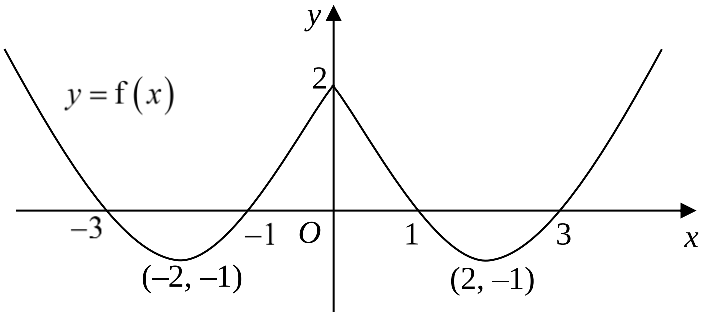
H2 Mathematics (9758)
Chapter 2 Transformations of Curves
Discussion Questions (Suggested Solutions)

Level 1

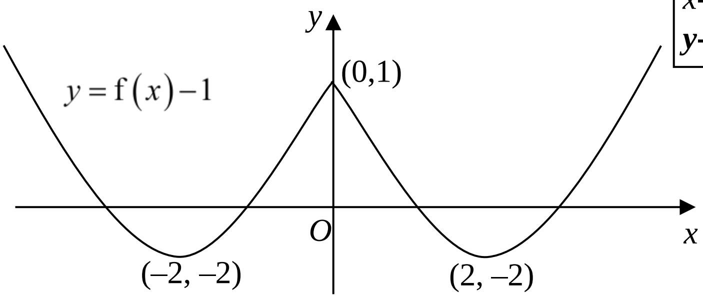
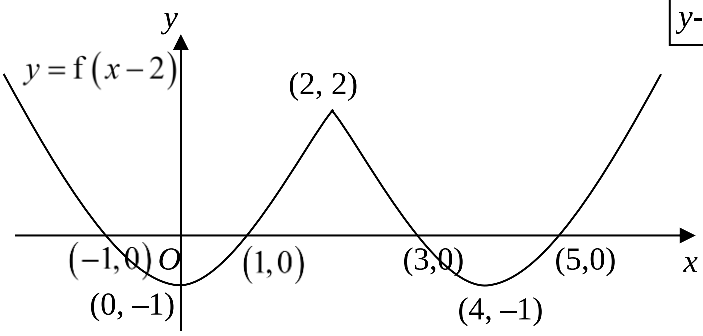
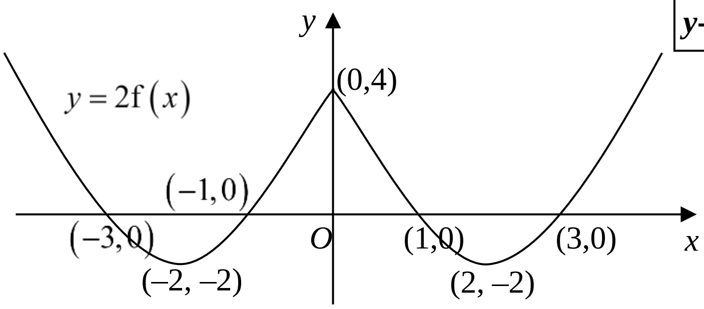
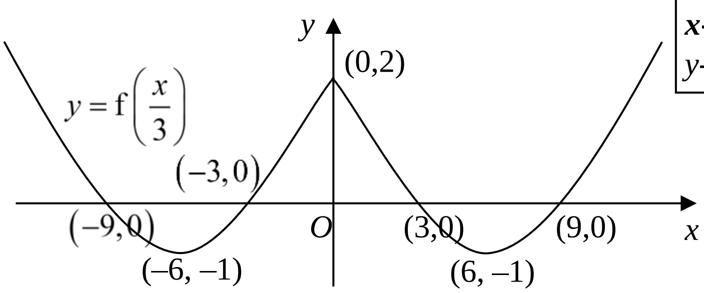
1 Describe a single transformation that will change $y=\mathrm{f}(x)$ into each of the following functions in part (a) – (f). The diagram shows the graph of $y=\mathrm{f}(x)$.

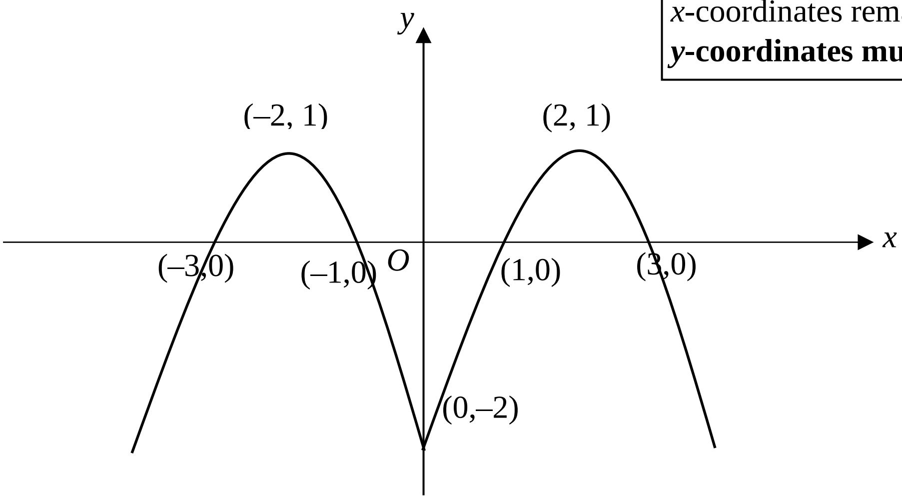
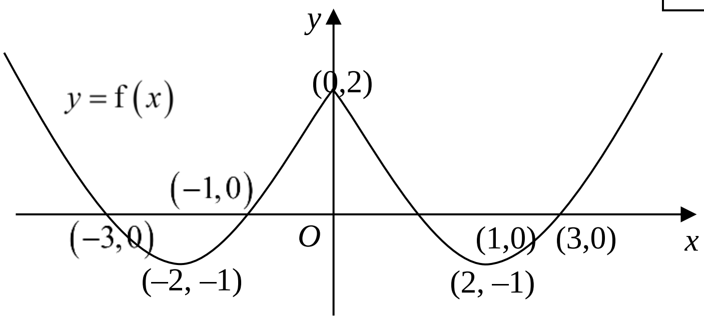
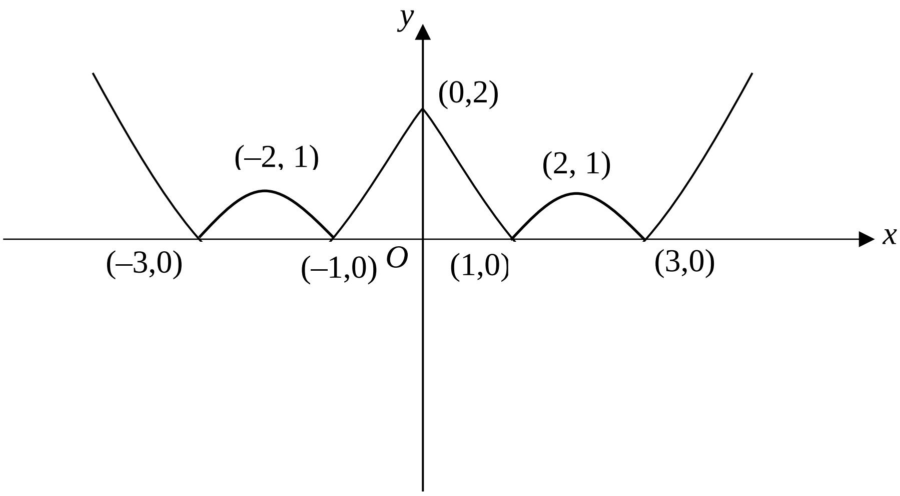


On separate diagrams, sketch the graphs of the following, showing clearly the coordinates of intersection with the axes.

(a) $y=\mathrm{f}(x)-1$ **(b)** $y=\mathrm{f}(x-2)$ **(c)** $y=2\mathrm{f}(x)$ **(d)** $y=\mathrm{f}\left(\dfrac{x}{3}\right)$

(e) $y=-\mathrm{f}(x)$ **(f)** $y=\mathrm{f}(-x)$ **(g)** $y=\left|\mathrm{f}(x)\right|$

1	**Solution**
(a)	$y=\mathrm{f}\left(x\right)-1$ Replace y by $y+1$ x-coordinates remain unchanged, **y-coordinates decreases by 1**  **Transformation:** Translation of 1 unit in the negative y-direction.
(b)	$y=\mathrm{f}\left(x-2\right)$ Replace x by $x-2$ **x-coordinates increases by 2**, y-coordinates remain unchanged  **Transformation:** Translation of 2 units in the positive x-direction.
(c)	$y=2\,\mathrm{f}\left(x\right)$ Replace y by $\dfrac{y}{2}$ x-coordinates remain unchanged, **y-coordinates multiplied by 2**  **Transformation:** Stretch of factor 2 parallel to the y-axis.
(d)	$y=\mathrm{f}\left(\dfrac{x}{3}\right)$ Replace x by $\dfrac{x}{3}$ **x-coordinates multiplied by 3**, y-coordinates remain unchanged  **Transformation**: Stretch of factor 3 parallel to the x-axis.

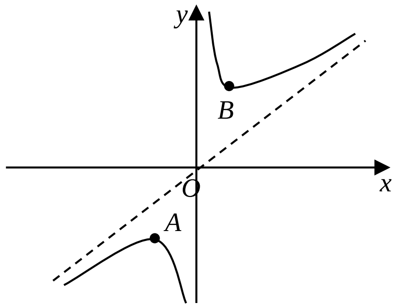
(e)	$y=-\mathrm{f}\left(x\right)$ Replace y by $-y$ x-coordinates remain unchanged, **y-coordinates multiplied by** -1  **Transformation**: Reflection in the x-axis.
(f)	$y=\mathrm{f}\left(-x\right)$ Replace x by $-x$ **x-coordinates multiplied by** -1, y-coordinates remain unchanged  **Transformation**: Reflection in the y-axis.
(f)	$y=\left\|\mathrm{f}\left(x\right)\right\|$ 

Level 2

2 The diagram shows the graph of $y = \mathrm{f}\left(x\right)$ with asymptotes $x = 0$ and $y = x$. The points *A*, *B*, *O* have coordinates $\left(-1,-2\right)$, $\left(1,2\right)$, $\left(0,0\right)$ respectively. Sketch on separate diagrams the graph of

(i) $y = 3\mathrm{f}\left(\frac{1}{2}x\right)$, **(ii)** $y = 2\mathrm{f}\left(x+4\right)$,

(iii) $y = \mathrm{f}\left(2x+1\right)$,



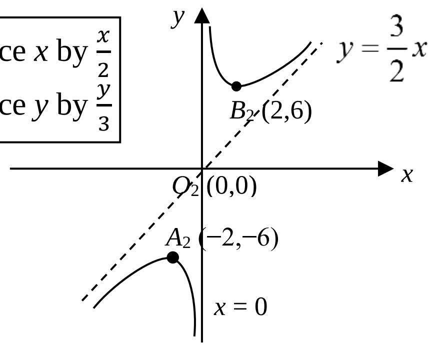
showing in each case the coordinates of the points corresponding to *A*, *B* and *O* and the equations of the asymptotes where applicable.

2 Suggested Solutions

(i) $y = 3\mathrm{f}\left(\frac{x}{2}\right)$

Replace *x* by $\frac{x}{2}$

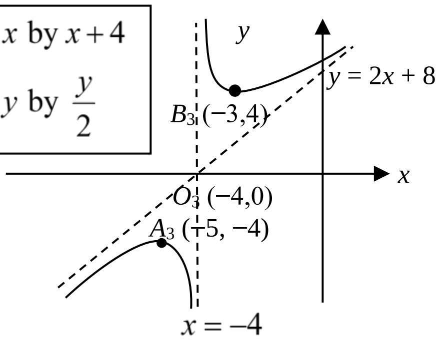
Replace *y* by $\frac{y}{3}$



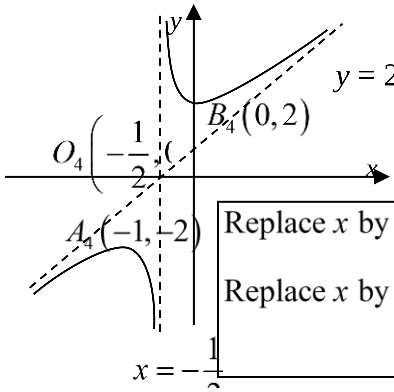
(ii) $y = 2\mathrm{f}\left(x+4\right)$

Replace x by $x+4$

Replace y by $\frac{y}{2}$



(iii) $y = \mathrm{f}\left(2x+1\right)$



Replace x by $x+1$

Replace x by $\frac{x}{\frac{1}{2}}$

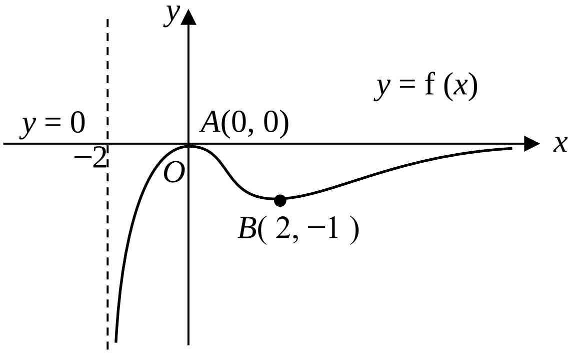
Note: For (ii), *y*-intercept of asymptote *y* = 2*x* +8 is (0, 8) and should be higher than turning point (3, −4) in your sketch

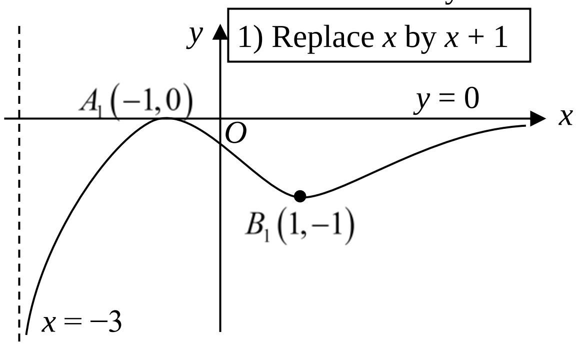
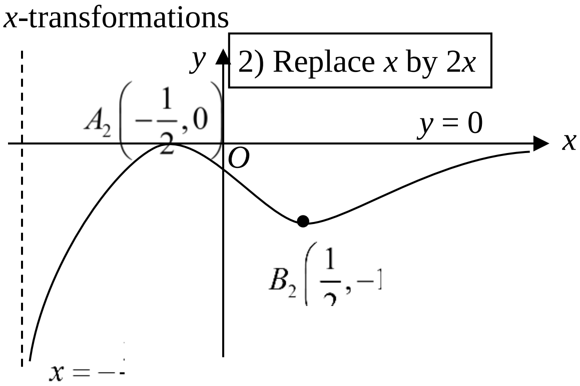
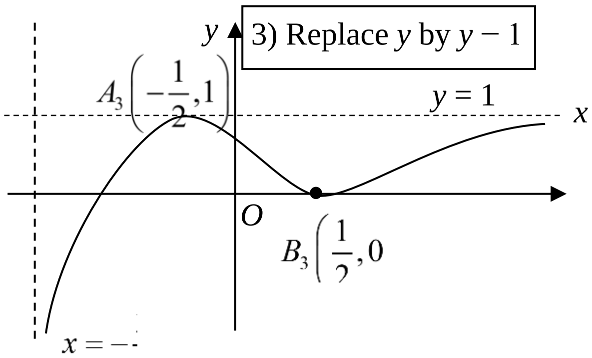
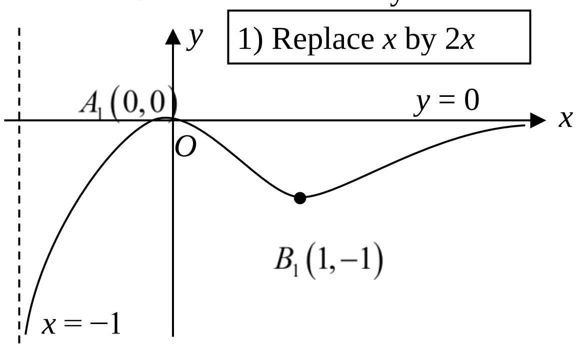
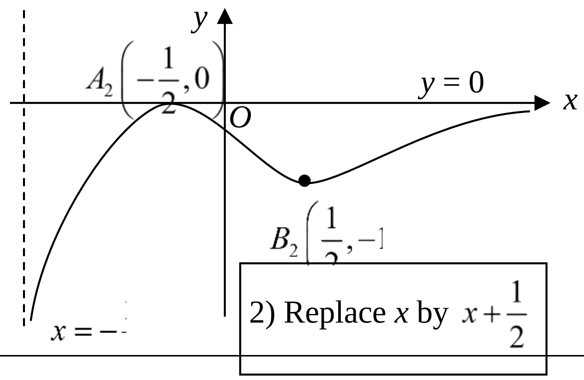
3 The diagram shows the graph of $y = \mathrm{f}(x)$. On separate diagrams, sketch the graphs of

(i) $y = \mathrm{f}(2x+1)+1$,

(ii) $\dfrac{y}{3} = -\mathrm{f}\left(|x|\right)$,

labelling each graph clearly, showing the asymptotes (if any) and the coordinates of the points corresponding to *A* and *B*.



3	Suggested Solutions
(i)	**Method 1:** Translation followed by stretch for *x*-transformations   
	Method 2: Stretch followed by translation for *x*-transformations  

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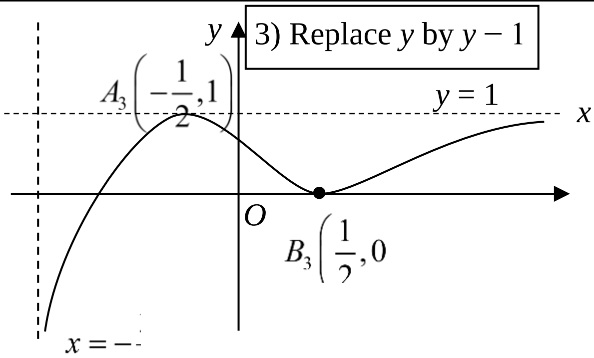

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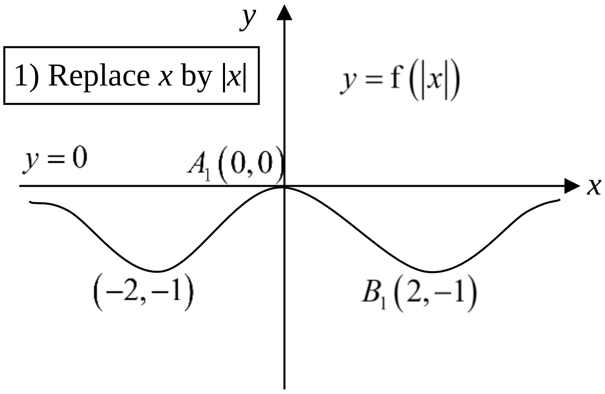
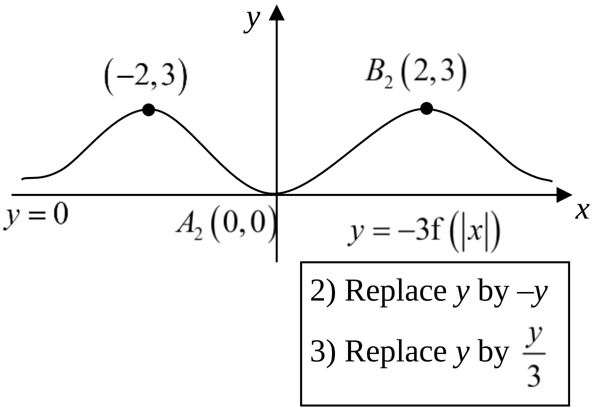
$\frac{y}{3}=-\mathrm{f}\left(|x|\right) \Rightarrow -\frac{y}{3}=\mathrm{f}\left(|x|\right)$





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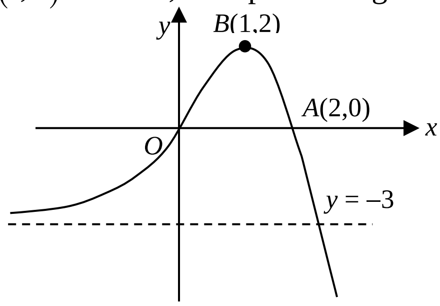
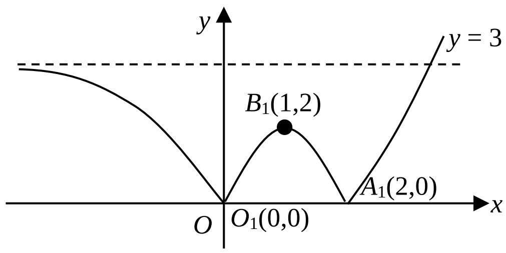
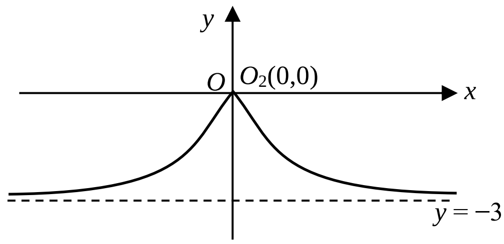
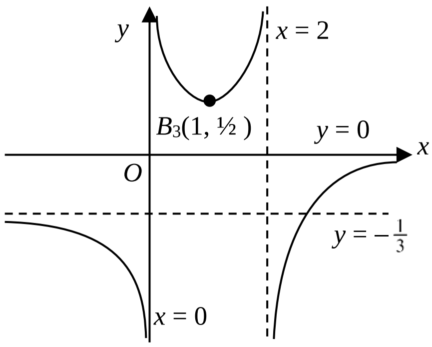
4 The diagram shows the graph of $y = \mathrm{f}\left(x\right)$. The curve crosses the x-axis at the origin O and the point $A\left(2,0\right)$, and has a maximum point at $B\left(1,2\right)$. Sketch, on separate diagrams, the graphs of



(i) $y = \left|\mathrm{f}\left(x\right)\right|$, **(ii)** $y = \mathrm{f}\left(-\left|x\right|\right)$,

(iii) $y = \dfrac{1}{\mathrm{f}\left(x\right)}$,

indicating in each case the coordinates of the axial intercepts and turning points, and the equation of the asymptotes where applicable.

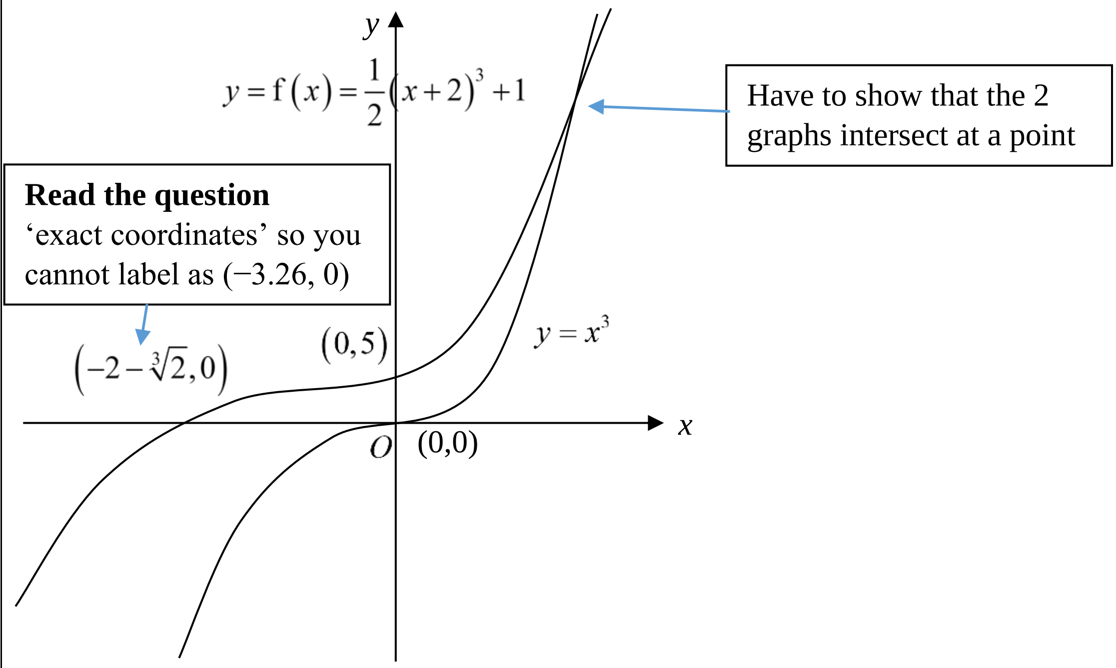
4	**Suggested Solutions**
(i) $y = \left\|\mathrm{f}\left(x\right)\right\|$ 	**(ii)** $y = \mathrm{f}\left(-\left\|x\right\|\right)$ 
(iii) $y = \dfrac{1}{\mathrm{f}\left(x\right)}$ 	**Note for (ii)** $y = \mathrm{f}\left(-\left\|x\right\|\right)$, we have to reflect in the y-axis first before we apply the transformation for |x|: $y = \mathrm{f}\left(x\right) \ \rightarrow \ y = \mathrm{f}\left(-x\right) \ \rightarrow$ **Incorrect order** if you apply the transformation for |x| followed by a reflection in the y-axis: $y = \mathrm{f}\left(x\right) \ \rightarrow \ y = \mathrm{f}\left(\left\|x\right\|\right) \rightarrow$ $y = \mathrm{f}\left(\left\|-x\right\|\right) = \mathrm{f}\left(\left\|x\right\|\right)$

5 2016/Specimen Paper/I/2

The curve C with equation $y = x^3$ is transformed onto the curve with equation $y = \mathrm{f}(x)$ by a translation of 2 units in the negative x-direction, followed by a stretch of factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 1 unit in the positive y-direction.

(i) Write down the equation of the new curve. [1]

(ii) Sketch C and the curve with equation $y = \mathrm{f}(x)$ on the same diagram, stating the exact values of the coordinates of the points where $y = \mathrm{f}(x)$ crosses the x- and y-axes. Find the x-coordinate(s) of the point(s) where the two curves intersect, giving your answer(s) correct to 3 decimal places. [4]

5	**Suggested Solutions**
(i)	$y = x^3$ $\downarrow$ Replace x by $x+2$ $y = (x+2)^3$ $\downarrow$ Replace y by $\dfrac{y}{\left(\frac{1}{2}\right)} = 2y$ $2y = (x+2)^3 \Rightarrow y = \frac{1}{2}(x+2)^3$ $\downarrow$ Replace y by $y-1$ $y-1 = \frac{1}{2}(x+2)^3 \Rightarrow y = \frac{1}{2}(x+2)^3 + 1$ $\therefore$ The equation of the curve is $y = \frac{1}{2}(x+2)^3 + 1$
(ii)	 x-coordinate = 7.722 (3 d.p.)

6 A curve $y=\mathrm{f}\left(x\right)$ undergoes, in succession, the following transformations.

A: A translation of magnitude 2 units in the positive *x*-direction.

B: A stretching parallel to the *x*-axis by a factor $\dfrac{1}{2}$.

C: A reflection in the *y*-axis.

D: A translation of magnitude 1 unit in the positive *y*-direction.

The resulting curve has equation $y=1+\mathrm{e}^{2(x+1)}$. Determine the equation of the curve before the four transformations were effected.

6	**Suggested Solutions**

A: Replace *x* by $x-2$

B: Replace *x* by $\dfrac{x}{\left(\frac{1}{2}\right)}$

C: Replace *x* by $-x$

D: Replace *y* by $y-1$

$y=1+\mathrm{e}^{2(x+1)}$

$\downarrow$ D': Replace *y* by *y* + 1

$y+1=1+\mathrm{e}^{2(x+1)} \Rightarrow y=\mathrm{e}^{2(x+1)}$

$\downarrow$ C': Replace *x* by $-x$

$y=\mathrm{e}^{2(-x+1)}$

$\downarrow$ B': Replace *x* by $\dfrac{x}{2}$

$y=\mathrm{e}^{2\left(-\frac{x}{2}+1\right)} \Rightarrow y=\mathrm{e}^{-x+2}$

$\downarrow$ A': Replace *x* by $x+2$

$y=\mathrm{e}^{-x}$

$\therefore$ The equation of the curve is $y=\mathrm{e}^{-x}$.

Learning points

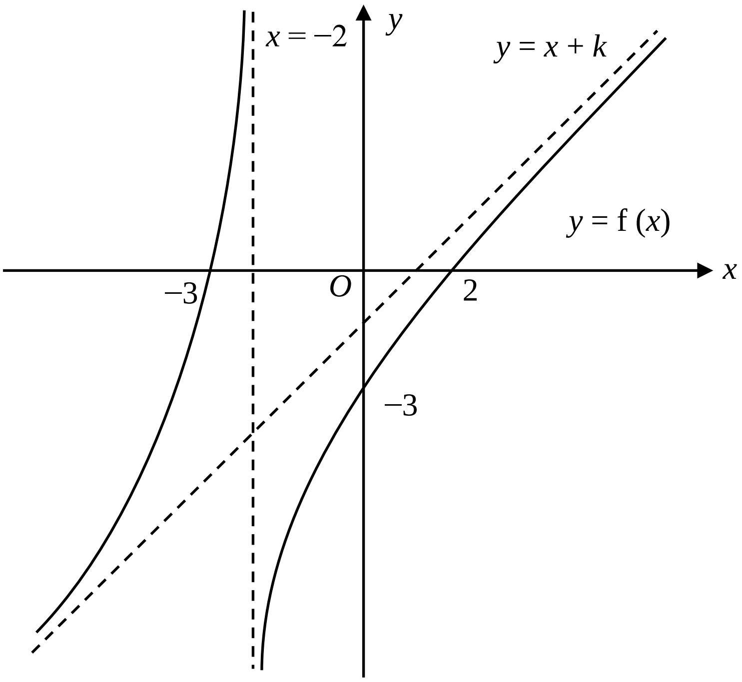
1. Question asks to **determine** the equation of the curve **before** the transformations, thus **start from the last** transformation and **undo each in order (i.e. *D′*→*C ′*→*B′*→*A′*)**

2. If *D*: Replace *y* by $y-1$
   To get D': Replace *y* by *y* + 1

3. **Check your final answer** by **working forward** using the transformations given to see if you are able to get the resulting curve

7 2013/SRJC Prelim/I/3 (Modified)

The graph of the function $y = \mathrm{f}(x)$ where $\mathrm{f}(x) = \frac{(x-a)(x+b)}{cx+d}$, $x \neq -\frac{d}{c}$, $a, b, c, d \in \mathbb{R}^+$, is shown below. The asymptotes are $y = x + k$ and $x = -2$ where k is a constant. The curve cuts the x-axis at -3 and 2 and the y-axis at -3.



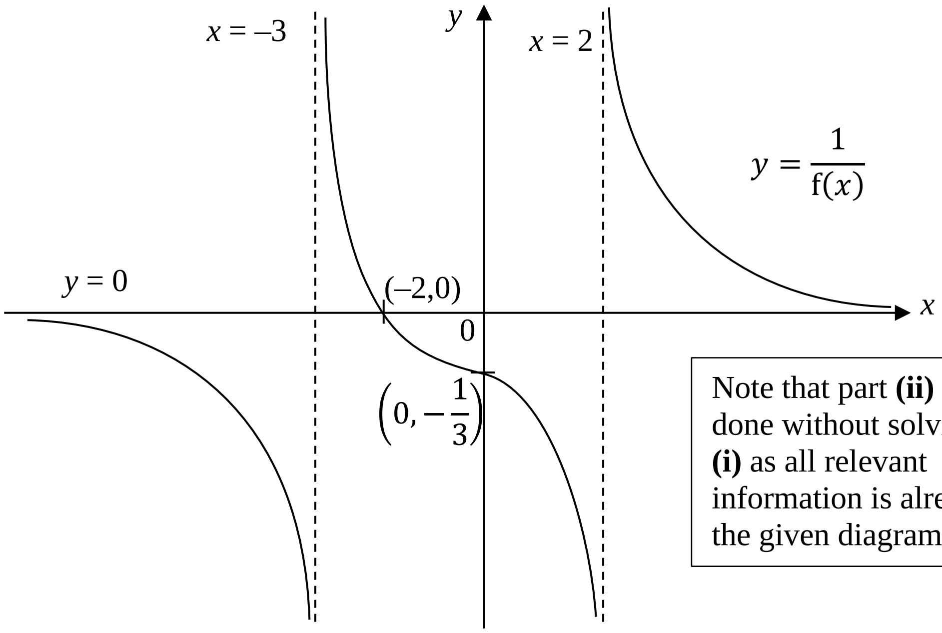
(i) Find the values of a, b, c, d and k. [5]

(ii) Sketch on a separate diagram, the graph of $y = \frac{1}{\mathrm{f}(x)}$ [3]

Your sketch should clearly show any axial intercepts and equations of asymptotes.

7	**Suggested Solutions**
(i)	As the x intercepts are $x = -3$ and $x = 2$, therefore $a = 2$, $b = 3$. Since oblique asymptote is $y = x + k$, by comparing coefficients of largest powers of x in numerator and denominator, $c = 1$. Since vertical asymptote is $x = -2$, $d = 2$. $y = \mathrm{f}(x) = \frac{(x-2)(x+3)}{x+2} = \frac{x^2+x-6}{x+2}$ **<u>Method 1 (Long Division)</u>** $\mathrm{f}(x) = \frac{x^2+x-6}{x+2} = x - 1 + \frac{-4}{(x+2)}$ Therefore, $k = -1$. **<u>Method 2 (Combine into single fraction)</u>** $\mathrm{f}(x) = x + k + \frac{A}{x+2} = \frac{(x+k)(x+2)+A}{x+2} = \frac{x^2+(k+2)x+2k+A}{x+2}$ By comparing coefficient of x in the numerator, $k = -1$.

(ii)



Note that part **(ii)** can be done without solving part **(i)** as all relevant information is already in the given diagram.

8 **2015(9740)/I/5**

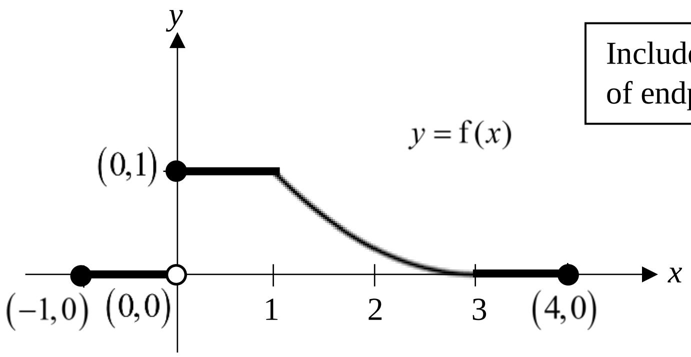
(i) State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = \frac{1}{4}(x-3)^2$. [3]

A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ \frac{1}{4}(x-3)^2 & \text{for } 1 \le x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

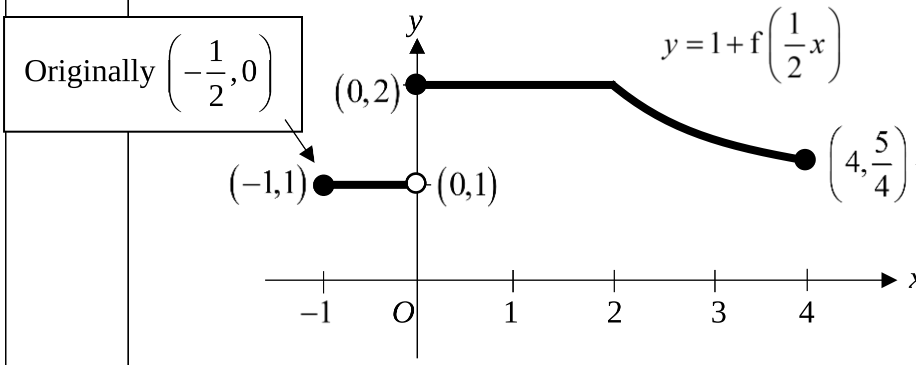
(ii) Sketch the curve for $-1 \le x \le 4$. [3]

(iii) On a separate diagram, sketch the curve with equation $y = 1 + \mathrm{f}\left(\frac{1}{2}x\right)$, for $-1 \le x \le 4$. [2]

8	**Suggested Solutions**
(i)	$y = x^2 \xrightarrow{\text{(1) replace } y \text{ by } \frac{y}{\frac{1}{4}}} y = \frac{x^2}{4} \xrightarrow{\text{(2) replace } x \text{ by } x-3} y = \frac{1}{4}(x-3)^2$ Check transformation using this diagram first! (1) Stretch parallel to *y*-axis by a factor of $\frac{1}{4}$. (2) Translate 3 units in the positive *x*-direction. *Any order is accepted. **<u>Alternative</u>** $y = x^2 \xrightarrow{\text{(1) replace } x \text{ by } \frac{x}{2}} y = \left(\frac{x}{2}\right)^2 = \frac{1}{4}x^2 \xrightarrow{\text{(2) replace } x \text{ by } x-3} y = \frac{1}{4}(x-3)^2$ (1) Scaling parallel to *x*-axis by a factor of 2. (2) Translate 3 units in the positive *x*-direction.
(ii)	 Include coordinates of endpoints.

(iii)

$$y=\mathrm{f}(x)\xrightarrow[\boxed{x\text{-coordinates }\times 2}]{\text{replace } x \text{ by } \frac{x}{2}} y=\mathrm{f}\left(\frac{x}{2}\right)\xrightarrow[\boxed{y\text{-coordinates }+1}]{\text{replace } y \text{ by } y-1} y=\mathrm{f}\left(\frac{x}{2}\right)+1$$



When $x=4$,

$$\begin{aligned} y&=1+\mathrm{f}\left(\frac{4}{2}\right)\\ &=1+\mathrm{f}(2)\\ &=1+\frac{1}{4}(2-3)^2\\ &=\frac{5}{4}\end{aligned}$$

Note: The part of the graph from (2,2) to (4, 5/4) must look like a curve**.**

<u>Level 3</u>

9 A curve C has equation $x^2+6x+5=\dfrac{(y-5)^2}{-4}$. Describe fully a sequence of three transformations which would transform C on to the curve $x^2+y^2=4^2$.

9	**Solution**

$$(x+3)^2-4=\frac{(y-5)^2}{-4} \quad\Rightarrow\quad (x+3)^2+\frac{(y-5)^2}{2^2}=2^2 \quad\Rightarrow\quad 2^2(x+3)^2+(y-5)^2=4^2$$

$$\frac{(x+3)^2}{\left(\frac{1}{2}\right)^2}+(y-5)^2=4^2$$

$\downarrow$ Replace y by $y+5$ $\Rightarrow$ (1) Translation of 5 units in the negative y-direction.

$$\frac{(x+3)^2}{\left(\frac{1}{2}\right)^2}+y^2=4^2$$

$\downarrow$ Replace x by $x-3$ $\Rightarrow$ (2) Translation of 3 units in the positive x-direction.

$$\frac{x^2}{\left(\frac{1}{2}\right)^2}+y^2=4^2$$

$\downarrow$ Replace x by $\dfrac{x}{2}$ $\Rightarrow$ (3) Stretch of factor 2 parallel to the x-axis.

$$x^2+y^2=4^2$$

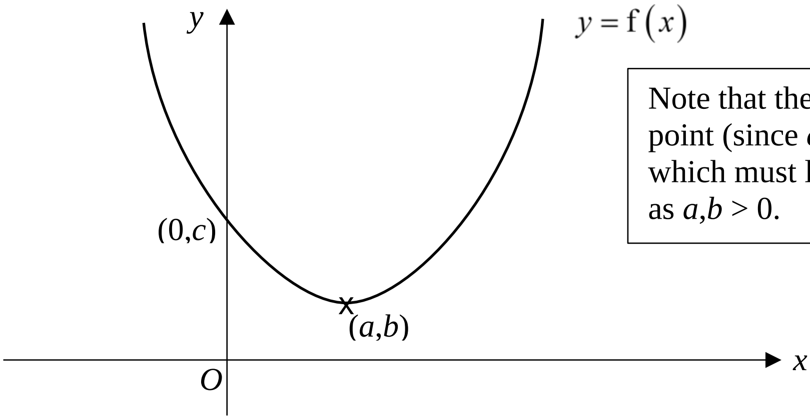
<u>Alternative</u>

(1) Translate by 5 units in the negative y-direction.
(2) Stretch of factor 2 parallel to the x-axis.
(3) Translate by 6 units in the positive x-direction.

10 2016(9740)/I/3

The curve $y = x^4$ is transformed onto the curve with equation $y = \mathrm{f}(x)$. The turning point on $y = x^4$ corresponds to the point with coordinates (a, b) on $y = \mathrm{f}(x)$. The curve $y = \mathrm{f}(x)$ also passes through the point with coordinates $(0, c)$. Given that $\mathrm{f}(x)$ has the form $k(x-l)^4 + m$ and that *a*, *b* and *c* are positive constants with $c > b$, express *k*, *l* and *m* in terms of *a*, *b* and *c*. [2]

By sketching the curve $y = \mathrm{f}(x)$, or otherwise, sketch the curve $y = \dfrac{1}{\mathrm{f}(x)}$. State, in terms of *a*, *b* and *c*, the coordinates of any points where $y = \dfrac{1}{\mathrm{f}(x)}$ crosses the axes and of any turning points. [4]

10	Suggested Solutions
	Method 1: $y = \mathrm{f}(x) = k(x-l)^4 + m$ Observe that the turning point is (l, m) And given that (a, b) is a turning point $\therefore l = a,\ m = b$ Sub $(0, c)$ into $y = \mathrm{f}(x)$: $k(0-a)^4 + b = c$ $\therefore\ k = \dfrac{c-b}{a^4}$ **Method 2:** $y = \mathrm{f}(x) = k(x-l)^4 + m$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4k(x-l)^3$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = l$ $\therefore l = a$ Sub (a, b) into $y = \mathrm{f}(x)$: $b = k(a-a)^4 + m$ $\therefore m = b$
	 Note that the curve has a minimum point (since $c > b$) which must lie in the first quadrant as $a,b > 0$.