RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 15B: Trapezium Rule and Simpson's Rule

SYLLABUS INCLUDES

H2 Further Mathematics:

- Approximation of integral of a function using the trapezium rule and Simpson's rule

PRE-REQUISITES

- Integration techniques
- GC skills in using the summation function

CONTENT

1 Approximate Methods for Evaluating a Definite Integral

The following are some reasons which may lead us to look for an approximation to a definite integral:

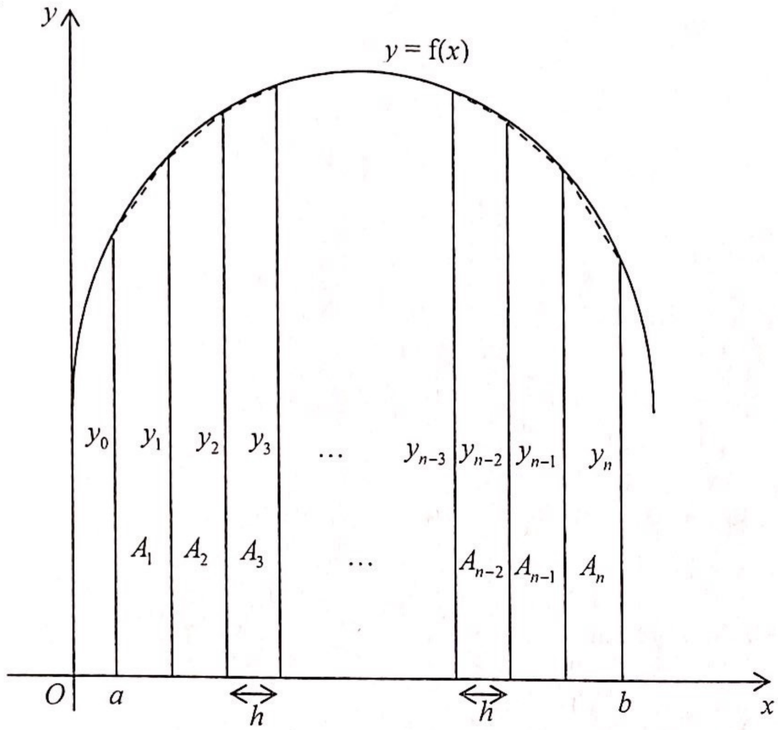
(a) we may not be able to integrate the given expression,
(b) the amount of computation needed to find the exact value of the integral may be excessive,
(c) we may not know the expression to be integrated but have only a set of its values.

As such, we may need to use numerical integration to help us get an approximation of a definite integral. There are two methods which we will study. They are: Trapezium Rule and Simpson's Rule.

1.1 Trapezium Rule

Consider the curve, $y = \mathrm{f}(x)$, where $\mathrm{f}(x) > 0$ in the interval $[a, b]$.

To obtain an approximate value of $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$, i.e. an approximate value of the area bounded by the curve $y = \mathrm{f}(x)$, $x = a$, $x = b$ and the x-axis, we can sum up the areas of the trapeziums as shown below:



So we have

$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ = Area under the curve from $x = a$ to $x = b$

$\approx$ Area of trapezium A_1 + Area of trapezium A_2 + … + Area of trapezium A_n

$= \frac{1}{2}h(y_0+y_1)+\frac{1}{2}h(y_1+y_2)+\frac{1}{2}h(y_2+y_3)+\ldots+\frac{1}{2}h(y_{n-2}+y_{n-1})+\frac{1}{2}h(y_{n-1}+y_n)$

$= \frac{1}{2}h(y_0+2y_1+2y_2+2y_3\ldots+2y_{n-1}+y_n)$

Hence, the **Trapezium Rule** for $(n + 1)$ ordinates $(\text{i.e. } y_0, y_1, y_2, \ldots, y_n)$ is given by:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{1}{2}h\left[y_0+2\left(y_1+y_2+y_3+\ldots+y_{n-2}+y_{n-1}\right)+y_n\right],$$

where $h=\dfrac{b-a}{n}$, $y_0=\mathrm{f}(a)$, $y_1=\mathrm{f}(a+h)$, $y_2=\mathrm{f}(a+2h)$, …, $y_n=\mathrm{f}(b)$.

Note:

(1) The smaller the value of h, the more accurate the approximation will be.

(2) $(n + 1)$ ordinates form n trapeziums.

(3) The trapezium rule may be used to obtain an approximation to the value of any definite integral which might be the value of a solid of revolution, length of an arc, etc.

1.2 Error in Approximation When Using Trapezium Rule

Consider the 2 curves, $y = \text{f}(x)$ and $y = \text{g}(x)$, where $\text{f}(x) > 0$ and $\text{g}(x) > 0$ in the interval $[a, b]$.

(a) When the curve, $y = \text{f}(x)$ is **concave downwards** (i.e. $\text{f}''(x) < 0$), then the approximation to $\int_a^b \text{f}(x)\,\text{d}x$ using the trapezium rule is **smaller** than the actual value $\int_a^b \text{f}(x)\,\text{d}x$, i.e. the estimate is an **under-estimate** (see Figure 1).

(b) When the curve, $y = \text{g}(x)$ is **concave upwards** (i.e. $\text{g}''(x) > 0$), then the approximation to $\int_a^b \text{g}(x)\,\text{d}x$ using the trapezium rule is **larger** than the actual value $\int_a^b \text{g}(x)\,\text{d}x$, i.e. the estimate is an **over-estimate** (see Figure 2).

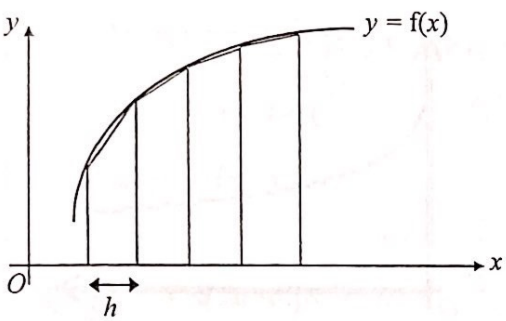


Figure 1

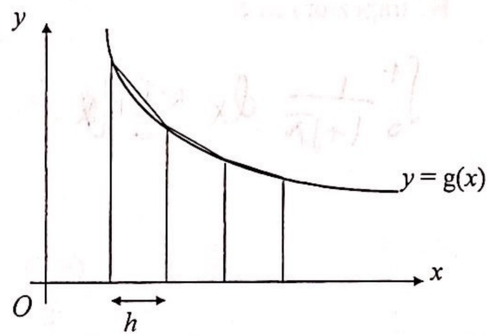


Figure 2

Example 1
Use the trapezium rule to obtain an approximate value (correct to 4 decimal places) to each of the following integrals, stating with a reason in each case, whether the value is more or less than the actual value.

(a) $\int_0^4 \frac{1}{1+\sqrt{x}}\,\text{d}x$ using 5 ordinates.

(b) $\int_0^{0.8} \text{e}^{x^2}\,\text{d}x$ using intervals of 0.2.

(c) $\int_0^{\pi} \sqrt{\sin x}\,\text{d}x$ using 4 equal strips.

Solution:

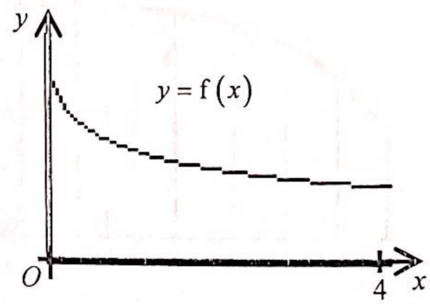
(a) For 5 ordinates, we have

Let $\mathrm{f}(x)=\dfrac{1}{1+\sqrt{x}}$ and so we have $y_0=\mathrm{f}(0)=1$,

$y_1 = f(1) = \frac{1}{2}$

By trapezium rule,

$\int_0^4 \frac{1}{1+\sqrt{x}}\,dx \approx \frac{1}{2}h[$



By looking at the graph of $y=\mathrm{f}(x)$, it can be seen that

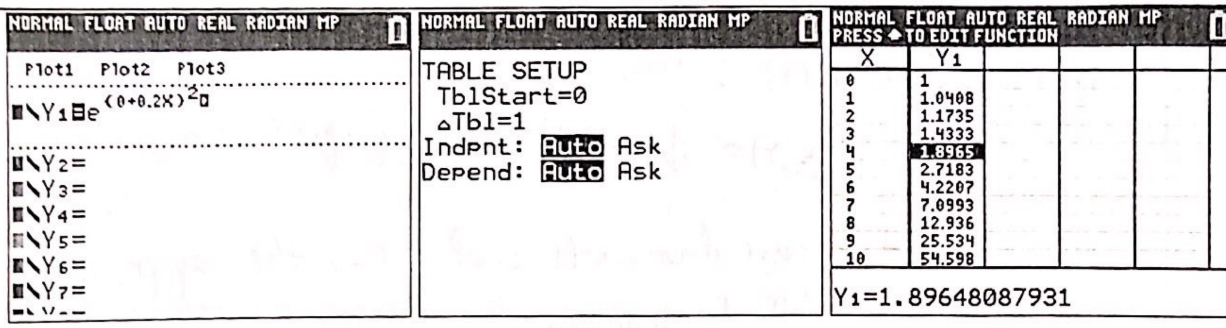
(b) For intervals of 0.2, we have

Let $\mathrm{g}(x)=\mathrm{e}^{x^2}$ and so $y_n=\mathrm{g}(a+nh)$, where $a=0$.

Thus we can have the following table:

n	y_n
$y_1=\mathrm{g}(0.2)=\mathrm{e}^{0.2^2}$,	
$y_2=\mathrm{g}(0.4)=\mathrm{e}^{0.4^2}$,	
$y_3=\mathrm{g}(0.6)=\mathrm{e}^{0.6^2}$,	
$y_4=\mathrm{g}(0.8)=\mathrm{e}^{0.8^2}$.	

Note that GC can be used to create the table of values as shown below:



By trapezium rule,

$$\int_0^{0.5} e^{x^2}\,dx \approx \frac{0.2}{2}\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right]$$

$$= 1.0192$$

Now, since $g(x) = e^{x^2} \Rightarrow g'(x) = 2xe^{x^2}$

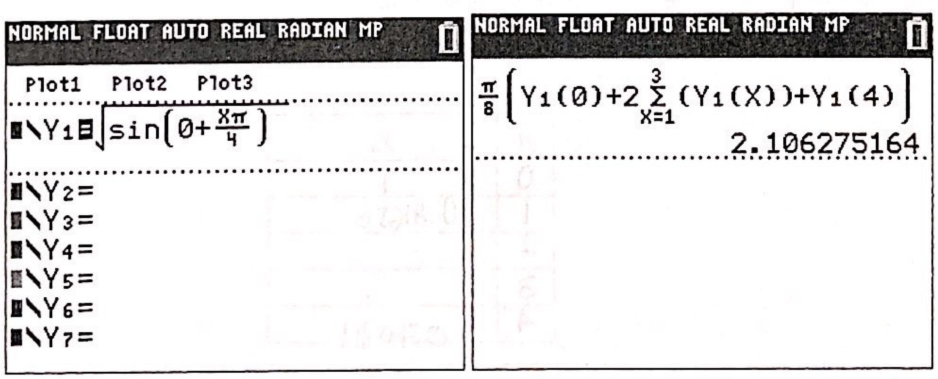
$$g''(x) = (2x)^2 e^{x^2} + (\ldots ?)$$

so $y = g(x)$ is

(c) For 4 equal strips, we have $n = 4$ and $h = \frac{\pi - 0}{4}$

Applying the trapezium rule using GC, $\int_0^{\pi} \sqrt{\sin x}\,dx = 2.1063$

Trapezium rule can be applied using GC in the way demonstrated below:



Let $\mathrm{k}(x)=\sqrt{\sin x}$. Now, since

$\mathrm{k}(x)=\sqrt{\sin x} \Rightarrow \mathrm{k}'(x)=\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\cos x)$

$\Rightarrow \mathrm{k}''(x)=-\frac{1}{4}(\sin x)^{-\frac{3}{2}}(\cos x)^2-\frac{1}{2}(\sin x)^{-\frac{1}{2}}(\sin x)<0$ for all $x \in (0,\pi)$

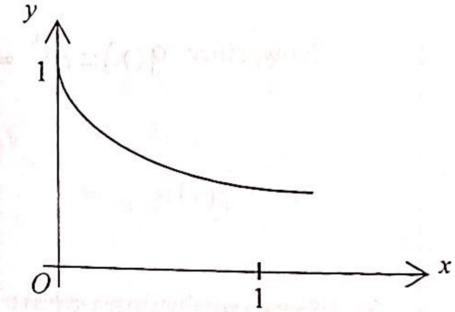
so $y=\mathrm{k}(x)$ is concave downwards and hence, the approximation is less than actual value

Example 2 [9205/1992/01/Q18](modified)

It is given that $\mathrm{f}(x)=\dfrac{1}{\sqrt{1+\sqrt{x}}}$ and the integral $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ is denoted by I.

(i) Using the trapezium rule, with four trapezia of equal width, obtain an approximation I_1 to the value of I, giving three decimal places in your answer.

(ii) A sketch of the graph of $y=\mathrm{f}(x)$ is given in the diagram. Use this diagram to justify the inequality $I<I_1$.



(iii) Evaluate I_2, where $I_2=\dfrac{1}{3}\sum_{r=1}^{3}\mathrm{f}\left(\dfrac{1}{3}r\right)$, giving three decimal places in your answer, and use the diagram to justify the inequality $I>I_2$.

Solution:

(i) For 4 trapezia, we have $n=4$ and $h=\dfrac{1-0}{4}=0.25$.

So we have the following table:

n	y_n
0	1
1	0.81650
2	
3	
4	0.70711

By trapezium rule, using GC, $I \approx I_1 = 0.79187$ (5dp) $= 0.792$ (3dp)

(ii) As can be seen from Figure A,

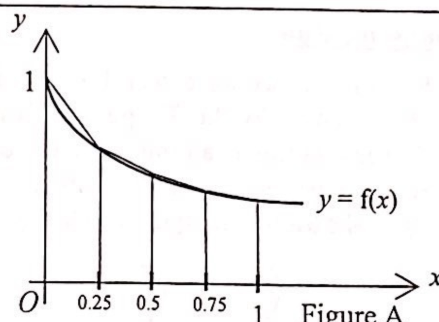


Figure A

$I_1 =$ Sum of the areas of 4 trapezia
> Area bounded by the curve, ... and x axis
≈ 1

(iii) $I_2 = \frac{1}{3}\sum_{r=1}^{3} f\left(\frac{1}{3}r\right)$

$= \frac{1}{3}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1)\right]$

⋮

$= 0.748$

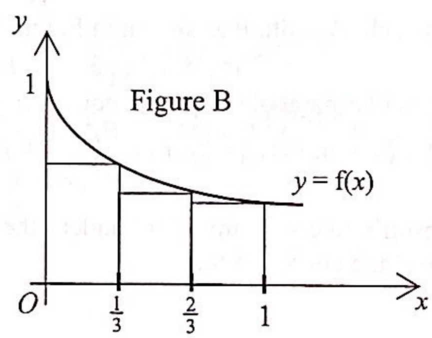
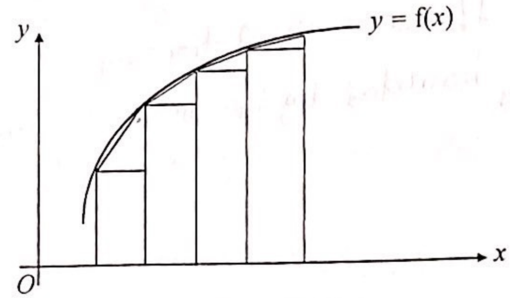


Figure B

As can be seen from Figure B,

1.3 Simpson's Rule

In Section 1.1, we have seen how we can use the areas of trapeziums to approximate the area "under" a curve by the **Trapezium Rule**, which uses **straight lines** to model a curve. We can easily see that this is an improvement over using rectangles to approximate the area "under" the same curve because the trapeziums cover more of the "missing" area that the rectangles cannot cover as shown by the figure below.



However, we seek an even better approximation for the area "under" a curve, using the **Simpson's Rule**, where we will use **parabolas** to approximate each part of the curve. This proves to be very efficient since it is generally more accurate than the other numerical methods.

Suppose that the area represented by $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is divided by the ordinates y_0, y_1 and y_2 into two strips each of width h as shown in Figure 3(a).

A particular parabola, say with equation $y = px^2 + qx + r$, can be found passing through the three points: $(x_1 - h, y_0)$, (x_1, y_1) and $(x_1 + h, y_2)$.

Simpson's rule uses the area "under" the parabola as an approximation for the value of the area "under" the curve $y = \mathrm{f}(x)$.

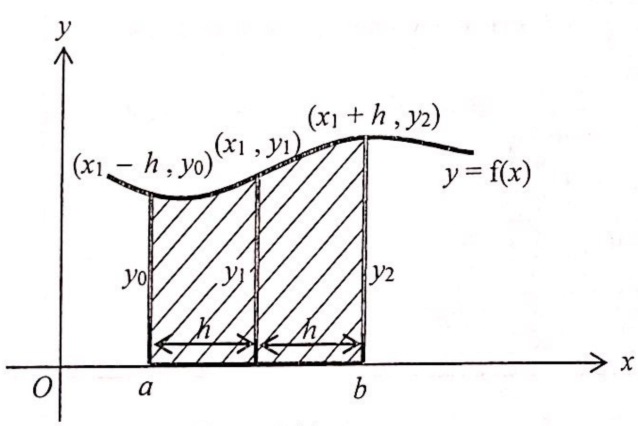


Figure 3(a)

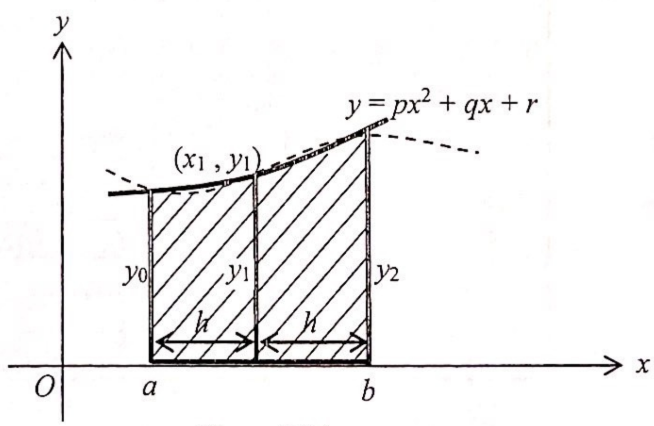


Figure 3(b)

So we have

$$\int_a^b f(x)\,dx \approx \int_{x_1-h}^{x_1+h} (px^2+qx+r)\,dx$$
$$= \left[\frac{px^3}{3}+\frac{qx^2}{2}+rx\right]_{x_1-h}^{x_1+h}$$
$$= \frac{p}{3}\left[(x_1+h)^3-(x_1-h)^3\right]+\frac{q}{2}\left[(x_1+h)^2-(x_1-h)^2\right]+r\left[(x_1+h)-(x_1-h)\right]$$
$$= \frac{p}{3}(6x_1^2h+2h^3)+\frac{q}{2}(4x_1h)+2rh$$
$$= \frac{h}{3}\left[p(6x_1^2+2h^2)+q(6x_1)+6r\right]$$

Since $y = px^2+qx+r$ is the parabola that passes through the coordinates as shown in Figure 3(b), then (x_1-h, y_0), (x_1, y_1) and (x_1+h, y_2) are on this parabola, which gives us

$$y_0 = p(x_1-h)^2+q(x_1-h)+r,$$
$$y_1 = px_1^2+qx_1+r,$$
$$y_2 = p(x_1+h)^2+q(x_1+h)+r.$$

Observe that

$$y_0+4y_1+y_2 = [p(x_1-h)^2+q(x_1-h)+r]+4[px_1^2+qx_1+r]+[p(x_1+h)^2+q(x_1+h)+r]$$
$$= p[(x_1-h)^2+4x_1^2+(x_1+h)^2]+q[(x_1-h)+4x_1+(x_1+h)]+6r$$
$$= p(6x_1^2+2h^2)+q(6x_1)+6r$$

Hence we arrived at

$$\int_{x_1-h}^{x_1+h}(px^2+qx+r)\,\mathrm{d}x = \frac{h}{3}\left[p(6x_1^2+2h^2)+q(6x_1)+6r\right]$$
$$= \frac{h}{3}(y_0+4y_1+y_2) \quad \text{------------- Formula (1)}$$

Thus we have $\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{h}{3}(y_0+4y_1+y_2)$.

However, if we divide the interval $[a, b]$ into four subintervals of equal width h, i.e. there are now 4 strips, and apply Formula (1) to each successive pair of subintervals, we can improve on the approximation of the area "under" the curve $y = \mathrm{f}(x)$ as shown in Figure 4 below.

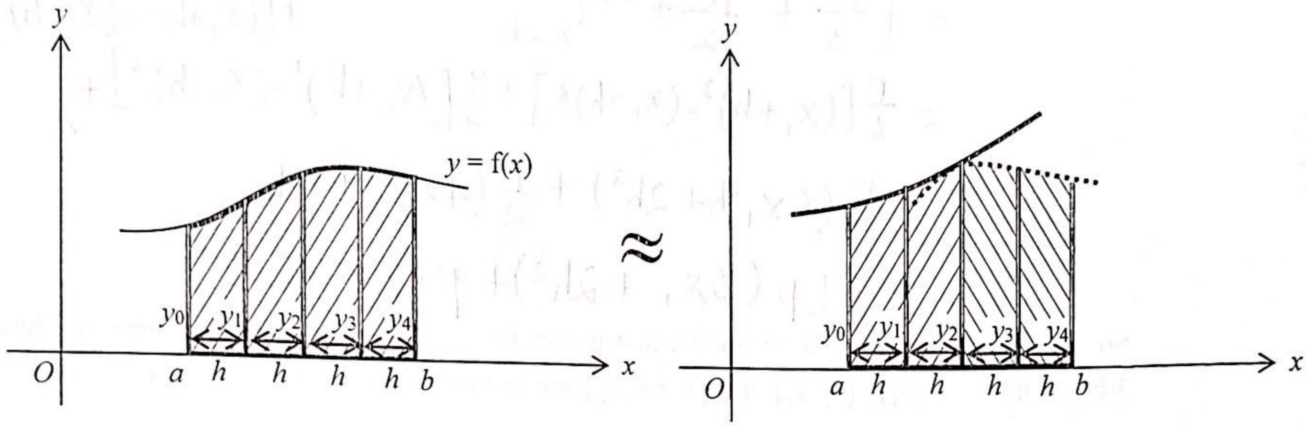


Figure 4

In other words, the area "under" the parabola through y_0, y_1 and y_2 is $\frac{h}{3}(y_0 + 4y_1 + y_2)$ and the area "under" the parabola through y_2, y_3 and y_4 is $\frac{h}{3}(y_2 + 4y_3 + y_4)$, which gives us a total area of $\frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$.

So, with five ordinates, Simpson's Rule is

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4).$$

In general, Simpson's Rule is obtained by dividing the interval $[a, b]$ into an <u>**even**</u> number of subintervals of equal width h and applying Formula (1) to approximate the area "under" the curve $y = \mathrm{f}(x)$ over successive pairs of subintervals. The sum of these approximations will therefore serve as an estimate of $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$.

More formally, let $[a, b]$ be divided into n subintervals of width $h = \dfrac{b-a}{n}$, where n is even. Let $y_0, y_1, \ldots, y_n$ be the values of $y = \mathrm{f}(x)$ at the subinterval endpoints $a = x_0, x_1, \ldots, x_n = b$.

By Formula (1),

the area "under" the curve $y = \mathrm{f}(x)$ over the first two subintervals is approximately

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right),$$

and the area over the second pair of subintervals is approximately

$$\frac{h}{3}\left(y_2 + 4y_3 + y_4\right),$$

and the area over the last pair of subintervals is approximately

$$\frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right).$$

Adding all the approximations and simplifying, we have the **Simpson's Rule** for $(n + 1)$ ordinates (n even):

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right),$$

where $h = \dfrac{b-a}{n}$, $y_0 = \mathrm{f}(a)$, $y_1 = \mathrm{f}(a+h)$, $y_2 = \mathrm{f}(a+2h)$, …, $y_n = \mathrm{f}(b)$.

Note:

(1) The use of Simpson's Rule requires an odd number of ordinates.

(2) For ease of computation, we can re-write Simpson's Rule by grouping it as follows:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{h}{3}\left[y_0 + 4\left(y_1 + y_3 + \ldots + y_{n-1}\right) + 2\left(y_2 + y_4 + \ldots + 2y_{n-2}\right) + y_n\right],$$

which gives us an easy way to remember the Simpson's Rule:

$$\int_a^b \mathrm{f}(x)\,\mathrm{d}x \approx \frac{h}{3}\left[(\text{first} + \text{last}) + 4(\text{sum of odds}) + 2(\text{sum of evens})\right].$$

Example 3

Use the Simpson's Rule with 5 ordinates to find an approximate value (correct to 4 decimal places) of $\int_0^{\pi} \sqrt{\sin x}\, dx$.

Solution:

With 5 ordinates, we have $n = 4$ and $h = \frac{\ }{4}$

Let $f(x) = \sqrt{\sin x}$ and so we have

$y_0 = f(0) = 0$

$y_1 = f\left(\frac{\pi}{4}\right) = 2^{-\frac{1}{4}}$

$y_2 = f\left(\frac{\pi}{2}\right) = 1$

$y_3 = f\left(\frac{3\pi}{4}\right) = 2^{-\frac{1}{4}}$

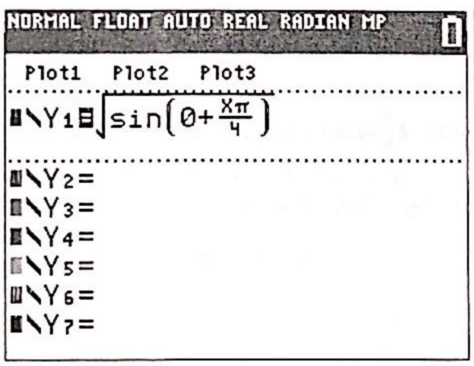
$y_4 = f(\pi) = 0$

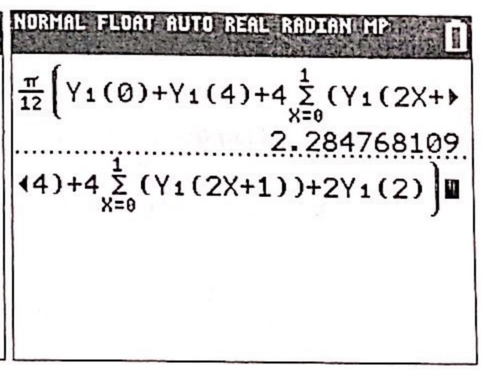
Using Simpson's Rule,

$$\int_0^{\pi} \sqrt{\sin x}\, dx \approx \frac{1}{3} h\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right]$$

$$= \frac{1}{3}\left(\frac{\pi}{4}\right)\left[0 + 4\left(2^{-\frac{1}{4}} + 2^{-\frac{1}{4}}\right) + 2\right]$$

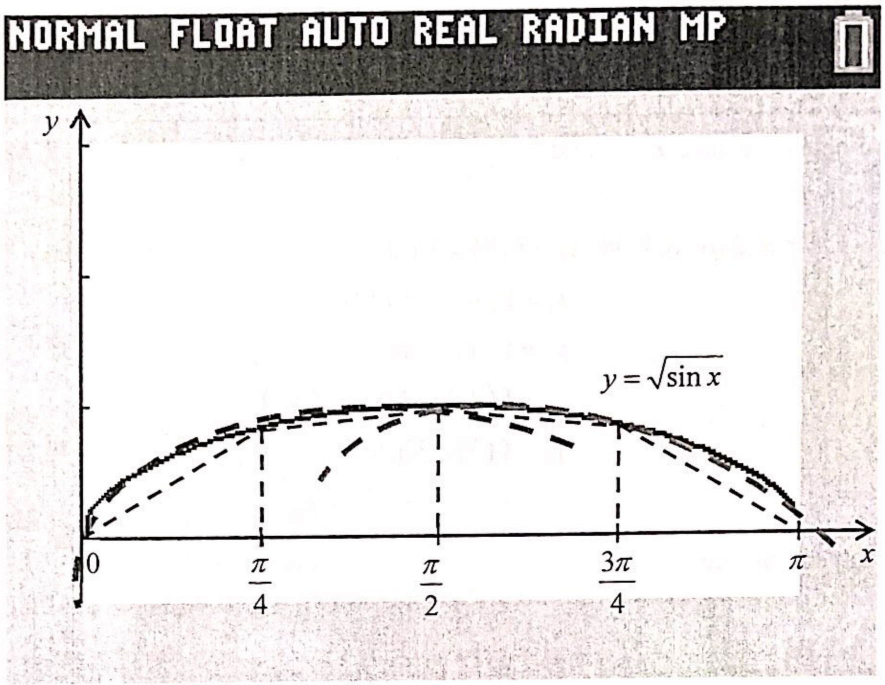
Note that we can use the GC to evaluate the above as shown below:





Note:

In Example 2(c), using the trapezium rule, $\int_0^{\pi} \sqrt{\sin x}\ \mathrm{d}x \approx 2.1063$ (4 dp), which was an underestimate of the actual value of $\int_0^{\pi} \sqrt{\sin x}\ \mathrm{d}x$. The value obtained above using the Simpson's Rule should be very accurate since the parabolas that the Simpson's Rule used fitted the curve of $y = \sqrt{\sin x}$ very well in the interval $[0, \pi]$ as shown in the diagram below.



Example 4 (Self Read)

Estimate, to 4 decimal places, $\int_3^5 x\ln x \, dx$, using 5 ordinates and applying

(a) the trapezium rule,
(b) the Simpson's Rule.

Using integration by parts, evaluate $\int_3^5 x\ln x \, dx$ and comment on the accuracy of your estimated values in **(a)** and **(b)**.

Solution:

With 5 ordinates, we have $n = 4$ and $h = \frac{5-3}{4} = 0.5$.

Let $f(x) = x\ln x$ and so we have

$$\begin{aligned} y_0 &= f(3) = 3\ln 3, \\ y_1 &= f(3.5) = (3.5)\ln(3.5), \\ y_2 &= f(4) = 4\ln 4, \\ y_3 &= f(4.5) = (4.5)\ln(4.5), \\ y_4 &= f(5) = 5\ln 5. \end{aligned}$$

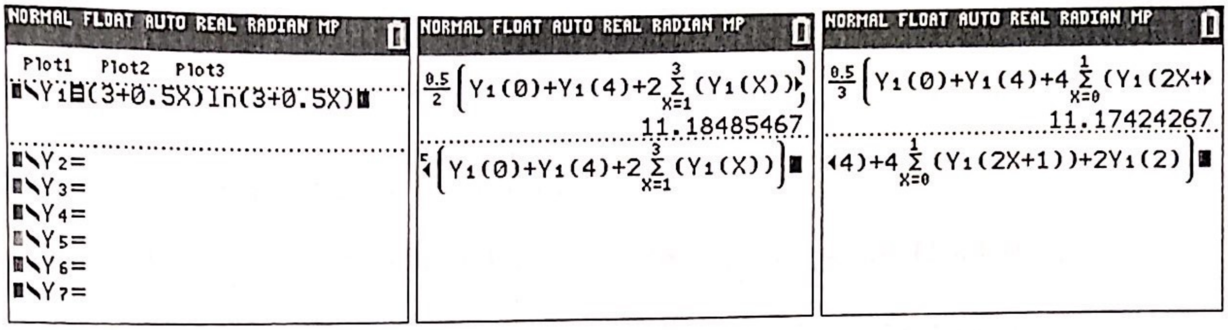
(a) By trapezium rule,

$$\begin{aligned} \int_3^5 x\ln x \, dx &\approx \frac{1}{2}h\left[y_0 + 2(y_1 + y_2 + y_3) + y_4\right] \\ &= \frac{0.5}{2}\left\{3\ln 3 + 2\left[(3.5)\ln(3.5) + 4\ln 4 + (4.5)\ln(4.5)\right] + 5\ln 5\right\} \\ &= 11.184855 \text{ (6 dp)} \\ &= 11.1849 \text{ (4 dp)}. \end{aligned}$$

(b) Using Simpson's Rule,

$$\begin{aligned} \int_3^5 x\ln x \, dx &\approx \frac{1}{3}h\left[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2\right] \\ &= \frac{0.5}{3}\left\{(3\ln 3 + 5\ln 5) + 4\left[(3.5)\ln(3.5) + (4.5)\ln(4.5)\right] + 2(4\ln 4)\right\} \\ &= 11.174243 \text{ (6 dp)} \\ &= 11.1742 \text{ (4 dp)}. \end{aligned}$$

Both **(a)** and **(b)** can be evaluated using the GC as shown below:



Using integration by parts,

$$\int_3^5 x\ln x \, dx = \left[\frac{x^2}{2}\ln x\right]_3^5 - \int_3^5\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right) dx$$

$$= \left(\frac{25}{2}\ln 5 - \frac{9}{2}\ln 3\right) - \int_3^5 \frac{x}{2}\, dx$$

$$= \left(\frac{25}{2}\ln 5 - \frac{9}{2}\ln 3\right) - \left[\frac{x^2}{4}\right]_3^5$$

$$= \left(\frac{25}{2}\ln 5 - \frac{9}{2}\ln 3\right) - \left(\frac{25}{4} - \frac{9}{4}\right)$$

$= 11.174219$ (6 dp)

$= 11.1742$ (4 dp).

The trapezium rule gave an estimate correct to 1 decimal place while the Simpson's Rule gave an estimate correct to 4 decimal places when both values are compared with the value obtained by integration by parts.

Moreover, with the percentage errors shown below, it can be concluded that the Simpson's Rule is more accurate.

Methods	Integration by parts	Trapezium Rule	Simpson's Rule
Value	11.174219 (6 dp)	11.184855 (6 dp)	11.174243 (6 dp)
Absolute Error	0	0.10636	0.000024
Percentage Error	0 %	0.952 % (3 sf)	0.000215 % (3 sf)

SUMMARY