River Valley High School
2023 JC2 Mathematics (9758)
Lecture Test 3 (Term 2)

Name	: ______________	**Index Number**	: ______________
Class	: ______________	**Date**	: **12 May 2023**
Duration	: **50 min**	**Max. No. of Marks**	: **30**

<u>Formulae from MF26</u>

Vectors

The point dividing AB in the ratio $\lambda : \mu$ has position vector $\dfrac{\mu\mathbf{a}+\lambda\mathbf{b}}{\lambda+\mu}$

Vector product:

$$\mathbf{a}\times\mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

[Answer all the questions on writing papers. **Up to 1 mark** will be deducted for poor presentation.]

1 The line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(-\mathbf{i} + \mathbf{k})$ where λ is a parameter.

The plane π_1 has equation $x + 2y + 2z = -2$ and the point A is at $(1, 1, 2)$.

(i) Find the acute angle between the plane π_1 and the line l. [2]

(ii) The plane π_2 contains the line l and is perpendicular to plane π_1. Show that the equation of plane π_2 is $2x - 3y + 2z = 3$. [2]

(iii) Find a vector equation of the line of intersection between π_1 and π_2. [2]

(iv) Find the shortest distance from the point A to the plane π_1. [3]

2 **(a)** The complex number $v = 1 + \mathrm{i}$ is a root of the equation $2z^2 - cz + 1 - \mathrm{i} = 0$ where c is a complex number.

(i) Without the use of a calculator, find the value of c. [4]

(ii) Without calculating the other root, explain why v^* is not the other root of the equation $2z^2 - cz + 1 + \mathrm{i} = 0$. [1]

(b) The complex number w is given by $w = 2\left(\cos\dfrac{\pi}{5} - \mathrm{i}\sin\dfrac{\pi}{5}\right)$.

Showing your working clearly, find the modulus and argument of $\dfrac{\mathrm{i}w^2}{2\mathrm{i} - 2\sqrt{3}}$. [6]

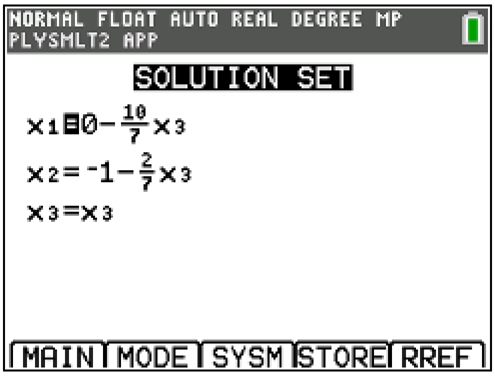
3 A disc with sectors labelled 1, 2 and 3 with a free spinning arrow and a fair coin are used in a game booth at a bazaar. When the arrow on the disc is spun, it comes to a stop pointing at the sector labelled 1 with probability of p while it has a probability of $\frac{2}{3}p$ pointing at the sector labelled 2. In a game, the arrow is spun, then the coin is tossed 1, 2 or 3 times according to the sector where the arrow points on the disc. The random variable X denotes the score of the game obtained by adding the total number of heads from the resulting coin toss(es).

(i) Show that $\mathrm{P}(X=1)=\frac{3}{8}+\frac{5}{24}p$. Construct a probability distribution table for X with the expressions of the probabilities in terms of p. [5]

(ii) Given that $\mathrm{E}(X)=\frac{5}{6}$, find the value of p. [2]

(iii) It is given that $\mathrm{Var}(X)=\frac{5}{9}$ and that X_1 and X_2 are two independent observations of X. Find $\mathrm{Var}\left(\frac{1}{2}X_1-3X_2\right)$. [3]

End of Paper

Question 1 [9] Vectors (Planes)		
(i)	Let the acute angle be θ. $\theta=\sin^{-1}\left\|\begin{pmatrix}1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}\right\|$ $=\sin^{-1}\left\|\dfrac{\begin{pmatrix}1\\2\\2\end{pmatrix}\bullet\begin{pmatrix}-1\\0\\1\end{pmatrix}}{\sqrt{1^2+2^2+2^2}\sqrt{2}}\right\|$ $=\sin^{-1}\left\|\dfrac{1}{3\sqrt{2}}\right\|$ $=13.6°$ (1 d.p.)	Some responses have mixed up the formula with others.
(ii)	The normal of plane π_1: $\mathbf{r}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=-2$ is parallel to the plane π_2. A normal vector to plane $\pi_2=\begin{pmatrix}1\\2\\2\end{pmatrix}\times\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}2\\-3\\2\end{pmatrix}$ And point A is on the plane. $\begin{pmatrix}1\\1\\2\end{pmatrix}\bullet\begin{pmatrix}2\\-3\\2\end{pmatrix}=2-3+4=3$. Therefore, the equation of plane π_2 is $2x-3y+2z=3$. (Shown)	Generally most responses are able to identify the 2 vectors to cross for the normal vector for π_2 and substituting the point $\overrightarrow{OA}$ for the constant.
(iii)	Using GC to solve the system of equations: $x+2y+2z=-2$ $2x-3y+2z=3$ 	Some responses crossed the normals of the 2 planes to get the direction of the line. However, many of these responses have difficulties getting a particular point on the line of intersection.

	$x=0-\frac{10}{7}t$ $y=-1-\frac{2}{7}t$ for some $t\in\mathbb{R}$. $z=t$ Line of intersection: $\mathbf{r}=\begin{pmatrix}0\\-1\\0\end{pmatrix}+t\begin{pmatrix}-\frac{10}{7}\\-\frac{2}{7}\\1\end{pmatrix},\ t\in\mathbb{R}$ or $\mathbf{r}=\begin{pmatrix}0\\-1\\0\end{pmatrix}+t\begin{pmatrix}10\\2\\-7\end{pmatrix},\ t\in\mathbb{R}$	
(iv)	Let $N(0,-1,0)$ be a point on π_1. $\overrightarrow{AN}=\overrightarrow{ON}-\overrightarrow{OA}=\begin{pmatrix}0\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\1\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\-2\end{pmatrix}$ Shortest distance from point A to plane π_1 $=\left\|\overrightarrow{AN}\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|$ $=\left\|\begin{pmatrix}-1\\-2\\-2\end{pmatrix}\bullet\frac{1}{\sqrt{1^2+2^2+2^2}}\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|$ $=\left\|\frac{-9}{\sqrt{9}}\right\|$ $=3$ units <u>Alternatively</u> Let l_n be a line passing through point A and perpendicular to π_1. l_n: $\mathbf{r}=\begin{pmatrix}1\\1\\2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\2\end{pmatrix},\ \mu\in\mathbb{R}$ Let point N be the intersection of l_n and π_1,	Some responses did not identify a point on the plane to get a vector from the plane to point A for the formula. Using $\overrightarrow{OA}$ for the formula gives the distance of point A to a plane containing the origin parallel to π_1.

	$\left[\begin{pmatrix}1\\1\\2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\2\end{pmatrix}\right]\bullet\begin{pmatrix}1\\2\\2\end{pmatrix}=-2$ $1(1+\mu)+2(1+2\mu)+2(2+2\mu)=-2$ $\mu=-1$ $\therefore \overrightarrow{ON}=\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}0\\-1\\0\end{pmatrix}$ Shortest distance from point A to plane π_1 $\left\lvert\overrightarrow{AN}\right\rvert=\left\lvert\overrightarrow{ON}-\overrightarrow{OA}\right\rvert$ $=\left\lvert\begin{pmatrix}0\\-1\\0\end{pmatrix}-\begin{pmatrix}1\\1\\2\end{pmatrix}\right\rvert$ $=\left\lvert\begin{pmatrix}-1\\-2\\-2\end{pmatrix}\right\rvert$ $=\sqrt{(-1)^2+(-2)^2+(-2)^2}$ $=3$ units	

Question 2 [11] Complex Numbers		
(ai)	Sub $z=1+\mathrm{i}$ into $2z^2-cz+1-\mathrm{i}=0$, $2(1+\mathrm{i})^2-c(1+\mathrm{i})+1-\mathrm{i}=0$ $2(1+2\mathrm{i}-1)-c(1+\mathrm{i})+1-\mathrm{i}=0$ $c(1+\mathrm{i})=1+3\mathrm{i}$ $c=\frac{1+3\mathrm{i}}{1+\mathrm{i}}\times\frac{1-\mathrm{i}}{1-\mathrm{i}}$ $=\frac{1-\mathrm{i}+3\mathrm{i}+3}{2}$ $=2+\mathrm{i}$	Most responses are able to substitute $z=1+\mathrm{i}$ to find c. Most are able to use operations to make c the subject.
(aii)	If v^* is also a root, then all coefficients of the quadratic equation must be real. However, that is not the case. Therefore, v^* is not a root.	Many responses fall short in quoting the theorem although their response shows sign of knowledge.
(b)	$\left\|2\mathrm{i}-2\sqrt{3}\right\|=4$ $\arg\left(2\mathrm{i}-2\sqrt{3}\right)=\frac{5\pi}{6}$ $\left\|\frac{\mathrm{i}w^2}{2\mathrm{i}-2\sqrt{3}}\right\|=\frac{\|\mathrm{i}\|\|w\|^2}{\left\|2\mathrm{i}-2\sqrt{3}\right\|}$ $=\frac{(1)(2)^2}{4}$ $=1$ unit Note that $w=2\left(\cos\frac{\pi}{5}-\mathrm{i}\sin\frac{\pi}{5}\right)=2\left(\cos\left(-\frac{\pi}{5}\right)+\mathrm{i}\sin\left(-\frac{\pi}{5}\right)\right)$. Therefore $\arg(w)=-\frac{\pi}{5}$ and $\|w\|=2$. $\arg\left(\frac{\mathrm{i}w^2}{2\mathrm{i}-2\sqrt{3}}\right)=\arg(\mathrm{i})+2\arg(w)-\arg\left(2\mathrm{i}-2\sqrt{3}\right)$ $=\frac{\pi}{2}+2\left(-\frac{\pi}{5}\right)-\frac{5\pi}{6}$ $=-\frac{11\pi}{15}$ rad	Many are not successful in getting the correct argument for $2\mathrm{i}-2\sqrt{3}$. Many are able to use the properties of modulus to get the correct answer. Few responses did not notice that w is not given in standard polar form so the argument is not $\frac{\pi}{5}$.

Question 3 [10] Discrete Random Variable		
(i)	$\mathrm{P}(X=1)=p\left(\frac{1}{2}\right)+\frac{2}{3}p\binom{2}{1}\left(\frac{1}{2}\right)^2+\left(1-\frac{5}{3}p\right)\binom{3}{1}\left(\frac{1}{2}\right)^3$ $=\frac{3}{8}+\frac{5}{24}p$ (shown) $\mathrm{P}(X=0)=p\left(\frac{1}{2}\right)+\frac{2}{3}p\left(\frac{1}{2}\right)^2+\left(1-\frac{5}{3}p\right)\left(\frac{1}{2}\right)^3=\frac{1}{8}+\frac{11}{24}p$ $\mathrm{P}(X=2)=\frac{2}{3}p\left(\frac{1}{2}\right)^2+\left(1-\frac{5}{3}p\right)\binom{3}{2}\left(\frac{1}{2}\right)^3=\frac{3}{8}-\frac{11}{24}p$ $\mathrm{P}(X=3)=\left(1-\frac{5}{3}p\right)\binom{3}{3}\left(\frac{1}{2}\right)^3=\frac{1}{8}-\frac{5}{24}p$	Many students are unable to cope with the complexity of the context. Some good response includes drawing a tree diagram to visualise all outcomes and calculate the probabilities based on the tree.
	$X=x$ | 0 | 1 | 2 | 3 $\mathrm{P}(X=x)$ | $\frac{1}{8}+\frac{11}{24}p$ | $\frac{3}{8}+\frac{5}{24}p$ | $\frac{3}{8}-\frac{11}{24}p$ | $\frac{1}{8}-\frac{5}{24}p$	Most responses missed out the outcome where there is no heads i.e. $X = 0$
(ii)	$\mathrm{E}(X)=\sum x\mathrm{P}(X=x)$ $(1)\left(\frac{3}{8}+\frac{5}{24}p\right)+2\left(\frac{3}{8}-\frac{11}{24}p\right)+(3)\left(\frac{1}{8}-\frac{5}{24}p\right)=\frac{5}{6}$ $\frac{3}{2}-\frac{4}{3}p=\frac{5}{6}$ $p=\frac{1}{2}$	Generally, those who did not manage to get the probability distribution table up are not able to do this part properly. Interesting observation is that those whose table missing out the "0" outcome can still get the correct answer for this part.
(iii)	$\mathrm{Var}\left(\frac{1}{2}X_1-3X_2\right)$ $=\left(\frac{1}{2}\right)^2\mathrm{Var}(X_1)+3^2\mathrm{Var}(X_2)$ $=\left(\frac{1}{4}\right)\left(\frac{5}{9}\right)+9\left(\frac{5}{9}\right)$ $=\frac{185}{36}=5\frac{5}{36}$	There are many responses where the variance is subtracted which produced a negative answer for variance. A simple check should reveal that it is impossible to have a negative variance.