix}+\begin{pmatrix}1\\-2\\8\end{pmatrix}=\begin{pmatrix}5\\-10\\14\end{pmatrix}$

The points D has position vector $\begin{pmatrix}1\\2\\d\end{pmatrix}$ and is such that $\left|\overrightarrow{AD}\right|=\left|\overrightarrow{BD}\right|$.

(b) Find the value of d. [2]

Solutions
$\overrightarrow{AD}=\overrightarrow{OD}-\overrightarrow{OA}=\begin{pmatrix}1\\2\\d\end{pmatrix}-\begin{pmatrix}-1\\2\\5\end{pmatrix}=\begin{pmatrix}2\\0\\d-5\end{pmatrix}$ $\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=\begin{pmatrix}1\\2\\d\end{pmatrix}-\begin{pmatrix}1\\-2\\8\end{pmatrix}=\begin{pmatrix}0\\4\\d-8\end{pmatrix}$

$$\left|\overrightarrow{AD}\right| = \left|\overrightarrow{BD}\right|$$

$$\sqrt{2^2+(d-5)^2} = \sqrt{4^2+(d-8)^2}$$

$$4+(d-5)^2 = 16+(d-8)^2$$

$$4+d^2-10d+25 = 16+d^2-16d+64$$

$$d^2-10d+29 = d^2-16d+80$$

$$6d = 51$$

$$d = \frac{17}{2} = 8.5$$

(c) Use a scalar product to find angle *ADB*. [3]

Solutions

$$\overrightarrow{DB} = \begin{pmatrix}1\\-2\\8\end{pmatrix} - \begin{pmatrix}1\\2\\8.5\end{pmatrix} = \begin{pmatrix}0\\-4\\-0.5\end{pmatrix}$$

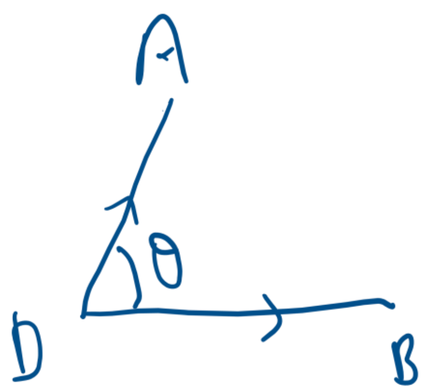
$$\overrightarrow{DA} = \begin{pmatrix}-1\\2\\5\end{pmatrix} - \begin{pmatrix}1\\2\\8.5\end{pmatrix} = \begin{pmatrix}-2\\0\\-3.5\end{pmatrix}$$

$$\cos\theta = \frac{\begin{pmatrix}0\\-4\\-0.5\end{pmatrix}\cdot\begin{pmatrix}-2\\0\\-3.5\end{pmatrix}}{\left|\begin{pmatrix}0\\-4\\-0.5\end{pmatrix}\right|\left|\begin{pmatrix}-2\\0\\-3.5\end{pmatrix}\right|}$$

$$= \frac{1.75}{16.25}$$

$$\theta = 1.5642$$

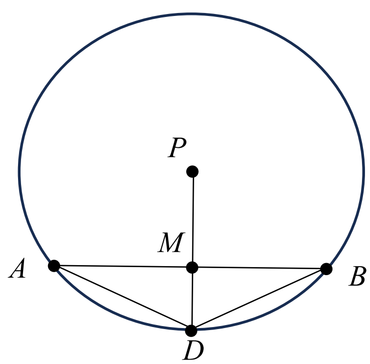
$$= 1.56 \text{ rad or } 83.8^\circ$$



(d) Find exactly the position vector of the point P, where P is the centre of the circle that passes through A, B and D. [5]

Solutions

<u>Method 1</u>



$$\overrightarrow{DA}=\overrightarrow{OA}-\overrightarrow{OD}=\begin{pmatrix}-1\\2\\5\end{pmatrix}-\begin{pmatrix}1\\2\\8.5\end{pmatrix}=\begin{pmatrix}-2\\0\\-3.5\end{pmatrix}$$

$$\overrightarrow{DB}=\overrightarrow{OB}-\overrightarrow{OD}=\begin{pmatrix}1\\-2\\8\end{pmatrix}-\begin{pmatrix}1\\2\\8.5\end{pmatrix}=\begin{pmatrix}0\\-4\\-0.5\end{pmatrix}$$

Let M be the midpoint of A and B,

$$\overrightarrow{DM}=\frac{\overrightarrow{DA}+\overrightarrow{DB}}{2}$$

$$=\frac{\begin{pmatrix}-2\\0\\-3.5\end{pmatrix}+\begin{pmatrix}0\\-4\\-0.5\end{pmatrix}}{2}$$

$$=\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$$

Equation of line DM is $\underset{\sim}{r}=\begin{pmatrix}1\\2\\8.5\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix},\lambda\in\mathbb{R}$

Since P is on line DM, $\overrightarrow{OP}=\begin{pmatrix}1\\2\\8.5\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$, for some $\lambda\in\mathbb{R}$

$$\overrightarrow{AP} = \begin{pmatrix} 1+\lambda \\ 2+2\lambda \\ 8.5+2\lambda \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2+\lambda \\ 2\lambda \\ 3.5+2\lambda \end{pmatrix}$$

$$\overrightarrow{DP} = \begin{pmatrix} 1+\lambda \\ 2+2\lambda \\ 8.5+2\lambda \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} = \begin{pmatrix} \lambda \\ 2\lambda \\ 2\lambda \end{pmatrix}$$

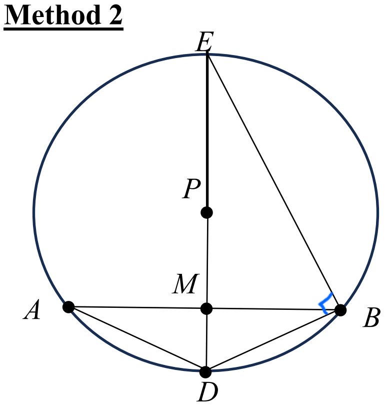
Since DP and AP are radius of the circle,

$$\left|\overrightarrow{AP}\right| = \left|\overrightarrow{DP}\right|$$

$$(2+\lambda)^2 + (2\lambda)^2 + (3.5+2\lambda)^2 = \lambda^2 + (2\lambda)^2 + (2\lambda)^2$$
$$2^2 + 4\lambda + \lambda^2 + 4\lambda^2 + 12.25 + 14\lambda + 4\lambda^2 = \lambda^2 + 8\lambda^2$$
$$18\lambda + 16.25 = 0$$
$$\lambda = -\frac{65}{72}$$

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} - \frac{65}{72}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$
$$= \frac{1}{72}\begin{pmatrix} 7 \\ 14 \\ 482 \end{pmatrix}$$

Method 2



Let M be the midpoint of A and B,

$$\overrightarrow{DM} = \frac{\overrightarrow{DA}+\overrightarrow{DB}}{2}$$

$$= \frac{\begin{pmatrix} -2 \\ 0 \\ -3.5 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \\ -0.5 \end{pmatrix}}{2}$$

$$= \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

Equation of line DM is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

Since P is on line DM, $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, for some $\lambda \in \mathbb{R}$

$$\overrightarrow{DE} = 2\overrightarrow{DP}$$

$$\overrightarrow{OE} - \overrightarrow{OD} = 2\left(\overrightarrow{OP} - \overrightarrow{OD}\right)$$

$$\overrightarrow{OE} = \overrightarrow{OD} + 2\left(\overrightarrow{OP} - \overrightarrow{OD}\right)$$

$$= 2\overrightarrow{OP} - \overrightarrow{OD}$$

$$= 2\left[\begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right] - \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + 2\lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

Note that $\measuredangle DBE = 90^\circ$

$\overrightarrow{DB} \bullet \overrightarrow{BE} = 0$

$$\begin{pmatrix} 0 \\ -4 \\ -0.5 \end{pmatrix} \bullet \left[\begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + 2\lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix} \right] = 0$$

$$\begin{pmatrix} 0 \\ -4 \\ -0.5 \end{pmatrix} \bullet \left[\begin{pmatrix} 0 \\ 4 \\ 0.5 \end{pmatrix} + 2\lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right] = 0$$

$$-16 - \frac{1}{4} - 16\lambda - 2\lambda = 0$$

$$\lambda = -\frac{65}{72}$$

$$\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} - \frac{65}{72} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{72} \begin{pmatrix} 7 \\ 14 \\ 482 \end{pmatrix}$$

Method 3

P is of equal distance from *A*, *B* and *D* and *P* lie in plane *ABD*.

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix}, \ \overrightarrow{AD} = \begin{pmatrix} 2 \\ 0 \\ 3.5 \end{pmatrix}$$

Normal vector of plane *ABD*,

$$\underset{\sim}{n} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ 3.5 \end{pmatrix}$$

$$= \begin{pmatrix} -14 - 0 \\ -(7-6) \\ 0 - (-8) \end{pmatrix}$$

$$= \begin{pmatrix} -14 \\ -1 \\ 8 \end{pmatrix}$$

$$\begin{pmatrix}-14\\-1\\8\end{pmatrix}\times\overrightarrow{AB}=\begin{pmatrix}-14\\-1\\8\end{pmatrix}\times\begin{pmatrix}2\\-4\\3\end{pmatrix}=\begin{pmatrix}29\\58\\58\end{pmatrix}=29\begin{pmatrix}1\\2\\2\end{pmatrix}$$

Let M be the midpoint of A and B,

$$\overrightarrow{OM}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$
$$=\begin{pmatrix}0\\0\\13/2\end{pmatrix}$$

Equation of line MP is

$$\underset{\sim}{r}=\begin{pmatrix}0\\0\\13/2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\2\end{pmatrix},\lambda\in\mathbb{R}\quad\text{---(1)}$$

$$\begin{pmatrix}-14\\-1\\8\end{pmatrix}\times\overrightarrow{AD}=\begin{pmatrix}-14\\-1\\8\end{pmatrix}\times\begin{pmatrix}2\\0\\2.5\end{pmatrix}=\begin{pmatrix}-3.5\\65\\2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-7\\130\\4\end{pmatrix}$$

Let N be the mid-point of AD.

$$\overrightarrow{ON}=\text{Mid-point of }AD=\left(0,2,\frac{27}{4}\right)$$

Equation of Line NP is

$$\underset{\sim}{r}=\begin{pmatrix}0\\2\\\frac{27}{4}\end{pmatrix}+\mu\begin{pmatrix}-7\\130\\4\end{pmatrix},\mu\in\mathbb{R}\quad\text{---(2)}$$

Solving equations (1) and (2):

$$\begin{pmatrix} 0 \\ 0 \\ 13/2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ \frac{27}{4} \end{pmatrix} + \mu \begin{pmatrix} -7 \\ 130 \\ 4 \end{pmatrix}$$

$$\begin{cases} \lambda + 7\mu = 0 \\ 2\lambda - 130\mu = 2 \\ 2\lambda - 4\mu = \frac{1}{4} \end{cases}$$

$$\lambda = \frac{7}{72}, \mu = -\frac{1}{72}$$

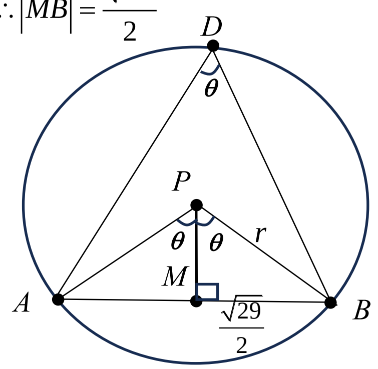
$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 0 \\ 13/2 \end{pmatrix} + \frac{7}{72} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{72} \begin{pmatrix} 7 \\ 14 \\ 482 \end{pmatrix}$$

Method 4

$$\overrightarrow{AB} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} \Rightarrow \left|\overrightarrow{AB}\right| = \sqrt{29}$$

$$\therefore \left|\overrightarrow{MB}\right| = \frac{\sqrt{29}}{2}$$



(Center angle = 2 × inscribed angle)

From (c): $\cos\theta == \frac{1.75}{16.25} \Rightarrow \sin\theta = \frac{\sqrt{261}}{16.25}$

From diagram: $\sin\theta = \dfrac{\frac{\sqrt{29}}{2}}{r}$

$$\therefore \frac{\frac{\sqrt{29}}{2}}{r} = \frac{\sqrt{261}}{16.25}$$

$$r = \frac{65}{24}$$

$$\overrightarrow{OP} = \overrightarrow{OD} + r\frac{\overrightarrow{DM}}{\left|\overrightarrow{DM}\right|}$$

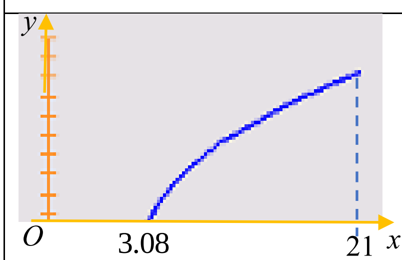
$$= \begin{pmatrix} 1 \\ 2 \\ 8.5 \end{pmatrix} + \frac{65}{24}\left(\frac{1}{3}\right)\begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}$$

$$= \frac{1}{72}\begin{pmatrix} 7 \\ 14 \\ 482 \end{pmatrix}$$

4 The curve C is defined by the parametric equations $x = 2t^2 + 3$ and $y = 5t - 1$ where $t \geq \frac{1}{5}$.

(a) Find the exact area between the curve C, the x-axis and the line $x = 21$. [3]

Solutions



$$\text{Area} = \int_{3.08}^{21} y \, dx$$

When $x = 21, t = 3$

When $x = 3.08, t = \frac{1}{5}$

$\frac{dx}{dt} = 4t$

$$= \int_{1/5}^{3} (5t-1) \left(\frac{dx}{dt}\right) dt$$

$$= \int_{1/5}^{3} (5t-1) (4t) dt$$

$$= 4\int_{1/5}^{3} (5t^2 - t) dt$$

$$= 4\left[\frac{5t^3}{3} - \frac{t^2}{2}\right]_{1/5}^{3}$$

$$= 4\left[\left(\frac{5(27)}{3} - \frac{9}{2}\right) - \left(\frac{5\left(\frac{1}{125}\right)}{3} - \frac{\frac{1}{25}}{2}\right)\right]$$

$$= \frac{12152}{75} \text{ units}^2$$

The curve D is defined by the parametric equations $x = 5u$ and $y = \frac{4}{u}$, where $u \neq 0$.

(b) Find a point of intersection, A, of the curves C and D. Show that there are no other points of intersection. [5]

$x = 2t^2 + 3$, $y = 5t - 1$ --- (1)

$x = 5u$, $y = \frac{4}{u}$ --- (2)

(1) = (2) :

$5u = 2t^2 + 3$---(3)

$\frac{4}{u} = 5t - 1$ --- (4)

Solving (3) and (4):

$$5\left(\frac{4}{5t-1}\right) = 2t^2 + 3$$

$$20 = 10t^3 + 15t - 2t^2 - 3$$

$$10t^3 - 2t^2 + 15t - 23 = 0 \quad \text{--- (*)}$$

Using GC, $t = 1$. $\therefore u = 1$

Point A (5, 4)

From (*):

$$10t^3 - 2t^2 + 15t - 23 = (t-1)(10t^2 + Bt + C)$$

By comparing coefficients of t : $15 = C - B$

By comparing constants: $C = 23$

Hence, $B = 8$

Consider $10t^2 + 8t + 23 = 0$

Discriminant $= 8^2 - 4(10)(23) = -856 < 0$

Since $10t^2 + 8t + 23 = 0$ has no real roots, hence $10t^3 - 2t^2 + 15t - 23 = 0$ has only one real root. Therefore, there is only one point of intersection at Point A.

(c) Find the coordinates of the point where the tangent to the curve C at the point A meets the curve D for a second time. [5]

$\frac{\mathrm{d}x}{\mathrm{d}t} = 4t, \frac{\mathrm{d}y}{\mathrm{d}t} = 5$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}$

$= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$

$= \frac{5}{4t}$

Tangent to C at A:

$y - 4 = \frac{5}{4(1)}(x - 5)$

$y = \frac{5}{4}x - \frac{9}{4}$

Substitute (2) into $y = \frac{5}{4}x - \frac{9}{4}$:

$\frac{4}{u} = \frac{5}{4}(5u) - \frac{9}{4}$

$16 = 25u^2 - 9u$

$u = -0.64$ or $u = 1$ (given)

Hence coordinates of the point is (-3.2, -6.25).

Section B: Probability and Statistics (60 marks)

5 A child's toy has 36 slots, numbered from 1 to 36. A child puts a ball into the toy, the ball falls into one of the 36 slots and the child's score is the number of that slot. The ball is equally likely to fall into any one of the slots.

Sadiq is investigating four possible events, A, B, C and D, which are defined as follows.

A	the score is odd
B	the score is even
C	the score is a multiple of 3
D	the score is a multiple of 6

(a) (i) State which pairs of the events, if any, are mutually exclusive. [1]

Solutions
A and B; A and D

(ii) Show that A and C are independent events, and state another pair of independent events. [2]

Solutions

$\mathrm{P}(A)=\frac{18}{36}=\frac{1}{2}$

$\mathrm{P}(C)=\frac{12}{36}=\frac{1}{3}$

$\mathrm{P}(A\cap C)=\mathrm{P}(\{3,9,15,21,27,33\})=\frac{6}{36}=\frac{1}{6}$

$\mathrm{P}(A)\mathrm{P}(C)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

Since $\mathrm{P}(A)\mathrm{P}(C)=\mathrm{P}(A\cap C)$, A and C are independent events.

$\mathrm{P}(B)=\frac{18}{36}=\frac{1}{2}$

$\mathrm{P}(B\cap C)=\mathrm{P}(\{6,12,18,24,30,36\})=\frac{6}{36}=\frac{1}{6}$

$\mathrm{P}(B)\mathrm{P}(C)=\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$

Since $\mathrm{P}(B)\mathrm{P}(C)=\mathrm{P}(B\cap C)$,

B and C are independent events.

Sadiq notices that a ball has become stuck in the slot labelled 36, and so balls put into the toy are now falling with equal probability, into one of only 35 slots, and the score can only be from 1 to 35.

(b) (i) State which pair of the four events, if any, are now mutually exclusive. [1]

Solutions
A and *B*; *A* and *D*

(ii) Determine whether *A* and *C* are now independent events. [1]

Solutions
$\mathrm{P}(A)=\frac{18}{36}=\frac{1}{2}$ $\mathrm{P}(C)=\frac{11}{36}$ $\mathrm{P}(A\cap C)=\mathrm{P}(\{3,9,15,21,27,33\})=\frac{6}{36}=\frac{1}{6}$ $\mathrm{P}(A)\mathrm{P}(C)=\frac{1}{2}\times\frac{11}{36}=\frac{11}{72}$ $\therefore \mathrm{P}(A)\mathrm{P}(C)\neq\mathrm{P}(A\cap C)$ *A* and *C* are not independent events.

6 A bag contains *r* red counters and *b* blue counters, where $r>12$ and $b>12$. Mei randomly removes 12 counters from the bag. The probability that there are 4 red counters among Mei's 12 counters is the same as the probability that there are 3 red counters.

(a) Show that $9r+5=4b$. [2]

Solutions
$\mathrm{P}(\text{4 of 12 counters are red and others are blue})=\mathrm{P}\begin{pmatrix}\text{3 of 12 counters are red}\\ \text{and others are blue}\end{pmatrix}$

$\left(\frac{r}{r+b}\right)\left(\frac{r-1}{r+b-1}\right)\left(\frac{r-2}{r+b-2}\right)\left(\frac{r-3}{r+b-3}\right)\left(\frac{b}{r+b-4}\right)\cdots\left(\frac{b-7}{r+b-11}\right)\left(\frac{12!}{4!8!}\right)$ $=\left(\frac{r}{r+b}\right)\left(\frac{r-1}{r+b-1}\right)\left(\frac{r-2}{r+b-2}\right)\left(\frac{b}{r+b-3}\right)\left(\frac{b-1}{r+b-4}\right)\cdots\left(\frac{b-7}{r+b-10}\right)\left(\frac{b-8}{r+b-11}\right)\left(\frac{12!}{3!9!}\right)$ $\frac{1}{4}(r-3)=\frac{1}{9}(b-8)$ $9r-27=4b-32$ $4b=9r+5$ (Shown)
Assuming that the counters are distinct, P(4 out of the 12 counters are red and the remaining are blue) = P(3 out of the 12 counters are red and the remaining are blue) $\frac{{}^{r}C_4\times{}^{b}C_8}{{}^{r+b}C_{12}}=\frac{{}^{r}C_3\times{}^{b}C_9}{{}^{r+b}C_{12}}$ $\frac{r!}{4!(r-4)!}\times\frac{b!}{8!(b-8)!}=\frac{r!}{3!(r-3)!}\times\frac{b!}{9!(b-9)!}$ $\quad\quad \frac{3!(r-3)!}{4!(r-4)!}=\frac{8!(b-8)!}{9!(b-9)!}$ $\frac{r-3}{4}=\frac{b-8}{9}$ $9r-27=4b-32$ $9r+5=4b$ (shown)

The probability that there are 3 red counters among Mei's 12 counters is $\frac{5}{3}$ times the probability that there are 2 red counters.

(b) Derive an equation similar to the equation in part (a) and hence find the probability that just one of the 12 counters removed is red. [6]

Solutions
$\text{P}(\text{3 of 12 counters are red and others are blue})=\frac{5}{3}\text{P}\left(\begin{matrix}\text{2 of 12 counters are red}\\ \text{and others are blue}\end{matrix}\right)$

$$\left(\frac{r}{r+b}\right)\left(\frac{r-1}{r+b-1}\right)\left(\frac{r-2}{r+b-2}\right)\left(\frac{b}{r+b-3}\right)\left(\frac{b-1}{r+b-4}\right)\cdots\left(\frac{b-8}{r+b-11}\right)\left(\frac{12!}{3!9!}\right)$$
$$=\frac{5}{3}\left(\frac{r}{r+b}\right)\left(\frac{r-1}{r+b-1}\right)\left(\frac{b}{r+b-2}\right)\left(\frac{b-1}{r+b-3}\right)\left(\frac{b-2}{r+b-4}\right)\cdots\left(\frac{b-8}{r+b-10}\right)\left(\frac{b-9}{r+b-11}\right)\left(\frac{12!}{2!10!}\right)$$
$$\frac{1}{3}(r-2)=\frac{5}{3}(b-9)\left(\frac{1}{10}\right)$$
$$2r-4=(b-9)$$
$$2r=b-5$$

Substitute $b=2r+5$ into the equation in (a):

$$9r+5=4(2r+5)$$

$$r=15$$

Hence, $b=35$.

$$\mathrm{P}(1\text{ of 12 counters are red and others are blue})$$
$$=\left(\frac{15}{50}\right)\left(\frac{35}{49}\right)\left(\frac{34}{48}\right)\cdots\left(\frac{25}{39}\right)\left(\frac{12!}{1!11!}\right)$$
$$=0.0516\text{ (to 3 sig fig)}$$

Assuming that the counters are distinct,

$$\mathrm{P}(3\text{ of 12 counters are red and others are blue})=\frac{5}{3}\mathrm{P}\begin{pmatrix}\text{2 of 12 counters are red}\\\text{and others are blue}\end{pmatrix}$$

$$\frac{{}^{r}C_3\times{}^{b}C_9}{{}^{r+b}C_{12}}=\frac{5}{3}\left(\frac{{}^{r}C_2\times{}^{b}C_{10}}{{}^{r+b}C_{12}}\right)$$

$$\frac{r!}{3!(r-3)!}\times\frac{b!}{9!(b-9)!}=\frac{5}{3}\left(\frac{r!}{2!(r-2)!}\times\frac{b!}{10!(b-10)!}\right) \qquad \frac{2!(r-2)!}{3!(r-3)!}=\frac{5}{3}\left(\frac{9!(b-9)!}{10!(b-10)!}\right)$$

$$\frac{r-2}{3}=\frac{5}{3}\left(\frac{b-9}{10}\right)$$

$$2r-4=b-9$$

$$2r+5=b$$

Sub $b=2r+5$ into the equation in (a):

$$9r+5=4(2r+5)$$

$$r=15$$

Hence, $b = 35$.

Required probability = $\dfrac{{}^{15}C_1 \times {}^{35}C_{11}}{{}^{50}C_{12}} = 0.0516$

7 The numbers of mobile phone subscriptions in Singapore, y million, for certain years from 2004 are given in the following table. The variable x is the number of years after a base year of 2000.

Year	2004	2006	2008	2010	2012	2014	2016	2018
Number of years after base year, x	4	6	8	10	12	14	16	18
Number of subscriptions, y million	3.99	4.79	6.41	7.38	8.07	8.10	8.46	8.39

Ling thinks that the number of mobile phone subscriptions in Singapore can be modelled by one of the formulae

$$y = ax + b, \qquad e^y = cx + d,$$

where a, b, c and d are constants.

(a) Find, correct to 4 decimal places, the value of the product moment correlation coefficient

(i) between x and y, [1]

(ii) between x and e^y. [1]

Solutions
(i) Using GC, product moment correlation coefficient between x and y, r = 0.9281 (ii) Using GC, product moment correlation coefficient between x and e^y, $r = 0.9697$

(b) Explain which of Ling's models, $y = ax + b$ or $e^y = cx + d$, gives a better fit to the data and find the equation of the regression line for this model. [3]

Solutions
Since the value of the product moment correlation coefficient of $e^y = cx + d$ (r = 0.9697) is closer to 1 than that of $y = ax + b$ (r = 0.9281), hence $e^y = cx + d$ gives a better fit to the data. $e^y = 375.615x - 1881.48$ $e^y = 376x - 1880$ (to 3 s.f.)

(c) Use the equation of the regression line to estimate the number of mobile phone subscriptions in 2023. Explain whether your estimate is reliable. [2]

$x = 2024 - 2000 = 24$ $e^y = 375.615(24) - 1881.48$ $y = 8.87$ Since x = 24 is not within the data range $4 \le x \le 18$, the estimate is not reliable.

8 **(a)** A company has a new machine designed to fill bags with, on average, 1 kg of granulated sugar. The production manager wishes to investigate if the machine is adjusted correctly. He intends to take a sample of bags and carry out a hypothesis test.

(i) State null and alternative hypotheses for the manager's test, defining any parameters you use. [2]

Solutions
Let Y be the random variable denoting the mass of a bag of granulated sugar in kg and μ kg be the population mean mass of bag of granulated sugar Null hypothesis, H_0: $\mu = 1$ Alternative hypothesis, H_1: $\mu \neq 1$

The production manager decides to take the first bag of sugar produced each morning and the first bag of sugar produced each afternoon, in a 5-day working week, to form a sample of 10 bags for the test.

(ii) Give two reasons why the production manager's sample is not suitable for z-test. [2]

Solutions

Since the manager does not know about the distribution of the mass of the granulated sugar, he needs to take a larger enough sample (i.e. at least 30) to be able to apply Central Limit Theorem to approximate the sample mean mass of the granulated sugar to follow normal distribution.

Also, the sample taken is not random. This may lead to bias, which invalidate the result of the test when applied to all other 1-kg bag of sugar produced by the machine.

Hence, z-test is not suitable.

(b) The company has a different machine which fills larger bags with, on average 2 kg of granulated sugar. One of the company's sales representatives has reported that some customers suspect the machine is no longer set correctly, and that the average mass of sugar in the bags may in fact be less than 2 kg. The production manager decides to carry out a hypothesis test at the 2.5% level of significance with a suitable sample of 40 bags of sugar. Summary data for the mass, x kg, of sugar in these bags is as follows.

$$n = 40 \qquad \sum x = 78.88 \qquad \sum x^2 = 155.6746$$

(i) State the hypotheses and find the critical region for this test.

Solutions
Let Y be the random variable denoting the mass of a bag of granulated sugar in kg and μ kg be the population mean mass of larger bag of sugar.

Test H_0: $\mu = 2$

Against H_1: $\mu < 2$ at 2.5% level of significance

Unbiased estimate of population mean, $\bar{x} = \frac{\Sigma x}{40} = \frac{78.88}{40} = 1.972$

Unbiased estimate of population variance, s^2

$$= \frac{1}{40-1}\left[\sum x^2 - \frac{\left(\sum x\right)^2}{40}\right]$$

$$= \frac{1}{39}\left[155.6746 - \frac{(78.88)^2}{40}\right] = 0.00316$$

Under H_0,

Since n = 40 is sufficiently large, by Central Limit Theorem,

Test statistic, $Z = \frac{\bar{X} - 2}{\frac{S}{\sqrt{n}}} \sim N(0,1)$ approximately.

To reject H_0:

Critical region: $z \leq -1.95996$

$$\frac{\bar{x} - 2}{\sqrt{\frac{0.00316}{40}}} \leq -1.95996$$

$$\bar{x} \leq 1.98$$

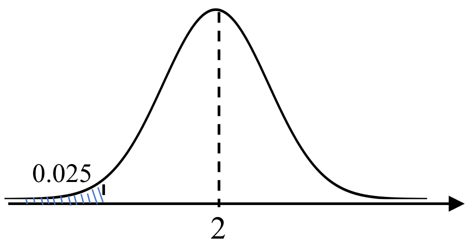
$\therefore \left\{\bar{x} \in \mathbb{R}^+ : \bar{x} \leq 1.98\right\}$ where $z \leq -1.95996$= - 1.96 (to 3 sf)

<u>Alternatively:</u>

Since n = 40 is sufficiently large, by Central Limit Theorem, $\bar{X}$ is approximately normal.

When H_0 is true, $\bar{X} \sim N\left(2, \frac{0.0316}{40}\right)$ approximately.

Critical region:



$\therefore \left\{ \bar{x} \in \mathbb{R}^+ : \bar{x} \le 1.98 \right\}$

(ii) State the conclusion of the test in the context of the question.

Solutions
Since $\bar{x} = 1.972 \le 1.98$, H_0 is rejected at 2.5% level of significance. There is sufficient evidence to conclude that the population mean mass of the larger bag of granulated sugar is less than 2 kg.

9. **In this question, you should state the parameters of any distributions you use.**
A company produces wooden planks of two different lengths, Regular and Long. The lengths, in metres, of the Long planks follow the distribution $N(1.82, 0.2^2)$.

(a) Find the probability that the length of a randomly chosen Long plank is less than 1.79m. [1]

Solutions
Let L be the length of a randomly chosen Long plank in metres. $L \sim N(1.82\ ,\ 0.2^2)$ $P(L<1.79) = 0.440$ (3 s.f)

(b) Find the probability that the total length of 8 randomly chosen Long planks is greater than 14.5m. [3]

Solutions
Let $X = L_1 + L_2 + \cdots + L_8$ $\text{E}(X) = 8\text{E}(L) = 8(1.82) = 14.56$ $\text{Var}(X) = 8\text{Var}(L) = 8(0.2)^2 = 0.32$ $\therefore X \sim \text{N}(14.56, 0.32)$ $\text{P}(X > 14.5) = 0.542$ (3 s.f)

The lengths, in metres, of the Regular planks follow the distribution $\text{N}(1.22, 0.3^2)$.

(c) Sylvio buys 120 of the Regular planks. Calculate the expected number of these planks that are longer than 1.25m. [2]

Solutions
Let R be the length of a randomly chosen Regular plank in metres. $R \sim \text{N}(1.22\ , 0.3^2)$ $\text{P}(R > 1.25) \times 120 = 0.46017 \times 120$ $= 55.22$ 55.2 pieces are expected to be longer than 1.25m.

(d) Find the probability that the total length of 10 randomly chosen Long plank differs by less than 0.65m from the total length of 16 randomly chosen Regular planks. [3]

Solutions
Let $D = (L_1 + L_2 + \cdots + L_{10}) - (R_1 + R_2 + \cdots + R_{16})$ $\text{E}(D) = 10\text{E}(L) - 16\text{E}(R) = 10(1.82) - 16(1.22) = -1.32$ $\text{Var}(D) = 10\text{Var}(L) + 16\text{Var}(R) = 10(0.2^2) + 16(0.3^2) = 1.84$ $D \sim \text{N}(-1.32\ , 1.84)$ $\text{P}(\|D\| < 0.65) = \text{P}(-0.65 < D < 0.65)$ $= 0.237$ (3 s.f)

The company finds that there is a demand for short planks. These planks are produced by cutting the Long planks into three exactly equal lengths, or the Regular planks into two exactly equal lengths.

(e) Find the probability that the length of a randomly chosen Short plank made from a Long plank is greater than the length of one made from a Regular plank. You should ignore any wastage caused by cutting the plank. [4]

Solutions
Let $F = \frac{1}{3}L - \frac{1}{2}R$ $\mathrm{E}(F) = \frac{1}{3}(1.82) - \frac{1}{2}(1.22) = -\frac{1}{300}$ $\mathrm{Var}(F) = \left(\frac{1}{3}\right)^2 (0.2^2) + \left(\frac{1}{2}\right)^2 (0.3^2) = \frac{97}{3600}$ $F \sim N\left(-\frac{1}{300}, \frac{97}{3600}\right)$ $\mathrm{P}(F > 0) = 0.492$ (3 s.f)

(f) Without doing any calculation, explain how your answers to part (e) would change if each cut of a plank caused a small amount of wastage. [1]

Solutions
Solution 1: Assuming the wastage for each cut is w. Since two cuts are needed for the Long plank, the length of each cut piece would be $\frac{1}{3}L - \frac{2}{3}w$. Whereas the length of each cut piece from the Short plank would be $\frac{1}{2}R - \frac{1}{2}w$. Thus, the difference between the two pieces would be $F = \frac{1}{3}L - \frac{1}{2}R - \frac{1}{6}w$. Consequently, the mean will be reduced and the answers to part (e) will reduce slightly.

10 **(a)** A small company makes 50 glass ornaments each working day. Some of the ornaments turn out to be faulty.

(i) State, in the context of the question, two assumptions needed for the number of faulty ornaments made in a day to be well modelled by a binomial distribution. [2]

Solutions
1. Whether an ornament is faulty or not is independent of whether any other ornament is faulty. 2. The probability that an ornament is faulty is constant for the entire production of the day.

Assume now that the number of faulty ornaments produced each day has the distribution $\mathrm{B}(50, 0.04)$.

(ii) Show that the numerical values of the mean and variance of this distribution differ by 0.08. [1]

Solutions

Let X be the number of faulty ornaments produced in a day.

$X \sim \mathrm{B}(50, 0.04)$

$\mathrm{E}(X) = (50)(0.04) = 2$

$\mathrm{Var}(X) = (50)(0.04)(0.96) = 1.92$

$\mathrm{E}(X) - \mathrm{Var}(X) = 2 - 1.92 = 0.08$ (shown)

(iii) Find the probability that no more than 2 faulty ornaments are produced on a randomly chosen working day. [1]

Solutions

$\mathrm{P}(X \le 2) = 0.67671$

≈ 0.677 (3s.f)

(iv) Find the probability that no more than 2 faulty ornaments are produced on at least 3 days in a randomly chosen 5-day working week. State the distribution you use. [3]

Solutions

Let Y be the number of days, out of 5, in which no more than 2 faulty ornaments are produced.

$\mathrm{P}(X \le 2) = 0.67671$

i.e. $Y \sim \mathrm{B}(5, 0.67671)$

$\mathrm{P}(Y \ge 3) = 1 - \mathrm{P}(Y \le 2)$

$= 0.80477$

≈ 0.805 (3s.f)

(v) Find the probability that no more than 10 faulty ornaments are produced in a randomly chosen 5-day working week. State the distribution you use. [2]

Solutions

Let W be the number of faulty ornaments produced in a 5-day working week.

$W \sim \mathrm{B}(250, 0.04)$

$$\mathrm{P}(W \leq 10) = 0.58306$$
$$\approx 0.583 \quad \text{(3s.f)}$$

(b) The company also makes pens which are sold in randomly packed boxes of one hundred pens. The probability of a pen being **not** faulty is p, where $0 < p < 1$.

For quality control purposes, a random sample of pens from each box is tested. Mr Lu and Mrs Ming carry out the tests but they use different methods.

Mr Lu tests a random sample of 6 pens from a box. If there are no faulty pens or only 1 faulty pen the box is accepted.

Mrs Ming tests a random sample of 3 pens from a box.

- If there are no faulty pens in her sample the box is accepted.
- If there are 2 or 3 faulty pens in her sample the box is rejected.
- If there is 1 faulty pen in her sample she takes a second random sample of 3 pens. She accepts the box if there are no faulty pens in this second sample.

Show algebraically that Mrs Ming accepts a greater proportion of boxes than Mr Lu does. [6]

Solutions
Let X be the number of faulty pens out of 6. $X \sim \mathrm{B}(6, 1-p)$ $\mathrm{P}(\text{box accepted by Mr Lu})$ $= \mathrm{P}(X=0) + \mathrm{P}(X=1)$ $= p^6 + 6(1-p)p^5$ $= p^6 + 6p^5 - 6p^6$ $= 6p^5 - 5p^6$ Let Y_1 and Y_2 be the number of faulty pens out of the first and second sample of 3 respectively. $Y_1 \sim \mathrm{B}(3, 1-p)$, $Y_2 \sim \mathrm{B}(3, 1-p)$ $\mathrm{P}(\text{box accepted by Mrs Ming})$ $= \mathrm{P}(Y_1 = 0) + \mathrm{P}(Y_1 = 1)\mathrm{P}(Y_2 = 0)$ $= p^3 + 3(1-p)p^2 . p^3$ $= p^3 + 3p^5 - 3p^6$ To show $p^3 + 3p^5 - 3p^6 > 6p^5 - 5p^6$, consider $\left(p^3 + 3p^5 - 3p^6\right) - \left(6p^5 - 5p^6\right)$.

$\left(p^3+3p^5-3p^6\right)-\left(6p^5-5p^6\right)$

$=p^3-3p^5+2p^6$

$=p^3\left(1-3p^2+2p^3\right)$

$=p^3\left(p-1\right)\left(2p^2-p-1\right)$

$=p^3\left(p-1\right)\left(p-1\right)\left(2p+1\right)$

$=p^3\left(p-1\right)^2\left(2p+1\right)$

Since $p^3>0$, $\left(p-1\right)^2>0$ and

$\left(2p+1\right)>0$ for $0<p<1$,

$\therefore\left(p^3+3p^5-3p^6\right)-\left(6p^5-5p^6\right)>0$

i.e. $p^3+3p^5-3p^6>6p^5-5p^6$ (shown)