Work, Energy and Power

Contents

- Work
- Kinetic Energy & Potential Energy
- Energy conversion and conservation
- Power and Efficiency

Learning Outcomes

Candidates should be able to:

a) define and use work done by a force as the product of the force and displacement in the direction of the force
b) calculate the work done in a number of situations including the work done by a gas which is expanding against a constant external pressure: $W = p\Delta V$
c) give examples of energy in different forms, its conversion and conservation, and apply the principle of energy conservation.
d) show an appreciation for the implications of energy losses in practical devices and use the concept of efficiency to solve problems.
e) derive, from the equations for uniformly accelerated motion in a straight line, the equation $E_k = \frac{1}{2}mv^2$.
f) recall and use the equation $E_k = \frac{1}{2}mv^2$.
g) distinguish between gravitational potential energy, electric potential energy and elastic potential energy.
h) deduce that the elastic potential energy in a deformed material is related to the area under the force-extension graph.
i) show an understanding of and use the relationship between force and potential energy in a uniform field to solve problems.
j) derive, from the definition of work done by a force, the equation $E_p = mgh$ for gravitational potential energy changes near the Earth's surface.
k) recall and use the equation $E_p = mgh$ for potential energy changes near the Earth's surface.
l) define power as work done per unit time and derive power as the product of a force and velocity in the direction of the force.

a) define and use work done by a force as the product of the force and displacement in the direction of the force

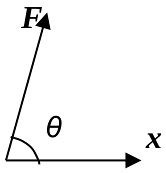
1 Work

Definition Work done by a force on the body is defined as the product of the force and the displacement in the direction of the force.

1.1 Work done by a constant force

For a constant force *F*,

$$\boxed{W = Fx\cos\theta}$$



where F is the magnitude of the constant force
x is the displacement
θ is the angle between the force and displacement

You can either view this as resolving ***F*** along the displacement ($F\cos\theta$) or resolving ***x*** along the force ($x\cos\theta$).

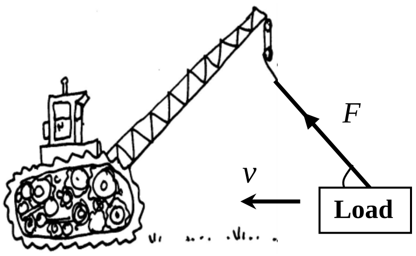
The S.I. unit for work done is joule (J).

Example 1(a)

Find the amount of work done by the tension of the cable when the crane pulls a load 20 m horizontally with a tension of 2000 N, making an angle of 30° with the horizontal.

Solution

$$\begin{aligned} W &= Fx\cos\theta \\ &= (2000)(20)\cos 30^\circ \\ &= 26000 \text{ J} \end{aligned}$$



Example 1(b)

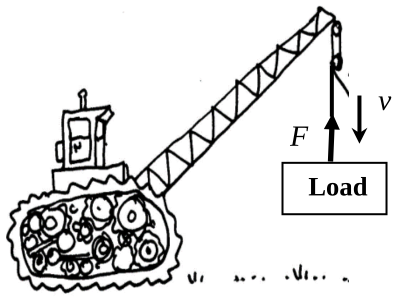
Find the amount of work done by the tension of the cable when the crane lifts a load 20 m vertically with a tension of 2000 N.

Solution

$$\begin{aligned} W &= Fx\cos\theta \\ &= (2000)(20)\cos 0^\circ \\ &= 40000 \text{ J} \end{aligned}$$

Example 1(c)

Find the amount of work done by the tension of the cable when the crane lowers a load vertically down 20 m with a tension of 2000 N.



Solution

$$\begin{aligned} W &= Fx\cos\theta \\ &= (2000)(20)\cos 180^\circ \\ &= -40000 \text{ J} \end{aligned}$$

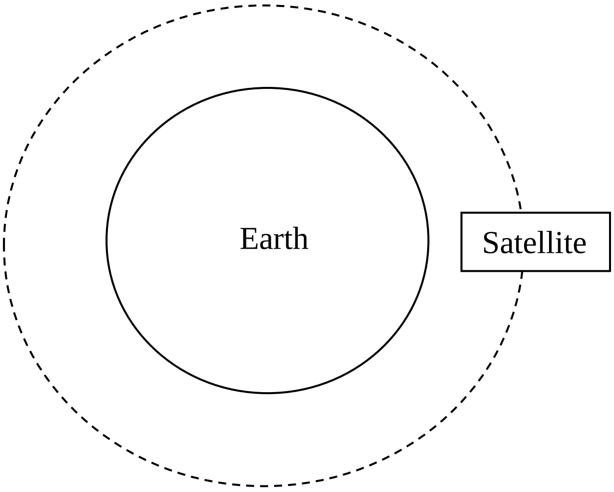
Discussion

Work can be <u>positive</u> or <u>negative</u>.

- Positive work is done by a force when a non-zero component of the force exerted is in the same direction as the displacement of the object.
- Negative work is done by a force when a non-zero component of the force exerted is in the opposite direction to the displacement of the object.

Example 2

A communications satellite moves in a circular orbit at a constant speed in response to gravity. Which of the following statements is correct?



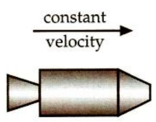
A The earth does positive work on the satellite.

B The earth does negative work on the satellite.

<u>C</u> The earth does no work on the satellite.

D Once a coordinate system is specified, the work changes sign every half orbit, so that the average work is zero.

Other cases where work done is zero:

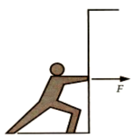


- Spacecraft travelling at constant velocity

  Work done by the net force on spacecraft = 0, because net force = 0

- Force exerted on an immovable object

  Work done by the force applied on the wall = 0, because displacement = 0

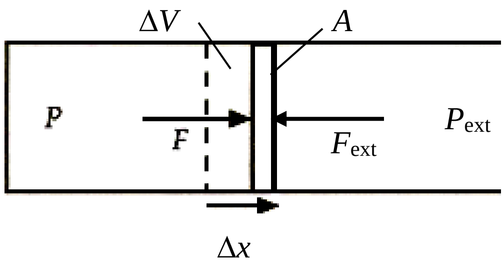


> b) *calculate the work done in a number of situations including the work done by a gas which is expanding against a constant external pressure:* $W = p\Delta V$

1.1.1 Work done by a gas expanding against constant pressure

There is another special example of work done by a constant force: work done by a gas expanding against constant pressure (to be covered in more detail in Thermal Physics).

Consider the expansion of a gas in a cylinder with a massless frictionless piston separating it from the surrounding. If the surrounding is a large entity (e.g. atmosphere), during the expansion, the external pressure P_{ext}, and hence the external force F_{ext} by the surrounding on the piston, is effectively constant.
F is the force by the gas on the surrounding.



A: cross-sectional area of the piston

ΔV: change of volume of the gas

Therefore, during this process when the piston moves over a distance Δx,

Work done by gas on surrounding (workdone by F) = $F\Delta x$

Work done by F is positive as the displacement and force is in the same direction

$$\begin{aligned}\text{Work done by surrounding on gas (Work done by } F_{ext}) &= -F_{ext}\Delta x \\ &= -P_{ext}A\Delta x \\ &= -P_{ext}\Delta V\end{aligned}$$

The work done is negative because the external force and the displacement of the piston are in opposite directions.

The expression above gives us the work done by the surrounding on the gas. The negative sign shows that energy is transferred from the gas to the surrounding. The more common term to describe this process is work done by the gas on the surrounding, which is simply the negative of work done by the surrounding on the gas.

Expanding against constant pressure,
Work done by a gas on surrounding = negative work done by surrounding on gas

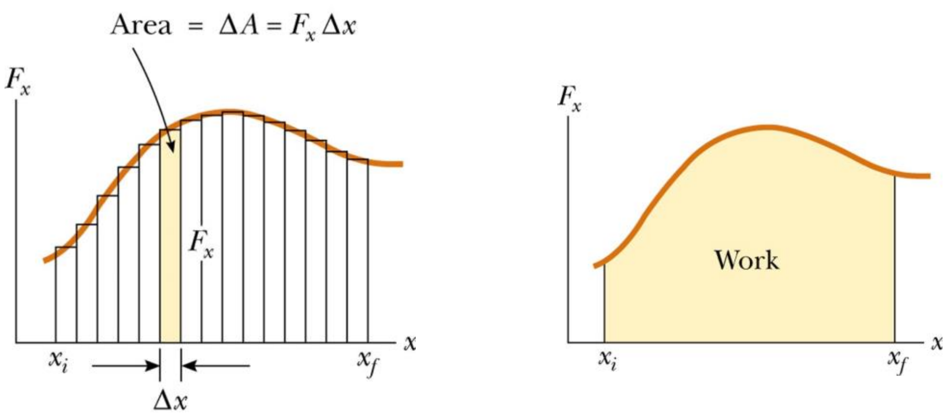
$$= -(-P_{ext}\,\Delta V)$$
$$= P_{ext}\,\Delta V$$

1.2 Work done by a force of varying magnitude

We have been considering situations where the force acting is of a constant magnitude in the direction of displacement. In many practical situations, the force acting on an object varies.

Suppose an object is being displaced along the x-axis under the action of a force F_x that acts in the x direction and varies with position. Let's imagine we were able to use a force sensor and a data logger to measure this variation, and obtain the plot shown below.

Consider the part when the object is displaced in the direction of increasing x from $x = x_i$ to $x = x_f$



The total displacement x can be calculated as the sum of many very small steps Δx. During each small step, the force is practically constant.

The total work done = sum of work done for each step

$$W = \Delta W_1 + \Delta W_2 + \Delta W_3 + \ldots\ldots$$

$$= F_1\Delta x + F_2\Delta x + F_3\Delta x + \ldots..$$

As the width of the small steps, Δx, gets really small, the sum of area of these strips will just be the area under the graph of F_x against x.

$$W = \int_{x_i}^{x_f} F_x dx$$

In other words, the work done by a variable force acting on an object that undergoes a displacement is equal to the area under the force–displacement graph.

(e) derive, from the equations for uniformly accelerated motion in a straight line, the equation $E_k = ½mv^2$

(f) recall and use the equation $E_k = ½mv^2$

2 Energy

Definition
Energy is defined as the stored ability to do work.

Energy is measured using the same unit as work done, namely joule (J).

Energy and work are two closely linked concepts. Work is a process of conversion of energy from one form to another form or the transfer of energy from one body to another by means of a force.

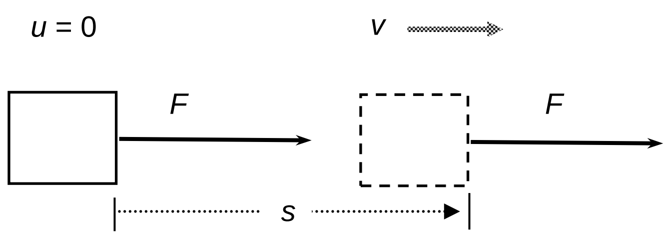
2.1 Kinetic Energy E_k

Derivation of the equation $E_k = \frac{1}{2}mv^2$

The starting point is to define that the kinetic energy of a body at velocity *v* is the work done on it by an external force *F* to bring it from rest to its final state of velocity *v*.

Change in kinetic energy, ΔE_k = work done by *F*

Consider a body of mass *m* at rest brought to velocity *v* over a distance *s* by a force *F*. For simplicity, assume *F* is constant throughout the motion.



Change in kinetic energy, $\Delta E_k = Fs$

By Newton's second law: $\Delta E_k = (ma)s$

Using the equations of motion for uniform acceleration:

$$v^2 = u^2 + 2as \Rightarrow s = \frac{v^2 - u^2}{2a}$$

Hence, $\Delta E_k = (ma)(\frac{v^2 - u^2}{2a})$

$$\Delta E_k = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$$

Since the mass started from rest (*u* = *0*), $E_k = \frac{1}{2}mv^2$

Example 3

Two blocks of ice, one twice as heavy as the other, are at rest on a frozen lake. A person pushes each block a distance of 5.0 m with a constant force (the same magnitude of force for each block). Assume that friction may be neglected. The kinetic energy of the lighter block after the push is

A smaller than that of the heavy block
B <u>equal to that of the heavy block</u>
C larger than that of the heavy block

<u>2.1.1 Kinetic Energy and Momentum</u>

Kinetic energy can be expressed in terms of the momentum of a body too.

$$\boxed{E_k = \frac{1}{2}mv^2 = \frac{1}{2m}m^2v^2 = \frac{p^2}{2m}}$$

This expression is very useful in problem-solving involving energy and momentum conservations.

Example 4

A bullet of mass 50 g is fired from a gun of mass 2.0 kg. If the total kinetic energy produced by the explosion is 3000 J, what are the kinetic energies of the bullet and the gun?

Strategy

Think about Conservation of Momentum and Energy

Solution

By the Principle of Conservation of momentum,

$$p_{initial} = p_{final}$$
$$0 = p_b + p_g$$
$$p_b = -p_g$$

Hence,

$$\text{Total } E_k = \frac{{p_b}^2}{2m_b} + \frac{{p_g}^2}{2m_g}$$
$$3000 = \frac{{p_b}^2}{(2)(50\times10^{-3})} + \frac{{p_b}^2}{(2)(2.0)}$$
$${p_b}^2 = 292.7 \text{ kg}^2 \text{ m}^2 \text{ s}^{-2}$$

Hence,

$$E_k \text{ of bullet} = \frac{{p_b}^2}{(2)(50\times10^{-3})} = \frac{292.7}{(2)(50\times10^{-3})} = 2.9\times10^3 \text{ J}$$
$$E_k \text{ of gun} = \frac{{p_b}^2}{(2)(2.0)} = \frac{292.7}{(2)(2.0)} = 73 \text{ J}$$

(g) distinguish between gravitational potential energy, electric potential energy and elastic potential energy

(h) deduce that the elastic potential energy in a deformed material is related to the area under the force-extension graph

2.2 Potential Energy E_p

Definition
Potential energy is the ability of a body to do work due to its position or shape.

The three common types of potential energy (PE) are gravitational PE, elastic PE and electric PE (learnt in later topic).

(j) derive, from the definition of work done by a force, the equation $E_p = mgh$ for gravitational potential energy changes near the Earth's surface

(k) recall and use the equation $E_p = mgh$ for gravitational potential energy changes near the Earth's surface

2.2.1 Gravitational Potential Energy

Gravitational potential energy of an object is the energy it possesses by virtue of its position in a gravitational field.

Consider a body of mass m being lifted from height h_1 to height h_2 on the surface of the Earth.

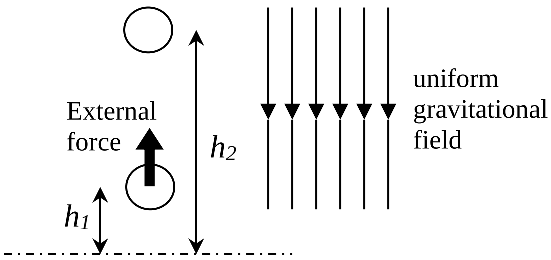
An external force F that is equal and opposite to mg is applied to the mass (This can be achieved by moving the object very slowly so that it can be modelled as in equilibrium with a constant velocity)

Change in gravitational potential energy, ΔE_P = work done by F
$= F\Delta h$

Work done on the body by F:

$$W = Force \times Displacement$$
$$= F(h_2 - h_1)$$
$$= mg(h_2 - h_1)$$
$$= mg\Delta h$$

$$\boxed{|F| = |mg|}$$



The work done represents a transfer of energy into the system (of the object and Earth) in the form of change in gravitational potential energy, ΔGPE. (To be covered in greater details under the topic 'Gravitation'.).
Therefore, we can identify the quantity mgh as the gravitational potential energy.

Example 5

A mass of weight 5.0 N is lifted by a vertical force of 7.0 N through a vertical distance of 2.0 m.

a) Calculate the work done by the lifting force.
b) Calculate the change in gravitational potential energy.
c) Assuming it was initially at rest, calculate the final speed of the mass.

Solution

a) Work done by lifting force = 7(2) = 14 J
b) Gain in GPE = (5)(2) = 10 J
c) By conservation of energy

Workdone by lifting force = gain in KE + gain in GPE

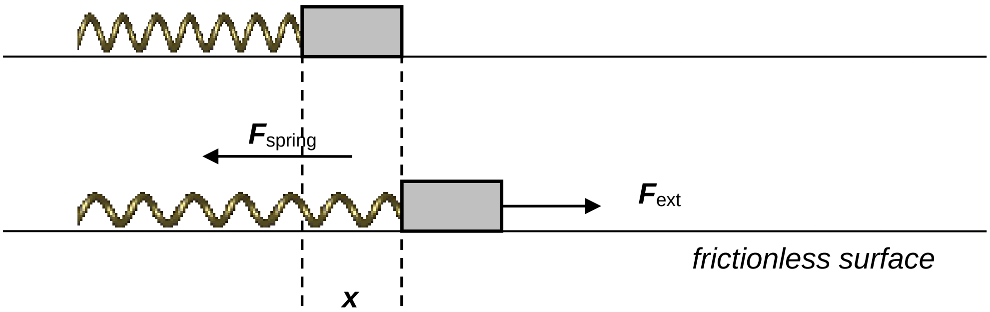
$$14 = 10 + \tfrac{1}{2}\left(\frac{5}{9.81}\right)v^2$$

$$v = 3.96 \text{ ms}^{-2}$$

2.2.2 Elastic Potential Energy

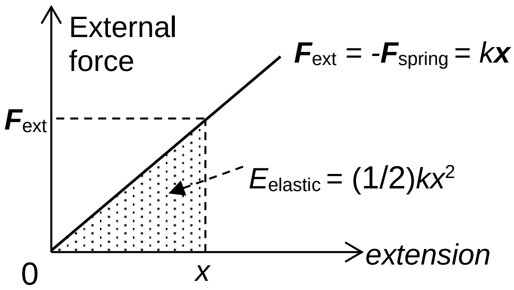
When a spring is compressed or stretched, energy is stored in the form of elastic potential energy.

Consider a light spring that is stretched through an extension $\boldsymbol{x}$ very slowly (at practically zero velocity) by applying an external force $\boldsymbol{F}_{ext}$. To just overcome the tension in the spring, $\boldsymbol{F}_{spring}$, the external force $\boldsymbol{F}_{ext} = -\boldsymbol{F}_{spring}$.



If the spring obeys Hooke's law, the tension in the spring $\boldsymbol{F}_{spring}$ is proportional to its extension $\boldsymbol{x}$, i.e. $\boldsymbol{F}_{spring} = -k\boldsymbol{x}$, where k is the spring constant.
(The negative sign shows that the force and extension are in opposite direction)

The work done by $\boldsymbol{F}_{ext}$ can be found by evaluating the area under $\boldsymbol{F}_{ext}$ – $\boldsymbol{x}$ graph.



$$W = \text{area under the external force - extension graph}$$

$$W = \frac{1}{2}(F_{ext})(x)$$

$$W = \frac{1}{2}(-F_{spring})(x)$$

$$W = \frac{1}{2}(kx)(x)$$

$$W = \frac{1}{2}kx^2$$

The work done by $\boldsymbol{F}_{ext}$ is stored in the spring as elastic potential energy:

$$\boxed{\Rightarrow E_{elastic} = \frac{1}{2}k\,x^2}$$

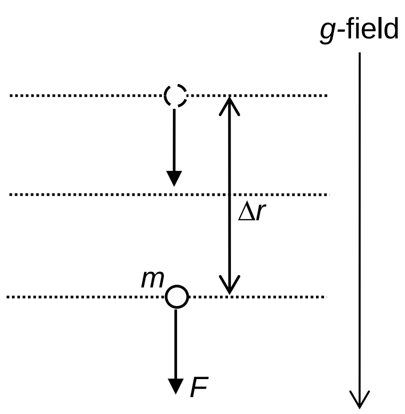
(i) show an understanding of and use the relationship between force and potential energy in a uniform field to solve problems

2.2.3 Relationship between Field Force and Potential Energy

We can associate a potential energy for a system with a force acting between members of the system. In general, the work done, W, by a force on the object that is a member of a system as the object moves from one position to another is equal to the initial value of the potential energy, U, of the system minus the final value:

$$W = U_i - U_f = -\Delta U$$

We will arrive at a general rule for the relationship between a potential energy function (how the potential energy varies with displacement) and the size of the force. We shall, for simplicity's sake, consider a mass falling in *uniform gravitational field.*



The mass falls freely under the pull of gravity *F* through a vertical distance Δr.

Work done by the gravitational force: $W = F\Delta r$

Compare it with the scenario described in the derivation of gravitational potential energy on page 8. As GPE is only a function of position, instead of a gain in GPE as on page 8, now a movement downwards results in a loss of GPE of the same magnitude. In other words, the work done by an internal force F within the system of the mass and Earth results in a loss of GPE (note: and a gain in KE. No energy is lost from the system as we are discussing an internal force).

$$W = -\Delta GPE$$
$$F\Delta r = -\Delta GPE$$
$$F = -\frac{\Delta GPE}{\Delta r}$$

Work done by gravity is positive as the gravitational force and displacement of the object are in the same direction, while ΔGPE is negative as GPE is reduced in the motion, hence the negative sign in the equation.

It turns out that this relationship can be generalised to any conservative force*, non-uniform field with its associated potential energy (U). The relationship takes the following form:

$$\boxed{F_{field} = -\frac{dU}{dx}}$$

The significance of this equation can be interpreted as follows:

- The force of a field at a certain position has the magnitude of the derivative of the potential energy of the object with respect to the displacement of the object at that position.
- The direction of the force is in the direction of decreasing potential energy, as indicated by the negative sign.

This relationship will be examined in more detail in later topics on Gravitation and Electric Field.
Conservative force (not in syllabus): a force is conservative if the work it does on an object moving between two points is independent of the path taken between the points and the work done moving through any closed loop is zero. Examples of conservative forces are gravitational force, electric force and force of an ideal spring (which means F_{field} will be F_{spring}).*

(c) give examples of energy in different forms, its conversion and conservation, and apply the principle of energy conservation

2.3 Principle of Conservation of Energy

The Principle of Conservation of Energy states that:

Energy may be transformed from one form to another but it cannot be created or destroyed.

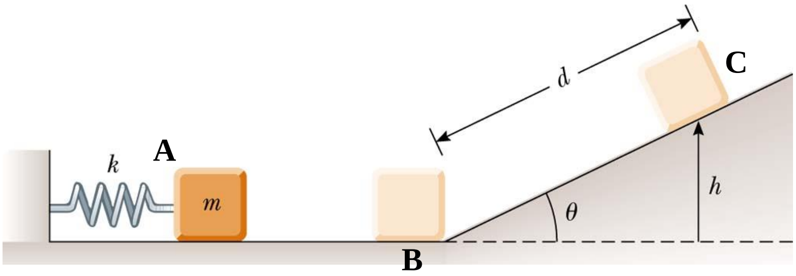
Example 6

A 0.50 kg block rests on a horizontal, frictionless surface. The block is pressed against a light spring having a spring constant of k = 80 N m^{-1}. The spring is compressed a distance of 2.0 cm to point A and released. Calculate

a) the speed of the block when it is at the bottom of the incline, position B.
b) the maximum distance, d, the block travels up the frictionless incline if the incline angle is 25°.

Strategy

Use Principle of Conservation of Energy.



Solution

a) By principle of conservation of energy at A and B

$$\text{Loss of EPE} = \text{Gain of KE}$$

$$\frac{1}{2}kx^2 = \frac{1}{2}mv_B{}^2$$

$$\frac{1}{2}(80)(2.0\times10^{-2})^2 = \frac{1}{2}(0.50)v_B{}^2$$

$$v_B = 0.25 \text{ m s}^{-1}$$

b) By principle of conservation of energy at A and C,

$$\text{Loss of EPE} = \text{Gain in GPE}$$

$$\frac{1}{2}kx^2 = mgh$$

$$\frac{1}{2}kx^2 = mg(d\sin 25°)$$

$$\frac{1}{2}(80)(2.0\times10^{-2}) = (0.50)(9.81)d\sin 25°$$

$$d = 0.0077 \text{ m}$$

Discussion

If there is energy loss, how would your solution be different?

(l)	define power as work done per unit time and derive power as the product of a force and velocity in the direction of the force.

3 Power

Definition

Power is the work done per unit time or rate of transfer of energy or the rate of doing work.

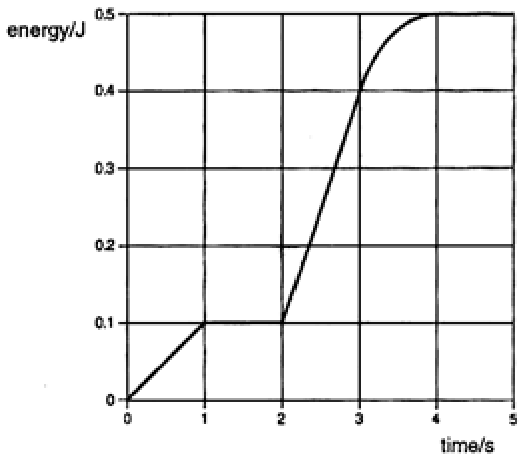
Instantaneous power $P = \dfrac{dE}{dt} = \dfrac{dW}{dt}$

Average power $<P> = \dfrac{\Delta E}{\Delta t} = \dfrac{\text{Total work done}}{\text{Total time taken}} = \dfrac{W}{\Delta t}$

The S.I. unit for power is joules per second or watt, i.e. 1 W = 1 J s^{-1}

Example 7

A bicycle dynamo is started at time zero. During the first 5 seconds, the total energy transformed by the dynamo increases as shown in the graph.



What is the maximum power generated at any instant during these first 5 s?

A. 0.10 W
B. 0.13 W
C. 0.30 W
D. 0.50 W

Solution

Power = $\frac{dE}{dt}$

Maximum power occurs at the steepest part of graph which is between 2s < t < 3s.

Maximum power = $\frac{0.4-0.1}{3} = \mathbf{0.30\ W}$

3.1 Relationship between Power, Force and Velocity

Consider a force *F* which moves a distance *x* at constant *v* in the direction of the force, in time *t*. The work done *W* by the force is given by

$$W = Fx$$

Dividing both sides by time t gives

$$\frac{W}{t} = F\frac{x}{t}$$

Now $\frac{W}{t}$ is the rate of doing work, i.e. the instantaneous power *P* and $\frac{x}{t} = v$.

Hence $P = Fv$

Instantaneous power, *P* = *force* x *velocity*

Example 8

a) What power must be developed by the engine of a 1600 kg car moving at a constant speed of 25 m s^{-1} on a level road if the total resistive force is 700 N?

Solution

To maintain the car at the same speed, force applied by the engine
= resistive force

Hence, $P_{engine} = F_{engine}v = (700)(25) = 17500\text{ W}$

Alternatively,

Power of engine must be equal to the rate at which the resistive force removes energy from the system (power of friction).

$$P_{engine} = P_{resistive} = (700)(25) = 17500\ W$$

b) What is the additional power needed if the car is to move with the same speed up a slope that makes an angle of 10° with the horizontal?

When the car goes up the slope,

$$\begin{aligned}
\text{Additional power needed} &= \text{Rate of increase of GPE} \\
&= \frac{d}{dt}(mgh) \\
&= mg\frac{dh}{dt} \\
&= mg\frac{d(x\sin\theta)}{dt} \\
&= mg\sin\theta\frac{dx}{dt} \\
&= mgv\sin\theta \\
&= (1600)(9.81)(25)(\sin 10°) \\
&= 68100\text{ W}
\end{aligned}$$

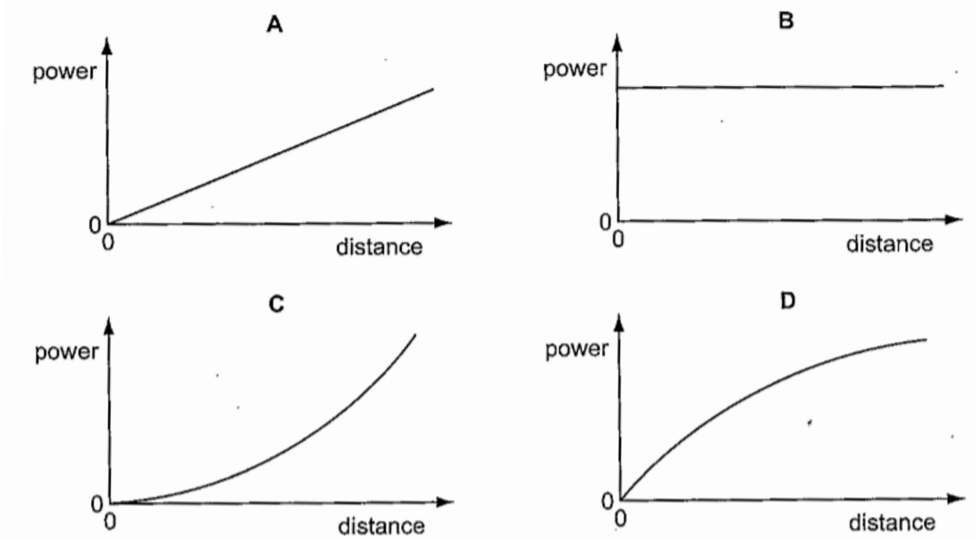
Discussion

Constant speed implies constant kinetic energy. Is it possible to have varying speed? If so, how would the solution change? (Discuss with your tutor)

Example 9: 2010/P1/Q11

A vehicle starts from rest and accelerates uniformly.

Which graph shows how the power output of the vehicle varies with distance travelled?



Solution

Instantaneous power, P = Fv ---------- (1)

Object accelerate uniformly, i.e F = ma => F =constant

v= instantaneous velocity is not constant. v is varying.

Using kinematics eqn, $v^2 = u^2 + 2as$

$$v = \sqrt{2as} \text{ ----- (2)}$$

Sub (2) into (1),

$P = F\sqrt{2a}\,(\sqrt{s})$

Ans: D

Note: F and a are constant

(d) show an appreciation for the implications of energy losses in practical devices and use the concept of efficiency to solve problems

3.2 Efficiency

No practical engine can transform energy from one form to another without some energy going to internal energy and other non-useful forms of energy

Energy *input* = *Useful energy output* + other forms of energy

$$\text{Efficiency} = \frac{\text{useful energy output}}{\text{energy input}} \times 100\%$$

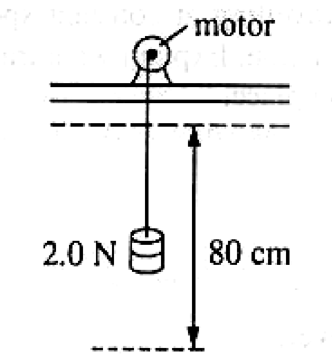
Alternatively,

$$\text{Efficiency} = \frac{\text{useful power output}}{\text{power input}} \times 100\%$$

Example 10

A small electric motor is used to raise a weight of 2.0 N through a vertical height of 80 cm in 4.0 s. The efficiency of the motor is 20 %.

What is the electrical power supplied to the motor?



A. 0.080W
B. 0.80 W
C. 2.0 W
D. 200 W

Solution

Electrical power = $\frac{2(080)}{4(0.2)} = 2.0\ W$