2019 H2 Math A Level Paper 1 Suggested Solutions

Q1

$2+i$ is a root

$\Rightarrow a(2+i)^3 + b(2+i)^2 + c(2+i) + d = 0$

$\Rightarrow (2+11i)a + (3+4i)b + (2+i)c + d = 0$

$$\Rightarrow \begin{cases} 2a+3b+c+d = 0 & \text{— ①} \\ 11a+4b+c=0 & \text{— ②} \end{cases}$$

-3 is a root

$\Rightarrow -27a + 9b - 3c + d = 0$ — ③

Solving ①, ②, ③ using GC, ← simult eqn solver

$b=-a, \quad c=-7a, \quad d=15a.$

Note :

1) when keying in the coefficients of a, b, c, d in GC, key in sequence that for b, c, d, then a. (Why?)

2) Though $2-i$ is also a root, we need not make use of it to form equations. (Why?)

Alternative method :

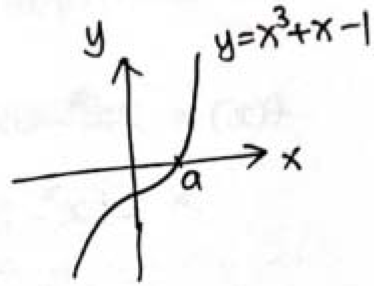
$$\begin{aligned} f(z) = az^3+bz^2+cz+d &= a(z-(2+i))(z-(2-i))(z+3) \\ &= a(z-2-i)(z-2+i)(z+3) \\ &= a((z-2)^2-i^2)(z+3) \\ &= a(z^2-4z+5)(z+3) \end{aligned}$$

Then compare coefficients : $b=-a, \ c=-7a, \ d=15a.$

Q2

(i)

$$x^3+x-1=0$$

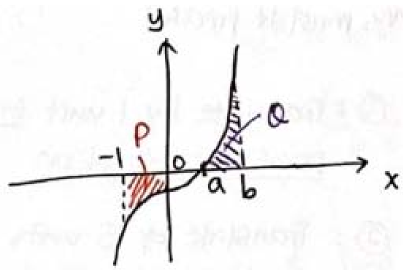


By GC, $a = 0.682328$

$= 0.682$ (to 3d.p.)

finding x-intercept of graph

(ii)



Area of Q = 2 × Area of P

$$\int_a^b y\,dx = 2\left|\int_{-1}^0 y\,dx\right|$$

$$\int_a^b x^3+x-1\,dx = 3.5 \text{ (by GC)}$$

$$\left[\frac{x^4}{4}+\frac{x^2}{2}-x\right]_a^b = 3.5$$

Rem to sub in $a = 0.682328$ (more s.f.)

$$\frac{b^4}{4}+\frac{b^2}{2}-b-\left(\frac{a^4}{4}+\frac{a^2}{2}-a\right) = 3.5$$

$$\frac{b^4}{4}+\frac{b^2}{2}-b-3.104647=0$$

plysmlt : poly root finder.

By GC, $b = 1.892$ or -1.353 (rej since $b>a$)

$b \approx 1.892$ (to 3d.p.)

Q3

(i)

$$f(x) = 2x^3 - 6x^2 + 6x - 12$$
$$= 2(x^3 - 3x^2 + 3x - 6)$$
$$= 2(x^3 - 3x^2 + 3x - 1 - 5)$$
$$= 2[(x-1)^3 - 5]$$

$\therefore$ $p = 2$, $q = -1$, $r = -5$

OR

$p = 2$ (by observation)

$$f(x) = 2x^3 - 6x^2 + 6x - 12$$
$$= 2(x+q)^3 + 2r$$
$$= 2(x^3 + 3x^2q + 3xq^2 + q^3) + 2r$$

$6q = -6 \Rightarrow q = -1$

$2q^3 + 2r = -12 \Rightarrow r = -5$

(ii)

$y = x^3$

$\downarrow$ ① replace x by $x-1$

$y = (x-1)^3$

$\downarrow$ ②

$y = (x-1)^3 - 5$

$\downarrow$ ③

$y = 2[(x-1)^3 - 5]$

①: Translate by 1 unit <u>in the positive x-direction</u>

②: Translate by 5 units <u>in the negative y-direction</u>

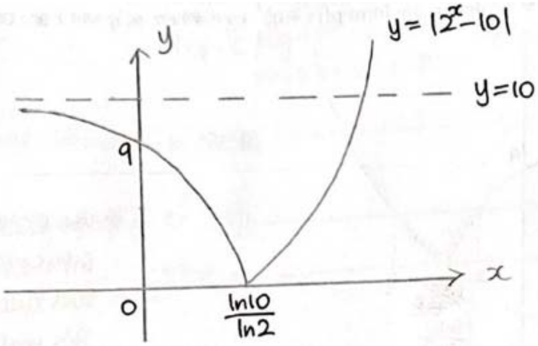
③: Scaling parallel to y-axis by <u>scale factor 2</u>

* not accepted : "translations along / in the x-axis"

"scaling by 2 units"

Q4

(i)



when $y=0$, $|2^x - 10| = 0$

$$2^x - 10 = 0$$

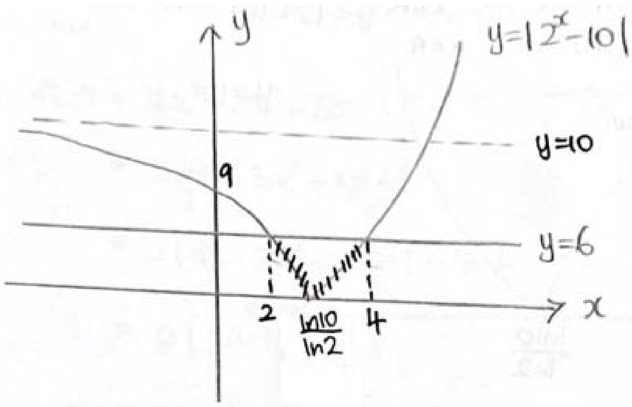
$$2^x = 10$$

$$x = \frac{\ln 10}{\ln 2}$$

* use GC to see shape & find y-intercept

** asymptote at $y=10$ (why?)

(ii)



* use GC to find intersection points. and check that it's really $x = 2, 4$

$|2^x-10| \le 6$

To find intersection points : $|2^x-10| = 6$

$2^x-10 = \pm 6$

$2^x = 16$ or 4

$x = 4$ or 2

From graph in (i) & adding in graph of $y=6$,

$2 \le x \le 4$

Q5

(i)

Let $y = f(x) = e^{2x} - 4$

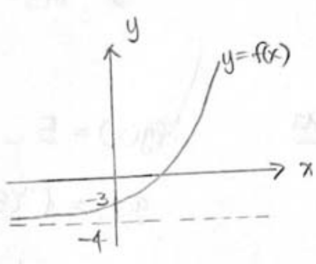
$y+4 = e^{2x}$

$\ln(y+4) = 2x$

$x = \frac{\ln(y+4)}{2}$

$f^{-1}(x) = \frac{1}{2}\ln(x+4)$

$D_{f^{-1}} = R_f = (-4, \infty)$



(ii)

M1

$$fg(x) = 5$$

$$f(x+2) = 5$$

$$e^{2(x+2)} - 4 = 5$$

$$e^{2x+4} = 9$$

$$2x+4 = \ln 9 = \ln 3^2$$

$$= 2\ln 3$$

$$x = \frac{2\ln 3 - 4}{2}$$

$$= \ln 3 - 2$$

M2

$$fg(x) = 5$$

$$g(x) = f^{-1}(5)$$

$$x+2 = \tfrac{1}{2}\ln(5+4) = \tfrac{1}{2}\ln 9$$

$$= \tfrac{1}{2}(\ln 3^2)$$

$$= \ln 3$$

$$\therefore \quad x = \ln 3 - 2$$

Q6

(i)

$$\frac{1}{4r^2-1} = \frac{1}{(2r+1)(2r-1)} = \frac{-\frac{1}{2}}{2r+1} + \frac{\frac{1}{2}}{2r-1} \rightarrow \text{cover-up rule}$$

$$= \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$\sum_{r=1}^{n} \frac{1}{4r^2-1} = \sum_{r=1}^{n} \frac{1}{2}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right)$$

$$= \frac{1}{2}\Big[\ \frac{1}{1} - \cancel{\frac{1}{3}} +$$

$$\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} +$$

$$\cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} +$$

$$\vdots$$

$$\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\ \Big]$$

$$= \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$$

(ii)

$$\sum_{r=11}^{\infty} \frac{1}{4r^2-1} = \underbrace{\sum_{r=1}^{\infty} \frac{1}{4r^2-1}}_{\text{let } n\to\infty} - \underbrace{\sum_{r=1}^{10} \frac{1}{4r^2-1}}_{\text{sub } n=10}$$

$$= \frac{1}{2}(1-0) - \frac{1}{2}\left(1 - \frac{1}{20+1}\right)$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{42}$$

$$= \frac{1}{42}$$

Q7

(i)

$$y = xe^{-x}$$

$$\frac{dy}{dx} = x(-e^{-x}) + e^{-x}$$

$$= e^{-x}(-x+1)$$

when $x = 1$,

$$\frac{dy}{dx} = e^{-1}(-1+1) = 0$$

$$y = 1e^{-1} = \frac{1}{e}$$

$\therefore$ eqn of tangent : $y = \frac{1}{e}$

when $x = -1$,

$$\frac{dy}{dx} = e(1+1) = 2e$$

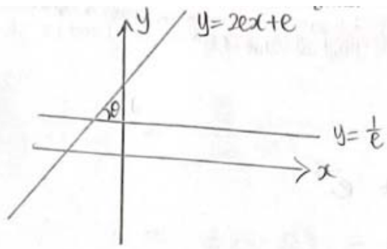
$$y = -1e = -e$$

$\therefore$ eqn of tangent : $y - (-e) = 2e(x - (-1))$

$$y + e = 2ex + 2e$$

$$y = 2ex + e$$

(ii)



let the acute ∡ between the tangents be θ

$$\tan\theta = 2e$$

$$\theta = \tan^{-1}(2e) \simeq 79.6° \text{ (to 1 dp)}$$

* in general, gradient of line = $\tan\theta$ where θ = ∡ that line makes w/ positive x-axis, $0° \leq \theta \leq 180°$

Q8

(a)

GP : $T_K = a(2^{K-1})$

AP : $S_{64} = \frac{64}{2}[2a + (64-1)(2a)]$

$$32(2\cancel{a} + 63(2\cancel{a})) = \cancel{a}(2^{K-1}) = \frac{\cancel{a}(2^K)}{2}$$

$$8192 = 2^K$$

$$2^K = 2^{13}$$

$$\therefore K = 13$$

(b)

$S_4 = 0$

$\frac{f(1-r^4)}{1-r} = 0$

$f(1-r^4) = 0$

$f = 0$ or $r = -1, 1$*

(reject since $f \neq 0$) (reject since $r \neq 1$)

$\therefore r = -1, \quad \underline{f \in \mathbb{R} \setminus \{0\}}$*

$$S_n = \frac{f(1-r^n)}{1-r} = \frac{f(1-(-1)^n)}{1-(-1)}$$

$$= \frac{1}{2} f(1-(-1)^n)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ f & \text{if } n \text{ is odd.} \end{cases}$$

(c)

$S_4 = 14$

$\frac{4}{2}(2a+3d) = 14$

$2(2a+3d) = 14$

$2a+3d = 7$

$a(a+d)(a+2d)(a+3d) = 0$

$\Rightarrow a = 0, \quad a = -d, \quad a = -2d, \quad a = -3d$

(reject since $a < 0$)

when $a=-d$, $-2d+3d=7 \Rightarrow d=7$, $a=-7$

when $a=-2d$, $-4d+3d=7 \Rightarrow -d=7$

$\Rightarrow d=-7$, $a=14$ (reject since $a<0$)

when $a=-3d$, $-6d+3d=7 \Rightarrow -3d=7$

$\Rightarrow d=-\frac{7}{3}$, $a=7$ (reject since $a<0$)

$T_{11} = a+10d = -7+10(7) = 63$

Q9

(i)(a)

$w = e^{i\theta}$

$$w+\frac{1}{w} = e^{i\theta}+e^{-i\theta}$$
$$= 2\cos\theta$$

$* \; e^{i\theta}+e^{-i\theta}$

$= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$

$= 2\cos\theta$

(i)(b)

$$\frac{w-1}{w+1} = \frac{\cos\theta+i\sin\theta-1}{\cos\theta+i\sin\theta+1}$$

$* \; \cos\theta = 2\cos^2\frac{\theta}{2}-1$

$= 1-2\sin^2\frac{\theta}{2}$

$$= \frac{-2\sin^2\frac{\theta}{2}+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}{2\cos^2\frac{\theta}{2}+i\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$$

$$= \frac{\sin\frac{\theta}{2}\left(-\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}$$

$* \; -1 = i^2$

$$= \tan\frac{\theta}{2}\,\frac{\left(i^2\sin\frac{\theta}{2}+i\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}$$

$$= \tan\frac{\theta}{2}\,\frac{i\left(i\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}$$

$$= i\tan\frac{\theta}{2}$$

$\therefore \quad k = i$

note : if choose to rationalize, should multiply top & bottom by $\cos\theta - i\sin\theta + 1$ (not $\cos\theta + i\sin\theta - 1$)

(ii)

Given $|z| = 1$

Let $z = x + iy$

$|z| = \sqrt{x^2+y^2} = 1 \Rightarrow x^2+y^2 = 1$

$$\left|\frac{z-3i}{1+3iz}\right| = \left|\frac{x+iy-3i}{1+3i(x+iy)}\right|$$

$$= \left|\frac{x+i(y-3)}{1-3y+i3x}\right|$$

$$= \frac{\sqrt{x^2+(y-3)^2}}{\sqrt{(1-3y)^2+(3x)^2}}$$

$$= \frac{\sqrt{x^2+y^2-6y+9}}{\sqrt{1-6y+9y^2+9x^2}}$$

$$= \frac{\sqrt{10-6y}}{\sqrt{10-6y}}$$

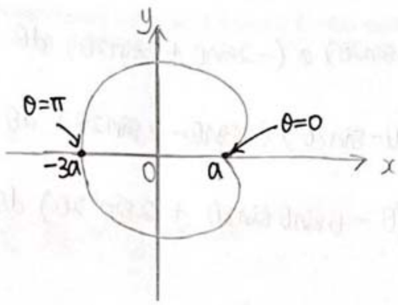
$$= 1$$

* common mistake:

- $|z-3i| = |z| - |3i|$

Q10

(i)



* Sub a=1 & graph using GC

* can trace to check x-intercepts ⇒ multiply by a

line of symmetry : $y=0$

(ii)

When $y=0$, $a(2\sin\theta - \sin 2\theta) = 0$

$2\sin\theta - \sin 2\theta = 0$

By GC, $\theta = 0, \pi, 2\pi$

graph eqn in function mode

* in fact, can trace the graph of C to check.

(iii)

$$\text{Area} = \int_{\theta=\pi}^{\theta=0} y \, dx \qquad \frac{dx}{d\theta} = a(-2\sin\theta + 2\sin 2\theta)$$

$$= \int_{\pi}^{0} a(2\sin\theta - \sin 2\theta)\, a(-2\sin\theta + 2\sin 2\theta)\, d\theta$$

$$= -\int_{\pi}^{0} a^2 (2\sin\theta - \sin 2\theta)(2\sin\theta - 2\sin 2\theta)\, d\theta$$

$$= \int_{0}^{\pi} a^2 \left(4\sin^2\theta - 6\sin\theta\sin 2\theta + 2\sin^2 2\theta\right) d\theta$$

$\therefore \theta_1 = 0, \ \theta_2 = \pi$

(iv)

$$\text{Area} = a^2 \int_0^{\pi} 4\left(\frac{1-\cos 2\theta}{2}\right) - 6\sin\theta\,(2\sin\theta\cos\theta) + 1-\cos 4\theta \; d\theta$$

$*\cos 2\theta = 1-2\sin^2\theta$

$$= a^2 \int_0^{\pi} 2-2\cos 2\theta - 12\cos\theta\sin^2\theta + 1 - \cos 4\theta \; d\theta$$

$$= a^2 \left[3\theta - \sin 2\theta - 12\,\frac{\sin^3\theta}{3} - \frac{\sin 4\theta}{4} \right]_0^{\pi}$$

$$= a^2\,(3\pi)$$

$$= 3\pi a^2$$

$\therefore$ Total area enclosed by $C = 2 \times 3\pi a^2$

$$= 6\pi a^2 \text{ units}^2$$

Q11

(i)(a)

$$\frac{d\theta}{dt} = -K(\theta-16) \text{ where } K>0$$

$$\int \frac{1}{\theta-16}\,d\theta = \int -K\,dt$$

$$\ln|\theta-16| = -Kt + C$$

$$|\theta-16| = e^{-Kt+C}$$

$$\theta-16 = \pm e^{-Kt}e^{C}$$

$$\theta = 16 + Ae^{-Kt} \text{ where } A = \pm e^{C}$$

when $t=0$, $\theta = 80$:

$$80 = 16 + A \Rightarrow A = 64$$

when $t=30$, $\theta = 32$:

$$32 = 16 + 64e^{-k(30)} \Rightarrow \frac{1}{4} = e^{-30k}$$

$$\Rightarrow e^{-k} = \left(\frac{1}{4}\right)^{\frac{1}{30}}$$

$$\therefore \theta = 16 + A(e^{-k})^t = 16 + 64\left(\frac{1}{4}\right)^{\frac{t}{30}}$$

(b)

when $t=45$,

$$\theta = 16 + 64\left(\frac{1}{4}\right)^{\frac{45}{30}} = 24\,\underline{°C}$$

(ii)

$\frac{dT}{dt} = \frac{m}{T}$ where m is a positive constant

$$\int T\,dT = \int m\,dt$$

$$\frac{T^2}{2} = mt + d$$

when $t=0$, $T=0$: $0 = d$

when $t=60$, $T=1$: $\frac{1}{2} = 60m \Rightarrow m = \frac{1}{120}$

$$\therefore \frac{1}{2}T^2 = \frac{1}{120}t$$

$$T^2 = \frac{1}{60}t$$

when $T=3$, $3^2 = \frac{1}{60}t \Rightarrow t = 540\ \underline{\text{min}}$

must have units!

Q12

(i)

$l_{PQ}: \ \underline{r} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}, \ \lambda \in \mathbb{R}$

$\Pi_{top}: \ \underline{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$

To find Q : $\begin{pmatrix} 2-2\lambda \\ 2-3\lambda \\ 4-6\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$

$\Rightarrow \ 2-2\lambda + 2-3\lambda + 4-6\lambda = 1$

$\Rightarrow \ \lambda = \frac{7}{11}$

$\therefore \ \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix} + \frac{7}{11} \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} \frac{8}{11} \\ \frac{1}{11} \\ \frac{2}{11} \end{pmatrix}$

$Q\left(\frac{8}{11}, \frac{1}{11}, \frac{2}{11}\right)$

$l_{RS}: \ \underline{r} = \begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix}, \ \mu \in \mathbb{R}$

$\Pi_{bottom}: \ \underline{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9$

To find R :

$\begin{pmatrix} -5-2\mu \\ -6-3\mu \\ -7-6\mu \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -9$

$-18 - 11\mu = -9 \ \Rightarrow \ 11\mu = -9$

$\Rightarrow \mu = -\frac{9}{11}$

$\overrightarrow{OR} = \begin{pmatrix} -5 \\ -6 \\ -7 \end{pmatrix} - \frac{9}{11} \begin{pmatrix} -2 \\ -3 \\ -6 \end{pmatrix} = \begin{pmatrix} -\frac{37}{11} \\ -\frac{39}{11} \\ -\frac{23}{11} \end{pmatrix}$

$\therefore \ R\left(-\frac{37}{11}, -\frac{39}{11}, -\frac{23}{11}\right)$

(ii)

$$\cos\theta = \frac{\left|\begin{pmatrix}-2\\-3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{49}\ \sqrt{3}} = \frac{11}{\sqrt{49}\sqrt{3}}$$

$$= \frac{11}{7\sqrt{3}}$$

$$\vec{QR} = \vec{OR} - \vec{OQ} = \begin{pmatrix}-37/11\\-39/11\\-23/11\end{pmatrix} - \begin{pmatrix}8/11\\1/11\\2/11\end{pmatrix} = \begin{pmatrix}-45/11\\-40/11\\-25/11\end{pmatrix}$$

$$= -\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}$$

$$\cos\beta = \frac{\left|\begin{pmatrix}9\\8\\5\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right|}{\sqrt{170}\ \sqrt{3}} = \frac{22}{\sqrt{510}}$$

(iii)

$$\text{Thickness} = QR\cos\beta$$

$$= \left|-\frac{5}{11}\begin{pmatrix}9\\8\\5\end{pmatrix}\right| \ \frac{22}{\sqrt{510}}$$

$$= \frac{5}{11}\sqrt{170}\ \frac{22}{\sqrt{170}\sqrt{3}}$$

$$= \frac{10}{\sqrt{3}} \text{ units.}$$

OR $\left|\vec{QR}\cdot\widehat{\begin{pmatrix}1\\1\\1\end{pmatrix}}\right|$

(iv)

$\cos\theta = \dfrac{11}{7\sqrt{3}}$, $\cos\beta = \dfrac{22}{\sqrt{510}}$

$\theta = 24.86995$ $\quad\beta = 13.04924$

$\sin\theta = k\sin\beta$

$k = 1.86$ (to 3s.f.)

(v)

If $\theta < \beta$,

$0 < \sin\theta < \sin\beta$

$\Rightarrow \dfrac{\sin\theta}{\sin\beta} < 1$

$\Rightarrow \quad k < 1$

* Try values to deduce :

eg $\theta = 45°$, $\beta = 60°$

$\sin 45° < \sin 60°$

$k = \dfrac{\sin 45°}{\sin 60°}$

$= 0.816 < 1$