Chapter 7 GRAVITATIONAL FIELD

Content

- Gravitational force between point masses
- Gravitational field
- Gravitational field of a point mass
- Gravitational field of a uniform sphere
- Gravitational field near to the surface of the Earth
- Gravitational potential energy and gravitational potential
- Escape velocity
- Rotation of Earth and circular orbits

Learning Outcomes

Candidates should be able to:

(a) show an understanding of the concept of a gravitational field as an example of field of force and define gravitational field strength at a point as the gravitational force exerted per unit mass placed at that point.

(b) recognise the analogy between certain qualitative and quantitative aspects of gravitational and electric fields *(to be taught in Year 6)*.

(c) recall and use Newton's law of gravitation in the form $F = \frac{Gm_1m_2}{r^2}$. ***(in H1)***

(d) derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass.

(e) recall and apply the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems.

(f) show an understanding that near the surface of the Earth g is approximately constant and equal to the acceleration of free fall.

(g) define the gravitational potential at a point as the work done per unit mass in bringing a small test mass from infinity to that point.

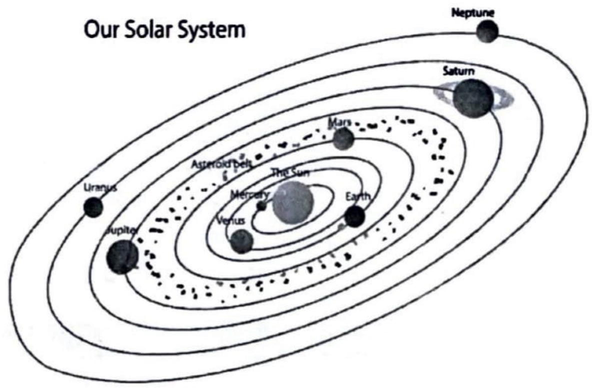
(h) solve problems using the equation $\phi = -\frac{GM}{r}$ for the gravitational potential in the field of a point mass.

(i) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes. ***(in H1)***

(j) show an understanding of geostationary orbits and their application. ***(in H1)***

7 Introduction

In 1687, Isaac Newton proposed in his ***Philosophiæ Naturalis Principia Mathematica*** that every mass attracts another mass with a force of gravity. According to him, two seemingly unrelated phenomena – the fall of an apple from an apple tree and the orbital motion of the planets around the Sun – are due to the same reason: gravitational attraction. He came up with the Newton's Law of Gravitation. This law is universally valid and applies to any planet in the solar system and even between distant galaxies.



7.1 Gravitational Force F

7.1.1 Newton's Law of Gravitation

Newton's Law of Gravitation

Definition

Newton's law of gravitation states that two point masses attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

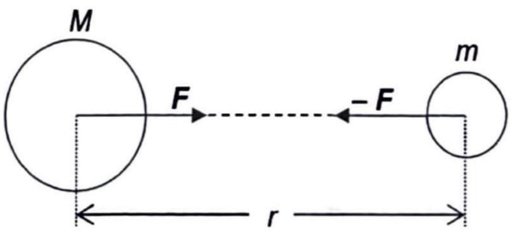
The magnitude of the gravitational force F between two particles of masses M and m which are separated by a distance r is given by

Formula

$$F = \frac{GMm}{r^2}$$

where G is the gravitational constant with a value of **6.67×10^{-11} N m^2 kg^{-2}**.

Gravitational force is a **vector** quantity.
S.I. unit for gravitational force is the **newton (N)**.



Note:

- This law is applicable only between point masses.
  - Every spherical object with constant density can be considered as a point mass at the centre of the sphere.
- The gravitational forces between two point masses are equal and opposite. They
  - constitute an action and reaction pair of forces.
  - are attractive in nature, and
  - always act along the line joining the two point masses.
- Newton's law of gravitation is an example of an inverse square law
  - because the magnitude of the force varies inversely with the square of the separation of the particles.
- Gravitational force is attractive in nature.
  - Some books indicate this with a negative sign in the formula.

Example 1

A man of mass 85.0 kg is standing on the surface of the Earth. The Earth has a mass of 5.98×10^{24} kg and a radius of 6.37×10^{6} m. Calculate the force that the Earth exerts on the man. Deduce the force that the man exerts on Earth.

Solution: Force that the Earth exerts on the man,

$$F_{man} = \frac{GM_E m}{r_E^{\,2}}$$

$$= \frac{(6.67 \times 10^{-11}) \times (5.98 \times 10^{24}) \times (85.0)}{(6.37 \times 10^{6})^2}$$

By Newton's third law,
the force that the man exerts on the Earth (F_{Earth})
is equal in magnitude to the
force that the Earth exerts on the man (F_{man}),
but in the opposite direction.

i.e. $|F_{Earth}| = |F_{man}|$

Example 2

[NJC/H2/ Prelims 2010/P1/16 – modified]

On the surface of the Earth, the gravitational force acting on an object is 45 N. When the object is at a height h above the surface, the gravitational force acting on it is 5 N. If R is the radius of the earth, calculate the value of h in terms of R.

Solution:

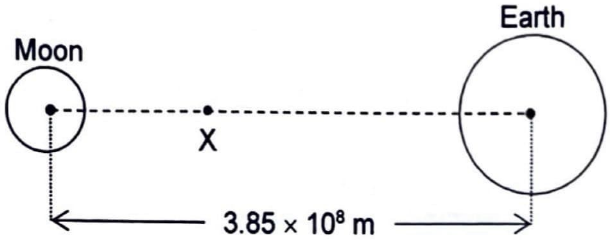
$$F_1 = \frac{GMm}{R^2} = 45\text{ N}$$

$$F_2 = \frac{GMm}{(R+h)^2} = 5\text{ N}$$

$$\frac{F_1}{F_2} = \left(\frac{R+h}{R}\right)^2 = 9$$

$$\Rightarrow h = 2R$$

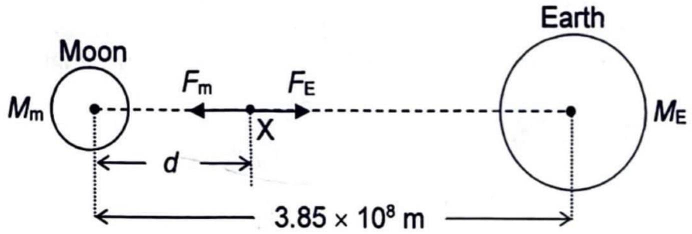
Example 3 The mass of the Earth is 5.98×10^{24} kg and that of the Moon is 7.35×10^{22} kg. A spacecraft travelling from the Earth to the Moon will reach a point X where it experiences no resultant gravitational force. The distance between the centre of the Earth and the centre of the Moon is 3.85×10^{8} m.



(a) Draw a free-body diagram of the spacecraft at X.
(b) Calculate the distance from X to the centre of the Moon.
(c) Describe the motion of the spacecraft before reaching X and after passing X.

Solution:

(a)



d is the distance from X to the centre of the Moon.
m is the mass of the spacecraft.
M_E is the mass of the Earth.
M_m is the mass of the Moon.
F_E is the force on the spacecraft by the Earth.
F_m is the force on the spacecraft by the Moon.

b)

$$F_m = F_E$$

$$\frac{GmM_m}{d^2} = \frac{GmM_E}{(3.85\times10^8 - d)^2}$$

$$\frac{7.35\times10^{22}}{d^2} = \frac{5.98\times10^{24}}{(3.85\times10^8 - d)^2}$$

$$d = 3.84\times10^7 \text{ m}$$

c) **(c)** Before reaching X, $F_E > F_M$.
Resultant force on the spacecraft opposes its motion, thus slowing it down.

7.2 Gravitational Field Strength g

7.2.1 Gravitational Field

Gravitational Field

Definition

A **gravitational field** is a region of space in which a mass placed in that region experiences a gravitational force.

Gravitational Field Lines

A region of gravitational field can be visualised as consisting of an array of imaginary field lines. The gravitational force on a mass placed at a point on a field line acts along the tangent at that point.

- The direction of the field lines indicates the direction of the gravitational field.
- The density of the field lines indicates its strength.
  - A region with a stronger gravitational field strength will have closer or denser field lines.
- For a point mass or a uniform spherical mass, the field lines are directed towards its centre.
- Fig. A shows the field lines around Earth, directed towards its centre. Zooming into a region near the surface of Earth, the field lines seem to be parallel to each other and evenly spaced as shown in Fig. B.
- Hence, near the surface of the Earth, we can consider the gravitational field to be approximately uniform.

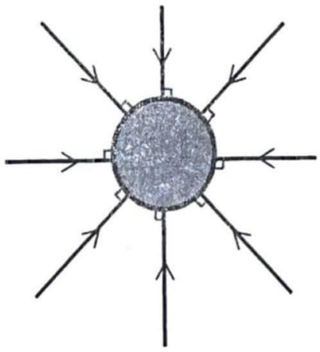


Fig. A: Field lines around Earth

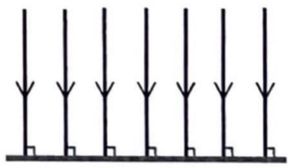


Fig. B: Field lines at surface of Earth

Gravitational Field Strength, g

Definition

The **gravitational field strength** at a point in space is defined as the gravitational force experienced per unit mass at that point.

Formula

$$g = \frac{F}{m}$$

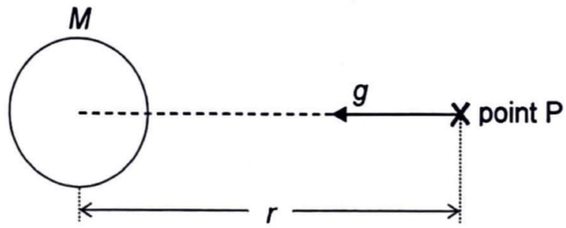
7.2.2 Gravitational Field Strength of a Point Mass

g for a Point Mass

Since the gravitational force between two point masses is given by $F = \frac{GMm}{r^2}$,

$$g = \frac{GM}{r^2}$$

Gravitational field strength is a **vector** quantity.
Gravitational field strength is also known as **gravitational acceleration** or **free fall acceleration.**
S.I. unit for gravitational field strength is **N kg^{-1}** or **m s^{-2}**.



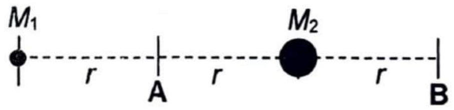
Note:

- Gravitational field strength points towards the mass which created it.
  - Some books indicate this by including a negative sign, just like gravitational force.

- The resultant field strength at a point due to more than one mass can be found from the **vector sum** of the individual gravitational field strengths due to each mass at that point.

Example 4

Determine the resultant gravitational field strength at points A and B due to the two masses, given that $M_2 > M_1$.



Solution:

At point A: due to M_1: $g_1 = \frac{GM_1}{r^2}$ (towards M_1)

due to M_2: $g_2 = \frac{GM_2}{r^2}$ (towards M_2)

Since $M_2 > M_1$, $g_A = g_2 - g_1 = \frac{GM_2}{r^2} - \frac{GM_1}{r^2}$ (towards M_2)

At point B: due to M_1: $g_1 = \frac{GM_1}{(3r)^2}$ (towards M_1)

due to M_2: $g_2 = \frac{GM_2}{r^2}$ (towards M_2)

$g_B = g_1 + g_2 = \frac{GM_1}{9r^2} + \frac{GM_2}{r^2}$ (towards M_2 and M_1)

7.2.3 Gravitational Field Strength of a Uniform Sphere

Shell Theorem

The **shell theorem** gives gravitational simplifications that can be applied to objects inside or outside a spherically symmetrical body.

Isaac Newton proved the shell theorem and stated that:

1. A spherically symmetric body affects external objects gravitationally as though all of its mass were concentrated at a point at its centre.
2. If the body is a spherically symmetric shell (i.e. a hollow ball), no net gravitational force is exerted by the shell on any object inside, regardless of the object's location within the shell.

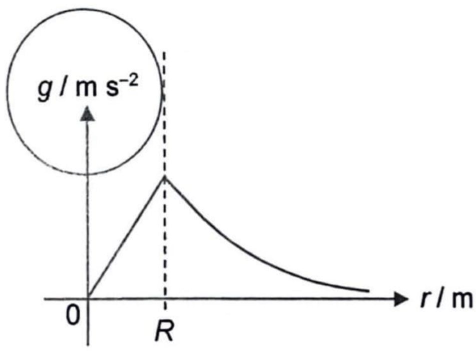
A result of the shell theorem is that inside a solid sphere of constant density, the gravitational force varies linearly with distance from the centre, becoming zero by symmetry at the centre of mass.

This can be seen as follows: take a point within such a sphere, at a distance from the centre of the sphere. You can then ignore all the shells of greater radius, according to the shell theorem. Only the mass of the sphere within this radius will have effect on the gravitational force at that point. *[Refer to Appendix A]*

These results were important to Newton's analysis of planetary motion.

For uniform spheres, planets and stars, we can use $F = \frac{GMm}{r^2}$ and $g = \frac{GM}{r^2}$ to calculate the gravitational forces and gravitational field strengths outside of them.

Variation of *g* with *r*



For a uniform solid sphere of radius R,

Inside the sphere ($r \le R$):

- The mass M of the inner sphere of radius r is $M = \rho V = \rho\left(\frac{4}{3}\pi r^3\right)$.
- The gravitational field strength due to the inner sphere varies linearly with distance r (from the shell theorem):

$$g = \frac{GM}{r^2} = \frac{G}{r^2}\left(\frac{4}{3}\rho\pi r^3\right) = \frac{4}{3}G\rho\pi r$$

Outside the sphere ($r \geq R$):

- The gravitational field strength due to a uniform sphere is identical to that of a point mass at the centre of the sphere. Therefore, $g = \frac{GM}{r^2}$.

In summary:

- The gravitational field strength inside a hollow sphere is zero.
- The gravitational field strength inside a uniform solid sphere varies linearly with distance r from its centre.
- The gravitational field strength outside a uniform solid sphere or hollow sphere varies inversely with r^2.
- The direction of gravitation field strength acts towards the centre of mass M.

7.3 Gravitational Potential Energy U Gravitational Potential ϕ

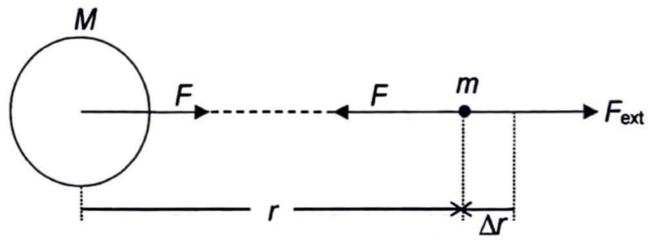
7.3.1 Gravitational Potential Energy U

Gravitational Potential Energy, *U*

Definition

The **gravitational potential energy** of a mass at a point in a gravitational field is defined as the work done by an external force in bringing the mass from infinity to that point.

Consider a system of two particles of masses *M* and *m*.



Mass *M* is fixed and mass *m* is free to move. The gravitational force *F* between the masses is attractive.

At infinity, the gravitational force due to mass *M* is zero. The **point at infinity** (i.e. $r = \infty$) is **defined to have zero gravitational potential energy**.

The work that must be done by an external force F_{ext} to move *m* from ∞ to *r* is

$$W = \int_{\infty}^{r} F_{ext} dr = \int_{\infty}^{r} \frac{GMm}{r^2} dr = \left[-\frac{GMm}{r}\right]_{\infty}^{r} = -GMm\left[\frac{1}{r} - \frac{1}{\infty}\right] = -\frac{GMm}{r}$$

RECALL: initial energy + work done by an external force = final energy

Since the initial gravitational potential energy of *m* at infinity is zero, the gravitational potential energy *U* of *m* at *r* must be given by $-\frac{GMm}{r}$. (Deduced from 0 + $W = U$)

Hence, the gravitational potential energy *U* of two particles of masses *M* and *m* separated by a distance *r* is given by

Formula

$$U = -\frac{GMm}{r}$$

Gravitational potential energy is a **scalar** quantity.
S.I. unit for gravitational potential energy is the **joule (J)**.

Note:

- Due to its definition, the **negative sign must be indicated**.
- For small distances *h* above the surface of the Earth, $U \approx mgh$.
  [Refer to Appendix B]

Relationship between Force and Potential Energy

We have learnt that for a field of force, the relationship between the force F and the potential energy U for one-dimensional motion is given by $F = -\frac{dU}{dx}$.

Hence, the gravitational force F acting on a mass m in the presence of another mass M is thus the **negative of the gravitational potential energy gradient.**

F and gravitational potential energy U is related by the following expression:

Formula

$$F = -\frac{dU}{dr}$$

- The magnitude of the force at point r is equal to the gradient of the potential energy curve at r.
- The negative sign indicates that the force points in the direction of decreasing potential energy.

Note: $-\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) = GMm\frac{d}{dr}\left(\frac{1}{r}\right) = GMm\left(-\frac{1}{r^2}\right) = -\frac{GMm}{r^2} = F$

7.3.2 Gravitational Potential ϕ

Gravitational Potential, ϕ

Definition

The **gravitational potential** at a point in a gravitational field is defined as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

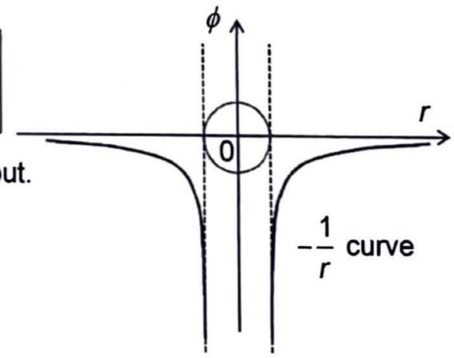
Formula

$$\phi = \frac{U}{m}$$

At a distance r from a point mass M, $U = -\frac{GMm}{r}$, therefore

Formula

$$\phi = -\frac{GM}{r}$$



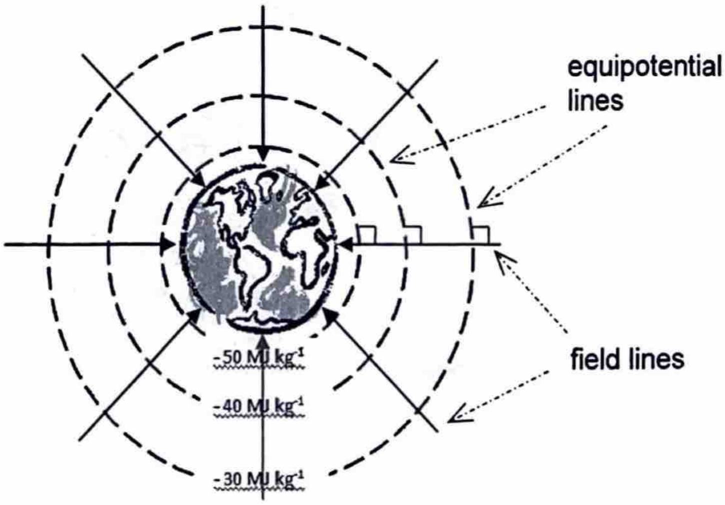
where the mass m of the small test mass cancels out.

Gravitational potential is a **scalar** quantity.
S.I. unit for gravitational potential is **J kg^{-1}**.

Note:

- The **negative sign** is not an indication of direction and it **cannot be omitted**.
- **At infinity, the gravitational potential is defined to be zero.**
  ⇒ This means that the potential has the highest value at this point (at ∞) and potential is negative at other locations.
  ⇒ As an object moves away from infinity towards mass M, the potential decreases (i.e. becomes more negative).

- The gravitational potential at a point due to two or more masses can be found by adding the individual potentials at that point due to each mass.
- Points at a fixed distance from a mass form an equipotential surface.
- Equipotential surfaces of equal interval are further apart as we move away from the mass.



NOTE!!

Why is gravitational potential (or potential energy) negative in value?

- Gravitational potential (or potential energy) at infinity is zero.
- Since gravitational force is attractive in nature, to bring a mass from infinity to a point in the gravitational field, the direction of the external force is opposite to the direction of displacement of the mass. (We can imagine the external force supporting the mass as it is being lowered towards the Earth.)
- This results in negative work done by the external force.
- Hence, based on its definition, gravitational potential (or potential energy) has a negative value.

Relationship between Field Strength and Potential

Formula

Gravitational field strength g is the **negative of the gravitational potential gradient.**

g and gravitational potential ϕ is related by the following expression:

$$g = -\frac{d\phi}{dr}$$

The negative sign indicates that the field strength points in the direction of decreasing potential.

Note: $F = -\frac{dU}{dr} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) \Rightarrow \frac{F}{m} = -\frac{d}{dr}\left(-\frac{GM}{r}\right) \Rightarrow g = -\frac{d\phi}{dr}$

Example 5

The mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^{6} m. Calculate the change in the gravitational potential as an object is moved from the surface of the Earth to a point 1200 m above the surface.

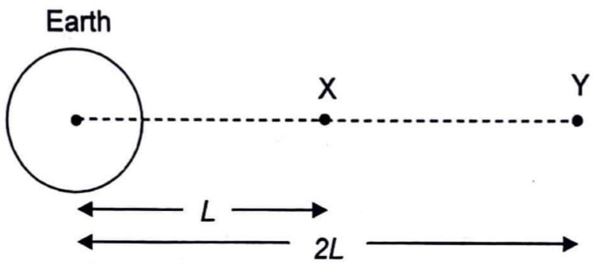
Solution:

$$\Delta\phi = \phi_f - \phi_i$$
$$= \left(-\frac{GM}{r}\right) - \left(-\frac{GM}{r_E}\right)$$
$$= GM\left(\frac{1}{r_E} - \frac{1}{r}\right)$$
$$= (6.67 \times 10^{-11})(5.98 \times 10^{24})\left[\frac{1}{6.37 \times 10^{6}} - \frac{1}{6.37 \times 10^{6} + 1200}\right]$$
$$= 1.18 \times 10^{4}\ \text{J kg}^{-1}$$

Example 6

[SAJC/H2/ Prelims 2010/P1/13 – modified]

The figure shows two points X and Y at distances L and $2L$ from the centre of the Earth. The gravitational potential at X is –8.0 kJ kg^{-1}.



(a) Determine the work done when a 2.0 kg mass is taken from X to Y.
(b) Explain whether the work done in **(a)** depends on the actual path taken from X to Y.

Solution:

(a) $\phi_X = -\frac{GM}{L} = -8.0\ \text{kJ kg}^{-1}$

$$\phi_Y = -\frac{GM}{2L} = -\frac{GM}{L}\left(\frac{1}{2}\right) = -4.0\ \text{kJ kg}^{-1}$$

work done on mass $= \Delta U = m\Delta\phi = m(\phi_Y - \phi_X)$

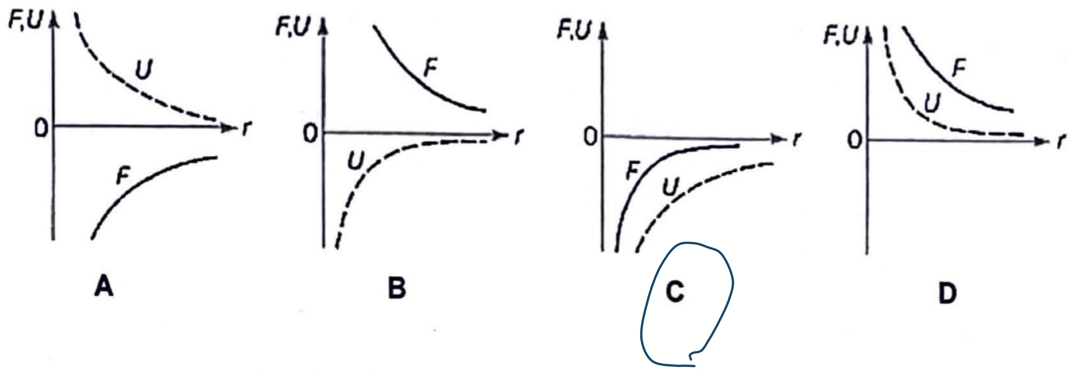
$$= 2[(-4.0) - (-8.0)] = +8.0\ \text{kJ}$$

(b) No, the work done does not depend on the actual path taken. It depends only on the initial and final positions in the gravitational field.

Example 7

[N95/I/7]

Which diagram shows the variation of gravitational force *F* on a point mass, and of gravitational potential energy *U* of the mass, with its distance *r* from another point mass?



Solution:

$F = -\frac{du}{dx}$

Simce the gradient at each point on the U-r graph is positive, F will always be negative.

7.3.3 ϕ–r Graph and g–r Graph between Two Masses

Potential and Field Strength between Two Masses

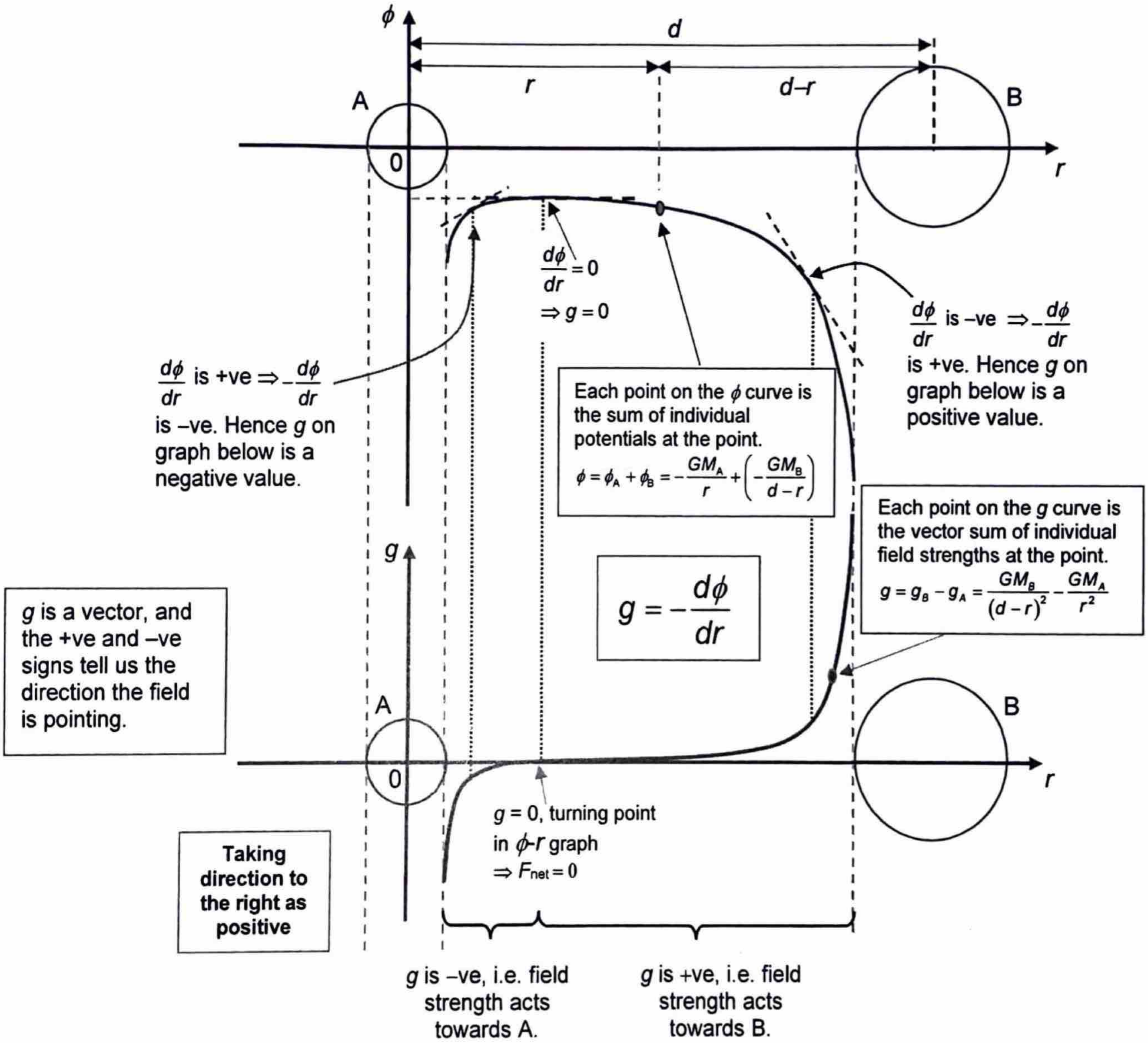
The potential ϕ at a point between two masses is the algebraic sum (i.e. scalar addition) of the individual potentials at that point due to each mass.

The field strength g at a point between two masses is the vector sum (i.e. vector addition) of the individual field strengths at that point due to each mass.

g and ϕ is related by: $g = -\frac{d\phi}{dr}$. The negative sign indicates that the field strength points in the direction of decreasing potential, as g is a vector.

Graphical Representation

The variation with displacement r, measured from the centre of A, of the total gravitational potential ϕ between the surfaces of masses A and B along the line joining their centres is shown below.



Corresponding variation with displacement r, measured from the centre of A, of the resultant gravitational field strength g between the surfaces of A and B along the line joining their centres is shown above.

7.4 Escape Velocity

Escape Velocity

Have you ever wondered if it is possible to throw an object into the air at such a high velocity that it escapes the Earth's gravitational pull and never falls back to Earth? This is actually plausible because the acceleration of free fall does not remain at 9.81 m s^{-2} but decreases as the object moves away from the Earth.

In order for the object to escape from the Earth's gravitational influence and reach infinity, it has to be thrown with a minimum speed.

NOTE!!

The **minimum speed** needed for the object to *just* escape from the gravitational influence of a massive body is called **escape velocity or escape speed.**

To determine the escape velocity of an object of mass *m*, let us consider the amount of energy the object needs in order to escape from the Earth's gravitational influence and *just* reach infinity.

At infinity, the object's G.P.E. (E_P) is defined as zero. If the object has sufficient energy to *just* reach infinity, its K.E. (E_K) at infinity is zero. So the total energy (E_T) of the object at infinity is given by $E_T = E_P + E_K = 0$.

NOTE!!

From the principle of conservation of energy, the total energy of an object should remain unchanged throughout its motion. This means that any object with a total energy of zero will be able to *just* reach infinity and stop there.

Any object with a total energy of more than zero will be able to go beyond infinity, and completely escape the Earth 's gravitational field.

Consider an object (e.g. a rocket) of mass *m* being projected vertically with a velocity *v* from the surface of the Earth.

Its initial G.P.E. and K.E. are:

$$E_P = U = -\frac{GMm}{R_{Earth}}$$

$$E_K = \frac{1}{2}mv^2$$

The energy required for the object to reach infinity and beyond is such that

Formula

$$E_T \geq 0$$

$$E_P + E_K \geq 0$$

$$-\frac{GMm}{R_{Earth}} + \frac{1}{2}mv^2 \geq 0$$

Therefore, the speed required for the object to reach infinity and beyond is

$$v \geq \sqrt{\frac{2GM}{R_{Earth}}}$$

$$v \geq \sqrt{2gR_{Earth}}$$

where the gravitational field strength at the Earth's surface $g = \frac{GM}{{R_{Earth}}^2}$

Hence the **escape velocity** or the minimum speed required for the object to *just* reach infinity is

$$v = \sqrt{\frac{2GM}{R_{\text{Earth}}}} = \sqrt{2gR_{\text{Earth}}}$$

Note:

- The escape velocity is independent of the mass of the object.
- The escape velocity for Earth is 11.2 km s^{-1}.

Example 8

[RI/H2/ Prelims 2010/P2/3 – modified]

The mass of the Earth is 5.98 × 10^{24} kg and its radius is 6370 km. Determine the minimum kinetic energy and escape speed required to project a spacecraft of mass 2550 kg from the surface of the Earth so that it completely escapes from the gravitational attraction of the Earth. Ignore air resistance.

Solution:

At infinity, $E_p = 0$ and $E_k = 0$ with minimum E_k.

By POCOE,

E_T on surface = E_T at infinity

$$E_s = E_\infty$$

$$\left(-\frac{GM_EM_s}{R_E}\right) = E_\infty$$

$$\left(-\frac{GM_EM_s}{R_E}\right) + E_{k\,min} = 0$$

$$\text{Min } E_k = \frac{GM_EM_s}{R_E} = \frac{(6.67\times10^{-11})(5.98\times10^{24})(2550)}{6370\times10^3} = 1.60\times10^{11}\ \text{J}$$

$$\frac{1}{2}M_s V_{min}^2 = 1.60\times10^{11}$$

Escape speed, V_{min}

$$= \sqrt{\frac{2(1.60\times10^{11})}{2550}} = 1.12\times10^4\ \text{ms}^{-1}$$

7.5 Circular Motion: Rotation of Earth & Obits

Circular Motion (Recap)

For any object of mass m to move in uniform circular motion of radius r, there must be a resultant force F_c acting on the object which is directed towards the centre of the circle. This is known as the centripetal force, which can be expressed as

$$F_c = ma_c = \frac{mv^2}{r} = mr\omega^2$$

where v is the linear velocity and ω is the angular velocity.

The time taken for the object to make one complete circle is known as the period T, which is given by

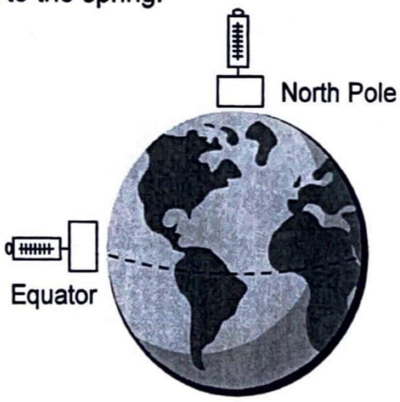
$$T = \frac{2\pi}{\omega} = \frac{2\pi r}{v}$$

7.5.1 Rotation of Earth: Effect on its Gravitational Field Strength

Apparent weight and free fall acceleration due to Earth's Rotation

The figure shows an object of mass m hanging from a spring balance at two places on the Earth – the Equator and the North Pole.

In both cases there are two forces acting on the object – gravitational force F_g (true weight) and force T due to the spring.



If the Earth is taken to be a uniform sphere, then F_g is the same at both the equatorial and polar regions. However, T will be different because the object undergoes circular motion at the Equator.

Resultant force F_R at the polar region:
By Newton's second law, resultant force

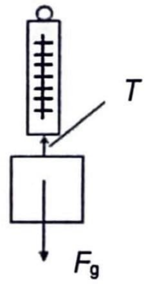
$$F_R = F_g - T$$

Since there is no circular motion,

$$F_R = 0$$

Therefore,

$$T = F_g$$



The spring balance indicates F_g , which is the true weight of the object.

Hence,

$$mg_{freefall} = mg$$
$$g_{freefall} = g$$

The free fall acceleration $g_{freefall}$ at the polar region is the true gravitational field strength g of the Earth at that region.

Resultant force F_R at the Equator:
By Newton's second law, resultant force

$$F_R = F_g - T$$

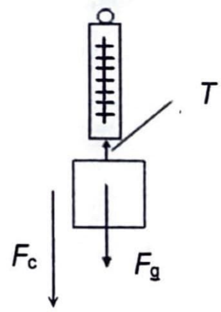
Since the object undergoes circular motion,

$$F_R = F_c = F_g - T$$

where F_c is the centripetal force.

Therefore,

$$T = F_g - F_c$$



The spring balance indicates a value <u>less</u> than the true weight F_g of the object. This value is known as the **apparent weight.**

Hence,

$$mg_{freefall} = mg - ma_c$$
$$g_{freefall} = g - a_c$$

where $g_{freefall}$ is the free fall acceleration,
a_c is the centripetal acceleration.

The free fall acceleration $g_{freefall}$ at the Equator is <u>less</u> than the true gravitational field strength g of the Earth at the Equator.

NOTE!!

At the Equator, gravitational force is responsible for the centripetal force to keep the object moving in a uniform circular motion and for accelerating the object towards Earth (acceleration of free fall).

Example 9

The radius of the Earth is 6370 km and the mass of the Earth is 5.98 × 10^{24} kg.

(a) Calculate the centripetal acceleration of an object placed at the Equator.

(b) If a 5.0 kg mass is placed on a weighing scale at the Equator, what would the scale read in newtons?

Solution:

(a) $a_c = r\omega^2$

$= r\left(\frac{2\pi}{T}\right)^2$

$= (6370\times10^3)\left(\frac{2\pi}{24\times60\times60}\right)^2$

$= 0.03369$

$= 0.0337 \text{ m s}^{-2}$

(b) Resultant force on mass = $F_g - N$. This resultant force provides for the <u>centripetal force</u> on the mass as it goes round in circular motion at the Equator.

g at the Equator $= \frac{GM_E}{R_E^{\,2}}$

$= \frac{(6.67\times10^{-11})(5.98\times10^{24})}{(6370\times10^3)^2} = 9.83 \text{ m s}^{-2}$

$F_g - N = ma_c$

$N = F_g - ma_c = mg - ma_c = 5.0(9.83 - 0.0337) = 48.98 = 49.0 \text{ N}$

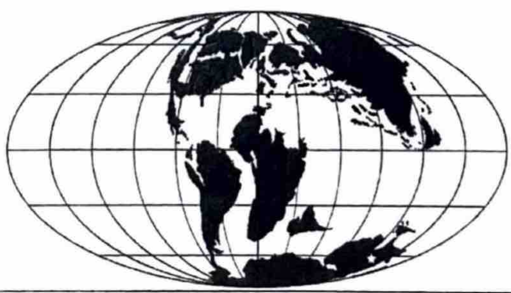
By Newton's third law:

$|\text{force on scale by mass}| = |\text{force on mass by scale}| = 49.0 \text{ N}$

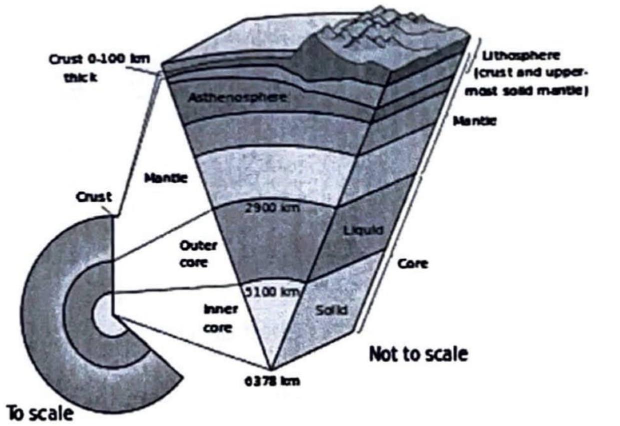
Other Factors Affecting Gravitational Field Strength of Earth

Near the Earth's surface, the gravitational field strength is approximately constant and is equal to the acceleration of free fall of 9.81 m s^{-2}, which is an average value. The precise strength of Earth's gravity varies with location.

1. The gravitational field strength over the Earth's surface varies as the result of its lack of spherical symmetry. Its polar diameter is about 40 km less than its equatorial diameter.



2. The Earth's density is not uniform. There are local variations in the Earth's gravitational field caused by mountains and trenches, as well as the presence of minerals and oil deposits.



Due to the combination of the non-spherical symmetry of Earth and the rotation of the Earth, the effective gravitational field strength at sea level increases from about 9.780 m s^{-2} at the Equator to about 9.832 m s^{-2} at the poles. So an object will weigh about 0.5 % more at the poles than at the Equator.

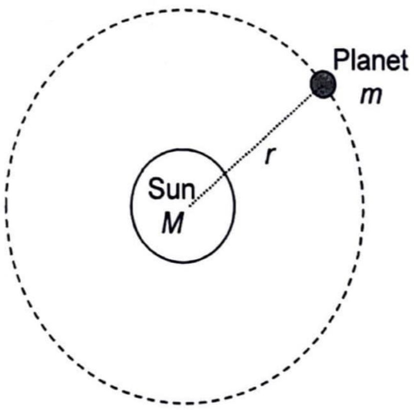
7.5.2 Circular Orbits

7.5.2.1 Planetary Motion

Kepler's Third Law

Kepler's third law shows that there is a precise mathematical relationship between a planet's distance from the Sun and the amount of time it takes to revolve around the Sun. It was this law that inspired Newton, who came up with three laws of his own to explain why the planets move as they do.

Consider a planet of mass m moving with linear speed v in a circular orbit of radius r about the Sun of mass M.



The only force acting on the planet is the gravitational force on the planet by the Sun. Hence, this **gravitational force F_g provides the centripetal force F_c** necessary for the circular motion of the planet round the Sun.

Applying Newton's second law of motion,

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = mr\omega^2$$

$$G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \frac{4\pi^2}{GM}r^3$$

$$T^2 \propto r^3$$

where T is the orbital period and r is the orbital radius.

This relationship is known as Kepler's third law which states that the square of the period of revolution of the planets are directly proportional to the cubes of their mean distances from the Sun.
The period is independent of the mass that is orbiting around the Sun.

Kepler's third law may also be applied to orbiting satellites. For planets or satellites describing circular orbits about the same central body, the square of the period is proportional to the cube of the radius of the orbit.

Example 10

[MI/H2/ Prelims 2010/P1/14d – modified]

Mercury is 5.79×10^{10} m from the Sun and it takes 0.241 Earth years for Mercury to make one revolution around the Sun. If Neptune is 450×10^{10} m from the Sun, calculate the period, in Earth years, of Neptune around the Sun.

Solution:

The gravitational force on the planet provides the centripetal force.

$$F_g = F_c$$

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\frac{GMm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$T^2 = \frac{4\pi^2}{GM}r^3$$

Since both Mercury and Neptune are orbiting about the same central body, the Sun,

$$T^2 \propto r^3$$

$$\left(\frac{T_N}{T_M}\right)^2 = \left(\frac{r_N}{r_M}\right)^3$$

$$\left(\frac{T_N}{0.241\text{ yr}}\right)^2 = \left(\frac{450\times10^{10}}{5.79\times10^{10}}\right)^3$$

$$T_N = 165\text{ yrs}$$

7.5.2.2 Satellite Motions

What is a satellite?

A satellite is a moon, planet or machine that orbits a planet or star. E.g, Earth is a satellite because it orbits the Sun. Likewise, the moon is a satellite because it orbits Earth. Usually, the word "satellite" refers to a machine that is launched into space and moves around Earth or another body in space.

Earth and the moon are examples of natural satellites. Thousands of artificial, man-made satellites orbit Earth. Some take pictures of the planet that help meteorologists predict weather and track hurricanes, others take pictures of other planets, the Sun, black holes, dark matter or faraway galaxies, which help scientists better understand the solar system and universe.

Other satellites are used mainly for communications, such as beaming TV signals and phone calls around the world. A group of more than 20 satellites make up the Global Positioning System, or GPS. If you have a GPS receiver, these satellites can help figure out your exact location.

Energy of a Satellite

Consider a satellite of mass *m* orbiting with a speed *v* round the Earth of mass *M* in a circular orbit of radius *r*.

Since the satellite is in circular orbit, **the gravitational force on it provides the centripetal force.**

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

The kinetic energy E_K of the satellite is

$$E_k = \frac{1}{2}mv^2$$

Formula

$$E_k = \frac{1}{2}\frac{GMm}{r}$$

The potential energy E_P of the satellite is

Formula

$$E_p = -\frac{GMm}{r}$$

The total energy E_T of the satellite is the sum of the kinetic energy and potential energy of the satellite.

$$E_T = E_P + E_K$$

$$E_T = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

Formula

$$E_T = -\frac{1}{2}\frac{GMm}{r}$$

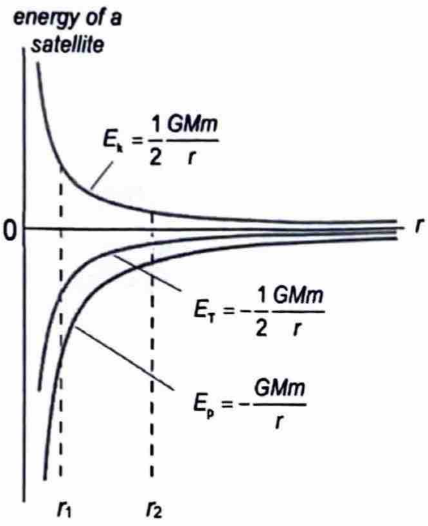
Objects that are bounded have a negative total energy.
Objects with $E_T \geq 0$ will be able to reach infinity and escape.

The graph shows the variation with r, the different forms of energy of a satellite in a circular orbit.

A satellite orbiting a massive object is a bound system.

Total energy in a bound system is negative, reflecting the fact that the satellite cannot escape.

Points on the graph along the dotted lines r_1 and r_2 indicate the corresponding values of the different energy forms that the satellite possesses at orbit radii of r_1 and r_2 respectively.



Example 11 The Earth is a uniform sphere of radius 6370 km and mass 5.98×10^{24} kg.

(a) An object orbits at an altitude of 300 km above the Earth. Determine its linear velocity.

(b) Briefly explain why the orbital plane of any satellite includes the centre of the Earth.

Solution:

a) F_G provides for F_C.

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{GM}{r}}$$

$$= \sqrt{\frac{(6.67\times10^{-11})(5.98\times10^{24})}{(6370\times10^3)+(300\times10^3)}}$$

$$= 7.33\times10^3 \text{ ms}^{-1}$$

b)
- The **gravitational force** on the satellite due to the Earth provides the required **centripetal force** for the satellite to move in a circular orbit around the Earth.
- Since the **centripetal force is directed towards the centre** of the Earth, the circular orbital plane must contain the centre of the Earth *(which is also the centre of the circular orbit).*

Geostationary Orbits

A **geostationary satellite** is one that remains at a fixed position in the sky as viewed from any location on the Earth's surface.

A geostationary satellite must satisfy the following conditions:

1. Its orbital period is the same as that of the Earth about its axis of rotation (24 hrs).

2. Its direction of rotation is the same as that of the Earth about its axis of rotation (from West to East).

3. Its plane of orbit lies in the same plane as the equator (i.e. the satellite is above the equator).

To determine the radius *r* of a geostationary orbit:
The gravitational force on the satellite provides the centripetal force.

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = mr\omega^2 \quad \Rightarrow \quad G\frac{Mm}{r^2} = mr\left(\frac{2\pi}{T}\right)^2$$

$$r = \left(\frac{T^2GM}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= \left(\frac{(24\times60\times60)^2\times(6.67\times10^{-11})\times(5.98\times10^{24})}{4\pi^2}\right)^{\frac{1}{3}}$$

$$= 42250\text{ km}$$

The height of the orbit above the Earth's surface = 42250 – 6370 = 35880 km.

Advantages of a geostationary satellite

1. There is continuous surveillance of the region under it.

2. It is easy for the ground station to communicate with it as it is permanently within view. Ground-based antennas can remain fixed in one direction. Hence, geostationary satellites are often used for communication purposes.

3. Due to the high altitude, the satellite can transmit and receive signals over a large area.

Disadvantages of a geostationary satellite

Its distance from the Earth's surface is large compared to the Low-Earth Orbit (LEO) satellites, which typically operate at an altitude of a few hundred kilometres.

This leads to

1. a significant loss of signal strengths,
2. poorer resolution in imaging satellites,
3. time-lag in telecommunication.

Example 12

Which of the following quantities are *not necessarily* the same for satellites that are in geostationary orbits around the Earth?

A. angular velocity
B. centripetal acceleration
C. kinetic energy
D. period of orbit
E. radius of orbit
F. mass
G. momentum

Solution: B, C, F, G

Example 13

[N99/III/2]

A satellite of mass 2400 kg is placed in a geostationary orbit at a distance of 4.23×10^7 m from the centre of the Earth.

(a) Calculate
1. the angular velocity of the satellite,
2. the speed of the satellite,
3. the acceleration of the satellite,
4. the force of attraction between the Earth and the satellite,
5. the mass of the Earth.

(b) Explain why a geostationary satellite
1. must be placed vertically above the equator,
2. must move from west to east.

(c) Explain why such satellites are often used for telecommunications.

Solution:

(a)

1. $\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{24 \times 60 \times 60} = 7.27 \times 10^{-5}$ rad s^{-1}

2. $v = r\omega = (4.23 \times 10^7) \times (7.27 \times 10^{-5}) = 3.08 \times 10^3$ m s^{-1}

3. $a = \dfrac{v^2}{r} = \dfrac{(3.08 \times 10^3)^2}{4.23 \times 10^7} = 0.224$ m s^{-2} OR $a = r\omega^2 = 0.224$ m s^{-2}

4. The gravitational force on the satellite provides the centripetal force.
$F_g = F_c = ma = (2400)(0.224) = 538$ N

5. $\dfrac{GMm}{r^2} = 538$

$M = \dfrac{538 \times (4.23 \times 10^7)^2}{(6.67 \times 10^{-11}) \times 2400} = 6.01 \times 10^{24}$ kg

(b) 1. Since the centripetal force on the satellite is directed towards the centre of the Earth, any circular orbit must have its centre at the centre of the Earth. If the satellite is not above the equator, the satellite must sometimes be over the northern hemisphere and sometimes over the southern hemisphere, and so cannot be geostationary.

2. For a satellite to stay above a fixed point, it must have the same direction of rotation as the Earth about its axis of rotation, i.e. from west to east.

(c) The earthbound transmitters and receivers can be aimed in a fixed direction, with no disruption to the signals.

7.5.2.3 Binary Star System

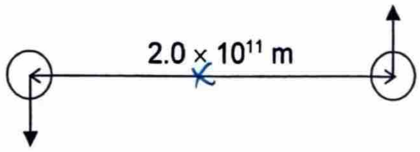
What is a binary star system?

A binary star system consists of two stars that are gravitationally bound. The two stars orbit around the center of mass of the system.

Astrophysicists find binary systems to be quite useful in determining the mass of the individual stars involved. When two objects orbit one another, their mass can be calculated very precisely by using Newton's calculations for gravity. The data collected from binary stars allows astrophysicists to extrapolate the relative mass of similar single stars.

Example 14

The figure below shows a binary star system which consists of two stars, each of mass 4.0×10^{30} kg, separated by 2.0×10^{11} m. The stars rotate about the centre of mass of the system.



(a) Label on the above diagram, with a letter X, a point where the gravitational field strength is zero. Explain why you have chosen this point.
(b) Determine the gravitational potential at X.
(c) Calculate the force on each star due to the other star.
(d) Calculate the linear speed of each star in the system.

Solution:

(a) X is at the mid-point between the centres of the two stars. At this point, the gravitational fields due to the stars have the same magnitude due to equal mass and equal distance from the centre of the stars. However, they act in opposite directions since gravitational field strength due to each star is attractive. Hence the vector sum of these two equal and opposite gravitational field strengths will give a resultant that is zero.

(b) $$\phi_X = \left(-\frac{GM}{r}\right) + \left(-\frac{GM}{r}\right) = 2\left(-\frac{(6.67\times10^{-11})(4.0\times10^{30})}{1.0\times10^{11}}\right) = -5.34\times10^{9}\ \text{J kg}^{-1}$$

(c) $$F = \frac{GM_1M_2}{r^2} = \frac{(6.67\times10^{-11})(4.0\times10^{30})^2}{(2.0\times10^{11})^2} = 2.668\times10^{28} = 2.67\times10^{28}\ \text{N}$$

(d) The centre of rotation is the centre of mass of the binary-star system.
The gravitational force on each star provides its centripetal force.

$$F = \frac{mv^2}{r}$$

$$2.668\times10^{28} = \frac{(4.0\times10^{30})v^2}{1.0\times10^{11}}$$

$$v = 2.58\times10^{4}\ \text{m s}^{-1}$$

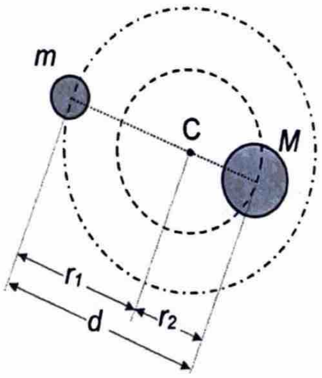
Example 15

[Serway]

Two stars of masses M and m, separated by a distance d, revolve in circular orbits about their centre of mass. Show that each star has a period given by

$$T^2 = \frac{4\pi^2 d^3}{G(M+m)}.$$

Solution:



$d = r_1 + r_2$

The centre of mass C is such that $mr_1 = Mr_2$.

Both stars revolve about C with the same angular velocity ω.

Gravitational force between the stars, $F_G = \frac{GMm}{d^2}$

The gravitational force on each star provides the centripetal force in each orbit.

$$\frac{GMm}{d^2} = m\omega^2 r_1 \quad \text{and} \quad \frac{GMm}{d^2} = M\omega^2 r_2$$

Since $mr_1 = Mr_2 \Rightarrow r_2 = \frac{m}{M} r_1$

And $d = r_1 + r_2 = r_1 + \frac{m}{M} r_1 = r_1\left(1 + \frac{m}{M}\right) \Rightarrow r_1 = \frac{d}{1 + \frac{m}{M}}$

$$\frac{GMm}{d^2} = m\omega^2 r_1 = m\left(\frac{2\pi}{T}\right)^2 r_1$$

$$T^2 = \frac{4\pi^2 d^2}{GM}\left(\frac{d}{1+\frac{m}{M}}\right) = \frac{4\pi^2 d^3}{G(M+m)}$$

Summary

- Gravitational force: $F = \frac{GMm}{r^2}$
- Gravitational field strength: $g = \frac{GM}{r^2}$
- Gravitational potential energy: $U = -\frac{GMm}{r}$
- Gravitational potential: $\phi = -\frac{GM}{r}$
- Relationship between F and g: $g = \frac{F}{m}$
- Relationship between U and ϕ: $\phi = \frac{U}{m}$
- Relationship between F and U: $F = -\frac{dU}{dr}$
- Relationship between g and ϕ: $g = -\frac{d\phi}{dr}$
- For a body to escape from the Earth's gravitational influence:

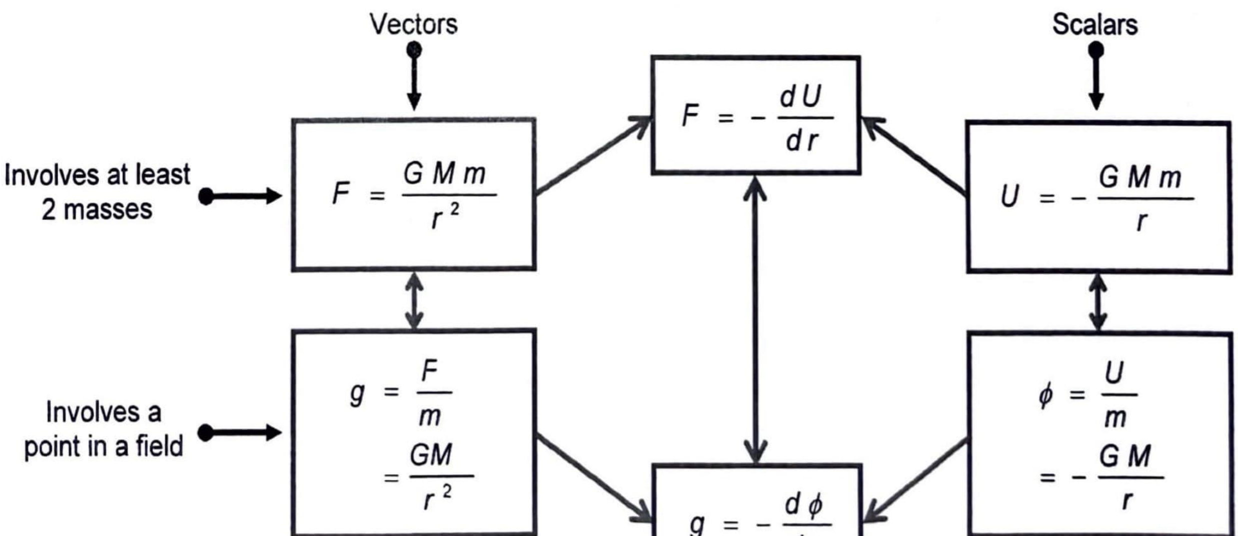
$$-\frac{GMm}{R_{\text{Earth}}} + \frac{1}{2}mv^2 \geq 0 \Rightarrow v_{\text{escape}} = \sqrt{\frac{2GM}{R_{\text{Earth}}}} = \sqrt{2gR_{\text{Earth}}}$$

- For a body to orbit around the Earth at radius r:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v_{\text{orbital}} = \sqrt{\frac{GM}{r}}$$

$$\frac{GMm}{r^2} = mr\omega^2 = mr\left(\frac{2\pi}{T}\right)^2 \Rightarrow T^2 = \frac{4\pi^2}{GM}r^3$$

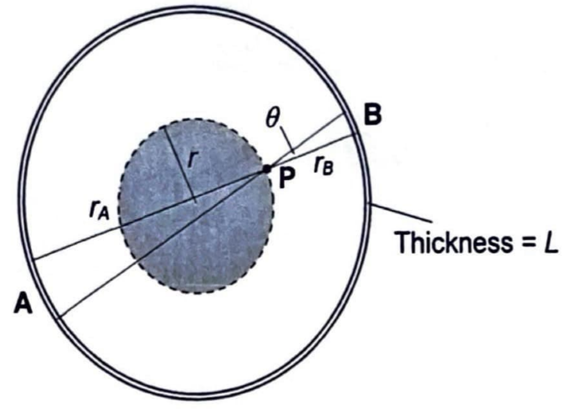
- These conditions will ensure that a geostationary satellite is always above the same point on the Earth's surface, i.e. its relative position with respect to the Earth remains unchanged:
    1. Same 24-hour orbital period as the Earth.
    2. Same West to East rotation as the Earth.
    3. Plane of orbit lies in the same plane as the equator.

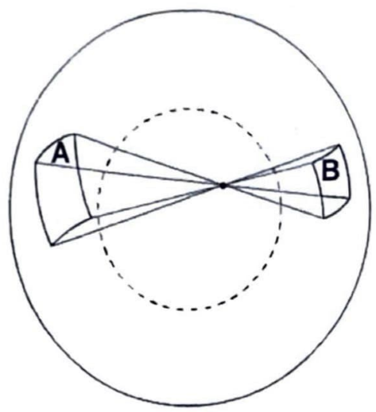


Appendix A

Gravitational Field Inside a Homogenous Spherical Shell of Mass

Cross-sectional view **3D view to show the areas of A and B**





Consider a point P located within a solid sphere a distance r from the centre of the sphere. We will prove that the "effective mass" consists of only the mass enclosed by the shaded sphere within dotted sphere of radius r. The approach is to prove that the rest of the mass (the unshaded portion) cancels out each other's gravitational effect such that their net force on any mass placed at P is zero.

First, consider a thin spherical shell of thickness L. P experiences the gravitational field from every portion of the shell. Two pieces of mass A and B are located on the opposite sides of P as shown. If the angle they subtend at P is θ, then their areas (considering a square of sides = arc length, $r\theta$) are given by $(r_A\theta)^2$ and $(r_B\theta)^2$ respectively. So their masses will be $(r_A\theta)^2L\rho$ and $(r_B\theta)^2L\rho$ respectively, where ρ is the density of the sphere. Hence, their gravitational field strength at P will be $G\frac{(r_A\theta)^2L\rho}{r_A^2}$ and $G\frac{(r_B\theta)^2L\rho}{r_B^2}$ respectively.

In both cases the r's cancel away and we get $G\theta^2L\rho$ for both. So the gravitational field strengths of A and B are equal in magnitude at P. It is also obvious that they are opposite in direction since A and B are on opposite sides of P. So the resultant field at P due to A and B is zero. Apply this argument to every direction around P, we find that the resultant field due to the whole spherical shell is zero.

A solid sphere can be imagined to be made up of layers of spherical shells. If every shell with radius > r produces zero resultant field at P then clearly their combined effect is still zero. So the effective mass is only those within the dotted shaded sphere.

Appendix B

Derivation of $\Delta U = mgh$

When an object of mass *m* is lifted through height *h* from point 1 to point 2 near the surface of the Earth, we usually use the formula *mgh* to determine the gain in gravitational potential energy *U*. This is a good approximation if *g* is constant between point 1 and point 2.

Proof:

Take point 1 to be the ground level and point 2 a distance *h* above it, such that $r_1 = R_E$ and $r_2 = R_E + h$. Then

$$\begin{aligned}
\Delta U &= \left(-\frac{GMm}{r_2}\right) - \left(-\frac{GMm}{r_1}\right) \\
&= \left(-\frac{GMm}{R_E + h}\right) - \left(-\frac{GMm}{R_E}\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E}{R_E + h} - 1\right) \\
&= -\frac{GMm}{R_E}\left(\frac{R_E - R_E - h}{R_E + h}\right) \\
&= \frac{GMm}{R_E}\left(\frac{h}{R_E + h}\right) \\
&\approx \frac{GMm}{R_E}\left(\frac{h}{R_E}\right) && \text{since } h \ll R_E \\
&= m\left(\frac{GM}{R_E^2}\right)h && \text{since } g = \frac{GM}{R_E^2} \\
&= mgh
\end{aligned}$$

Tutorial 7 GRAVITATIONAL FIELD

Self-Check Questions

S1 What is a gravitational field?

S2 What does Newton's law of gravitation state? What kind of objects can it be applied to?

S3 What is gravitational field strength? State the formula for the gravitational field strength due to a point mass.

S4 What is gravitational potential? State the formula for the gravitational potential due to a point mass.

S5 Derive the expression for the total energy of a satellite bound to a planet.

S6 What is a geostationary satellite? Why is the geostationary orbit unique?

S7 Derive the expressions for the escape velocity of an object from the surface of Earth and the orbital velocity of the same object around Earth. What are the differences in the two expressions?

Self-Practice Questions

SP1 The SI base units of the gravitational constant *G* is

A $m\ s^{-2}$ **B** $N\ m^{-2}\ kg^{-2}$ **C** $m^3\ kg^{-1}\ s^{-2}$ **D** $m^2\ kg^{-2}$

SP2 Outside a uniform sphere of mass *M*, the gravitational field strength is the same as that of a point mass *M* at the centre of the sphere. The Earth may be taken to be a uniform sphere of radius *r*. The gravitational field strength at its surface is *g*.

What is the gravitational field strength at a height *h* above the ground?

A $\dfrac{gr^2}{(r+h)^2}$ **B** $\dfrac{gr}{(r+h)}$ **C** $\dfrac{g(r-h)}{r}$ **D** $\dfrac{g(r-h)^2}{r^2}$

SP3 A planet has a mass of 5.0×10^{24} kg and a radius of 6.1×10^6 m. The energy needed to lift a mass of 2.0 kg from its surface into outer space is

A 1.8×10^1 J **B** 5.5×10^7 J **C** 1.1×10^8 J **D** 2.2×10^8 J

SP4 X and Y are two points at respective distances *R* and 2*R* from the centre of the Earth, where *R* is greater than the radius of the Earth. The gravitational potential at X is –800 kJ kg^{-1}.

When a 1 kg mass is taken from X to Y, the work done on the mass is

A –400 kJ **B** –200 kJ **C** +200 kJ **D** +400 kJ

SP5 An Earth satellite is moved from one stable circular orbit to another stable circular orbit at a greater distance from the Earth.

Which one of the following quantities increases for the satellite as a result of the change?

A gravitational force
B centripetal acceleration
C linear speed in the orbit
D angular velocity
E gravitational potential energy

SP6 A planet of mass P moves in a circular orbit of radius R round a sun of mass S with period T.

Which one of the following correctly shows how T depends on P, R, S?

A $T \propto R^{1/2}$ **B** $T \propto R^{3/2}$ **C** $T \propto S^{1/2}$ **D** $T \propto P$

SP7 A body of mass m is projected from the Earth's surface. At the point of launch, the acceleration of free fall is g and the radius of the Earth is R.

To escape from the gravitational field of the Earth, the speed of the body must be at least

A $\sqrt{gR}$ **B** mgR **C** $\sqrt{2gR}$ **D** $\dfrac{mg}{2R}$

SP8 Which quantity is not necessarily the same for satellites that are in geostationary orbits around the Earth?

A angular velocity
B centripetal acceleration
C kinetic energy
D orbital period

SP9 **[J90/II/2 – modified]**
A planet of mass m orbits the Sun of mass M in a circular path of radius r.

(a) Write down an expression, in terms of G, m, M and r, for the force F exerted by the Sun on the planet. [1]

(b) Use this expression to determine the angular velocity of the planet in the orbit. [2]

(c) Deduce the expression for the time taken to complete one orbit of the Sun. [2]

(d) The Earth is 1.50×10^{11} m from the centre of the Sun and takes exactly one year to complete one orbit. The planet Jupiter takes 11.9 years to complete an orbit around the Sun. Calculate the radius of Jupiter's orbit. [3]

SP10 A communications satellite of mass 3000 kg is to be put into an equatorial orbit with an angular velocity equal to that at which the Earth rotates, in order for the satellite to remain above the same point on the Earth's surface. Assume that the mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^{6} m.

(a) What is the angular velocity of the Earth's rotation about its axis? Give your answer in rad s^{-1}. [1]

(b) Determine the altitude of the satellite's orbit. [3]

(c) List the characteristics of a geostationary satellite. [2]

Discussion Questions

D1 **[N90/II/2]**

(a) Give an expression for *Newton's law of Gravitation*, explaining the symbols you use. [2]

(b) Show that *g*, the gravitational field strength a height *h* above the surface of a uniform planet of mass *M* and radius *R*, is given by

$$g = \frac{GM}{(R+h)^2}$$

[2]

(c) Information related to the Earth and the Moon is given below.

$$\frac{\text{Radius of Earth}}{\text{Radius of Moon}} = 3.7 \qquad \frac{\text{Mass of Earth}}{\text{Mass of Moon}} = 81$$

Distance of Moon from Earth $= 3.84 \times 10^{8}$ m.

Gravitational field strength due to the Earth at its surface $= 9.8$ N kg^{-1}.

(i) Using these data, calculate the gravitational field strength due to the Moon at its surface. [2]

(ii) There is a point on the line between the Earth and the Moon at which their combined gravitational field strength is zero. Calculate the distance between this point and the centre of the Earth. [3]

D2 [J96/III/2 – part]

(a) The Earth may be considered to be a uniform sphere of radius 6370 km, spinning on its axis with a period of 24 hours. The gravitational field at the Earth's surface is identical to that of a point mass of 5.98×10^{24} kg at the Earth's centre. For a 1.00 kg mass situated at the Equator,

(i) calculate, using Newton's law of Gravitation, the gravitational force on the mass, [2]

(ii) determine the force required to maintain the circular path of the mass, [2]

(iii) deduce the reading on an accurate newton-meter (spring balance) supporting the mass. [2]

(b) Using your answers to **(a)**, state what would be the acceleration of the mass at the Earth's surface due to

(i) the gravitational force alone, [1]

(ii) the force as measured on the Newton-meter. [1]

(c) A student, situated at the Equator, releases a ball from rest in a vacuum and measures its acceleration towards the Earth's surface. He then states that this acceleration is the 'acceleration due to gravity'. Comment on his statement. [2]

D3 [N2000/III/2]

(a) (i) Define *angular velocity* for an object travelling in a circle. [1]

(ii) Calculate the angular velocity of the Earth in its orbit around the Sun. Assume that the orbit is circular and give your answer in terms of the SI unit for angular velocity. [3]

(b) In order to observe the Sun continuously, a satellite of mass 425 kg is at point X, a distance of 1.60×10^{9} m from the centre of the Earth as shown in Fig. 3.1.

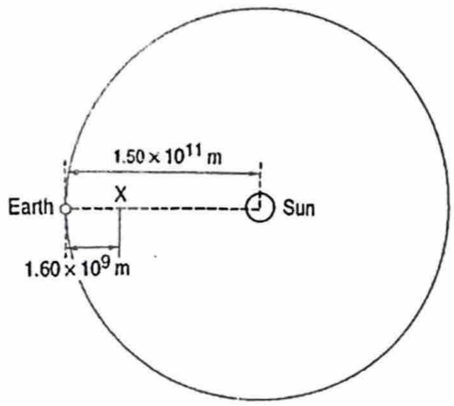


Fig. 3.1

mass of Sun $= 1.99 \times 10^{30}$ kg

mass of Earth $= 5.98 \times 10^{24}$ kg

Earth-Sun distance $= 1.50 \times 10^{11}$ m

(i) Calculate, using the data given,
1. the pull of the Earth on the satellite, [1]
2. the pull of the Sun on the satellite. [2]

(ii) Using Fig. 3.1 as a guide, draw a sketch to show the relative positions of the Earth, the Sun and the satellite. On your sketch, draw arrows to represent the two forces acting on the satellite. Label the arrows with the magnitude of the forces. [2]

(iii) Calculate
1. the magnitude and direction of the resultant force on the satellite, [2]
2. the acceleration of the satellite. [1]

(iv) The satellite is in a circular orbit around the Sun. Calculate the angular velocity of the satellite. [3]

(v) Using your answer to **(a)(ii)**, describe the motion of the satellite relative to the Earth.
Suggest why this orbit around the Sun is preferable to a satellite orbit around the Earth. [3]

(vi) Suggest two disadvantages of having a satellite in this orbit. [2]

D4 [N2016/III/9]
A uniform sphere of radius R has mass M. The mass of the sphere may be assumed to be a point mass at the centre of the sphere, as illustrated in Fig. 4.1.

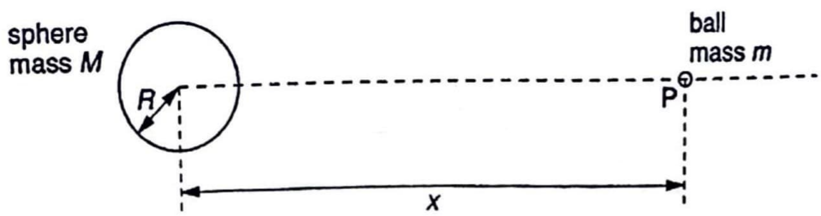


Fig. 4.1

A small ball of mass m is situated at point P, a distance x from the centre of the large mass. The sphere and the ball may be considered to be isolated in space.

(a) State expressions (one in each case) in terms of M, m and x and the gravitational constant G for

(i) the gravitational field strength g_P at point P, [1]

(ii) The potential energy E_P of the small ball at point P. [1]

(b) (i) The gravitational field strength at the surface of the sphere illustrated in Fig. 4.1 is g_s.

On the axes of Fig. 4.2, sketch a graph to show the variation with distance x of the gravitational field strength g of the sphere of mass M for values of x from $x = R$ to $x = 4R$.

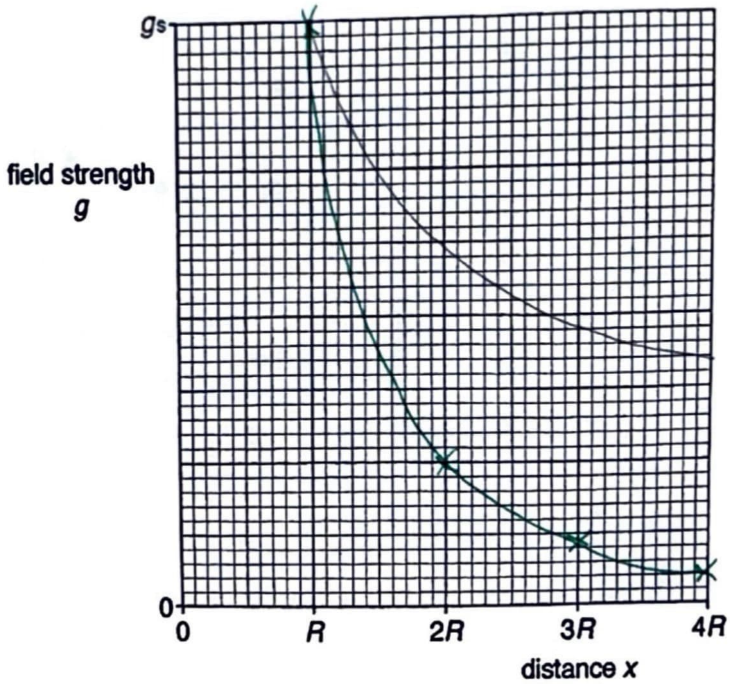


Fig. 4.2

[3]

(ii) State and explain the effect, if any, on the graph you have sketched on Fig. 4.2 when mass is lost uniformly from the surface of the sphere. [2]

(c) A binary star consists of two stars A and B. The two stars may be considered to be isolated in space. The centres of the two stars are separated by a constant distance d, as illustrated in Fig. 4.3.

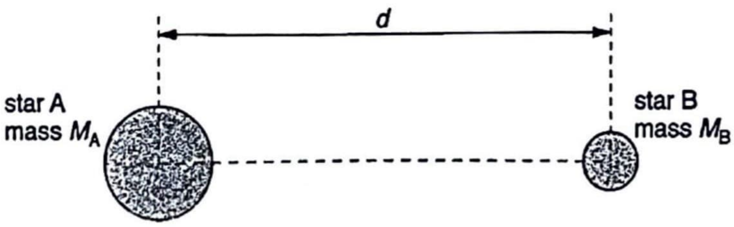


Fig. 4.3

Star A, of mass M_A, has a larger mass than star B of mass M_B, such that $\frac{M_A}{M_B} = 3.0$.

The stars are in circular orbits about each other such that the centre of their orbits is at a fixed point.

Viewed from Earth, over a period of time equal to the period T of the orbits, the appearance of the stars is shown in Fig. 4.4.

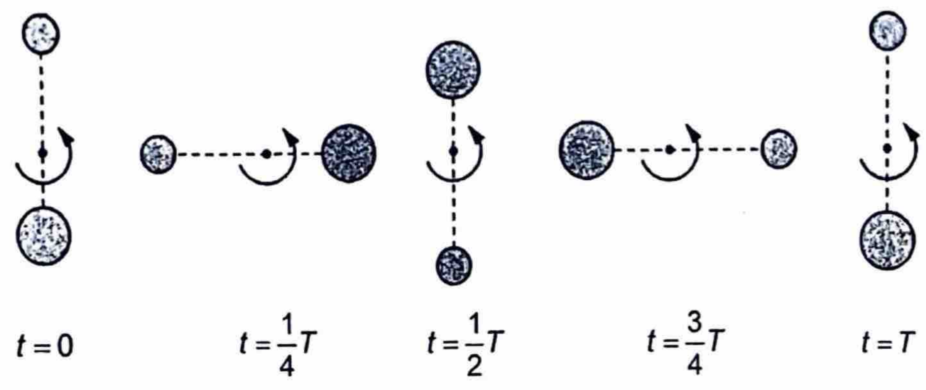


Fig. 4.4

The period T of each orbit is 4.0 years.
The separation d of the centres of the stars is 3.0×10^{11} m.

(i) Explain why the centripetal forces acting on the two stars are equal in magnitude. [2]

(ii) Calculate the angular speed ω of star A. (Give your answer in rad s^{-1}.) [2]

(iii) Determine the radius of the orbit of star A. Explain your working. [4]

(d) Use data from **(c)** and your answers in **(c)** to determine the mass of each star. [3]

(e) The plane of the orbits of the binary star in **(c)** is normal to the line of sight from Earth to the binary star.

A second binary star has the plane of its orbits parallel to the line of sight from Earth. This binary star is so far from Earth that the individual stars cannot be distinguished.

Suggest and explain what observation can be made to determine the period of the orbits of the stars. [2]

D5 (a) A small satellite is in a stable circular orbit of radius 7000 km around a planet of mass 5.7×10^{24} kg and radius 6500 km. Calculate

(i) the orbital speed of the satellite, [2]

(ii) the escape velocity from the surface of the planet. [2]

(b) By what factor would the escape velocity be reduced if the linear dimensions of the planet were 10^3 smaller, its mean density remaining unchanged? [2]

(c) In light of your answer explain why many small planets do not have a gaseous atmosphere. [1]

D6 (a) Calculate the Earth's gravitational field strength at a height of 0.12×10^6 m above the Earth's surface. Assume mass and radius of Earth are 5.98×10^{24} kg and 6.37×10^6 m respectively. [2]

(b) Explain briefly why an astronaut in a spacecraft orbiting the Earth at this altitude may be described as weightless. [2]

(c) The value of the gravitational potential ϕ at a point in the Earth's field is given by the equation

$$\phi = -\frac{GM}{r},$$

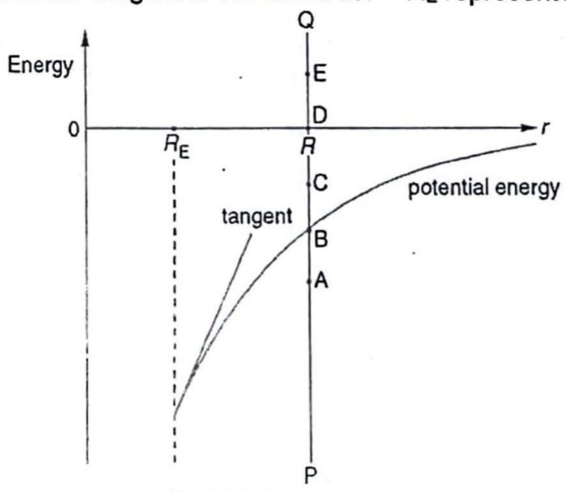
where M is the mass of the Earth and r is the distance of the point from the centre of the Earth (r is greater than the radius of the Earth). Explain

(i) what is meant by the term gravitational potential, [1]
(ii) why the potential has a negative value. [2]

(d) Use the expression given in **(c)** to calculate the gain in the potential energy of a satellite of mass 3000 kg between its launch and when it is at a height of 0.12×10^6 m above the Earth's surface. [2]

D7 [J85/II/9]
The curve below shows the way in which the gravitational potential energy of a body of mass m in the field of the Earth depends on r, the distance from the centre of the Earth, for values of r greater than the Earth's radius R_E.

What does the gradient of the tangent to the curve at $r = R_E$ represent?



The body referred to above is a rocket which is projected vertically upwards from the Earth. At a certain distance R from the centre of the Earth, the total energy of the rocket (its gravitational potential energy plus kinetic energy) may be represented by a point on the line PQ. Five points A, B, C, D, E have been marked on this line.

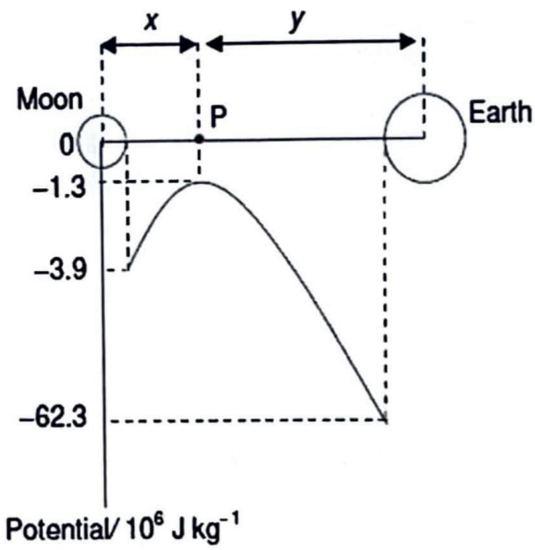
Which point (or points) could represent the total energy of the rocket

(a) if it were momentarily at rest at the top of its trajectory at $r = R$, [2]
(b) if it were falling towards the Earth at $r = R$, [2]
(c) if it were moving away from Earth at $r = R$, with sufficient energy to reach an infinite distance? [2]

In each case, explain briefly how you arrive at your answer.

D8 [J87/II/8]

A point mass m is at a distance r from the centre of the Earth. **Write down an expression**, in terms of m, r, the Earth's mass m_E and the gravitational constant G, for the gravitational potential energy U of the mass (consider only the values of r greater than the Earth's radius).



Certain meteorites (tektites) found on the Earth have a composition identical to that of lunar granite. It is thought that they may be debris from a volcanic eruption on the Moon. The figure above (not drawn to scale), shows how the gravitational potential between the surface of the Moon and the surface of the Earth varies along the line of centres. At the point P, the gravitational potential is a maximum.

Given: mass of the Earth = 6.0×10^{24} kg
mass of the Moon = 7.4×10^{22} kg

(a) By considering the separate contributions of the Earth and the Moon to the gravitational potential, explain qualitatively why the graph has a maximum and why the curve is asymmetrical. [2]

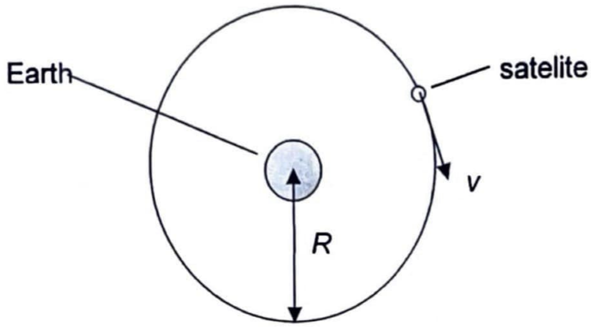
(b) State how the resultant gravitational force on the tektite at any point between the Moon and the Earth could be deduced from the graph. [1]

(c) When a tektite is at P, the gravitational forces on it due to the Moon and the Earth are F_M and F_E respectively. State the relation which applies between F_M and F_E. Hence find the value of x/y, where x and y are the distances of P from the centre of the Moon and the centre of the Earth respectively. [2]

(d) If a tektite is to reach the Earth, it must be projected from a volcano on the Moon with a certain minimum speed v_o. Making use of appropriate values form the graph, find this speed. Explain your reasoning. [3]

(e) Discuss very briefly whether a tektite will reach the Earth's surface with a speed less than, equal to or greater than the speed of projection. (Neglect atmospheric resistance.) [2]

D9 [N04/II/3]
A satellite orbits the Earth in a circular path, as illustrated below.



Both the Earth and the satellite may be considered to be point masses with their masses concentrated at their centres. The satellite has speed v and the radius of its orbit about the Earth is R.

(a) (i) Show that the speed v is given by the expression

$$v^2 = \frac{GM}{R},$$

where M is the mass of the Earth and G is the gravitational constant. [2]

(ii) The mass of the satellite is m. Determine the expression for the kinetic energy E_k of the satellite in terms of G, M, m and R. [2]

(b) (i) State an expression, in terms of G, M, m and R for the gravitational potential energy E_p of the satellite. [1]

(ii) Hence, show that the total energy E_t of the satellite is given by

$$E_t = -\frac{GMm}{2R}.$$ [2]

(c) As the satellite orbits the Earth, it gradually loses energy because of air resistance.

(i) State whether the total energy E_t becomes more or less negative. [1]

(ii) Hence, state and explain the effect of this change on

1. the radius of the orbit, [2]
2. the speed of the satellite. [2]

D10 [N08/II/3]
A satellite of mass m orbits a planet of mass M and radius R_p. The radius of the orbit is R. The satellite and the planet may be considered to be point masses with their masses concentrated at their centres. They may be assumed to be isolated in space.

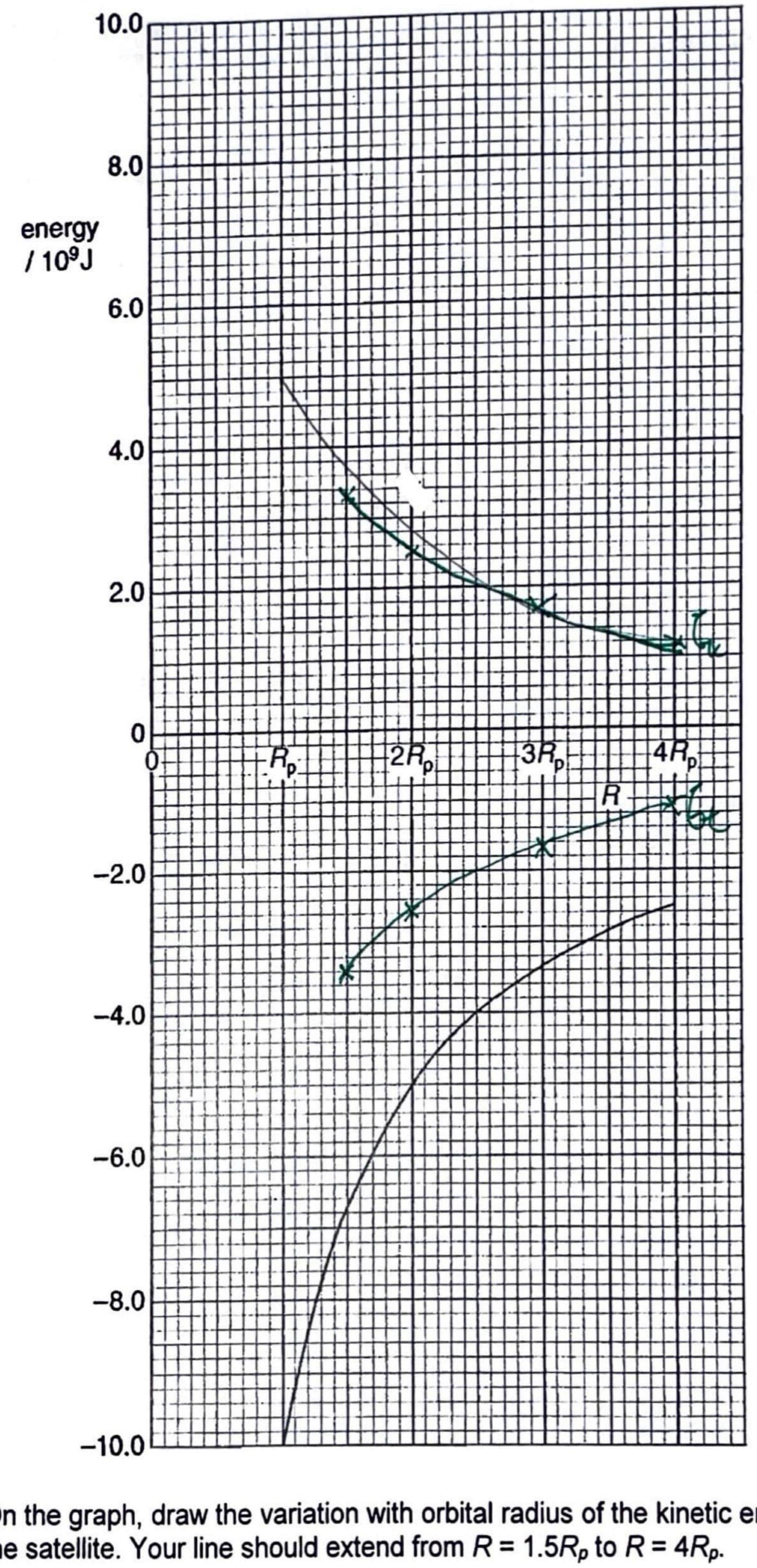
(a) (i) Derive an expression, in terms of M, m and R, for the kinetic energy of the satellite. Explain your working. [2]

(ii) Show that, for the satellite in orbit, the ratio

$$\frac{\text{gravitational potential energy of satellite}}{\text{kinetic energy of satellite}}$$

is equal to –2. [1]

(b) The variation with orbital radius R of the gravitational potential energy of the satellite is shown in the graph below.



(i) On the graph, draw the variation with orbital radius of the kinetic energy of the satellite. Your line should extend from $R = 1.5R_p$ to $R = 4R_p$. [2]

(ii) The mass m of the satellite is 1600 kg.
The radius of the orbit of the satellite is changed from $R = 4R_p$ to $R = 2R_p$.
Use the graph above to determine the change in orbital speed of the satellite. [5]

Challenging Questions

C1 A point P is inside a uniform solid sphere of radius R and mass M. If point P is at a distance r from the centre of the sphere, show that the gravitational potential at point P is

$$\phi = \frac{GMr^2}{2R^3} - \frac{3GM}{2R}.$$

How much work is done by the gravitational force in bringing a particle of mass m from the surface of the sphere to its centre?

Answers

D1 **(c)** **(i)** 1.66 N kg^{-1}
(ii) 3.46×10^8 m

D2 **(a)** **(i)** 9.83 N
(ii) 0.0337 N
(iii) 9.80 N
(b) **(i)** 9.83 m s^{-2}
(ii) 9.80 m s^{-2}

D3 **(a)** **(ii)** 1.99×10^{-7} rad s^{-1}
(b) **(i)** **1.** 0.0662 N, **2.** 2.56 N
(iii) **1.** 2.50 N towards the Sun, **2.** 5.87×10^{-3} m s^{-2}
(iv) 1.99×10^{-7} rad s^{-1}

D4 **(c)** **(ii)** 4.98×10^{-8} rad s^{-1}
(iii) 7.50×10^{-10} m
(d) $M_A = 7.54 \times 10^{29}$ kg, $M_B = 2.51 \times 10^{29}$ kg

D5 **(a)** **(i)** 7.37×10^3 m s^{-1}
(ii) 1.08×10^4 m s^{-1}

D6 **(a)** 9.47 N kg^{-1}
(d) 3.47×10^9 J

D7 **(a)** B
(b) C
(c) D, E

D8 **(c)** 0.11
(d) 2.3×10^3 m s^{-1}

D10 **(b)** **(ii)** 493 m s^{-1}

C1 $\dfrac{GMm}{2R}$

Tutorial 7 Gravitational Field Suggested Solutions

S1 A gravitational field is a region of space in which a mass placed in that region experiences a gravitational force.

S2 Newton's law of gravitation states that two point masses attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

It can be applied to point masses or uniform spherical objects (where their separation is taken to be the distance between their centres of mass).

S3 The gravitational field strength at a point in space is defined as the gravitational force experienced per unit mass at that point.

$$g = \frac{F}{m} = \frac{GM}{r^2}$$

S4 The gravitational potential at a point in a gravitational field is defined as the work done per unit mass by an external force in bringing a small test mass from infinity to that point.

$$\phi = \frac{U}{m} = -\frac{GM}{r}$$

S5 The gravitational force on the satellite provides the centripetal force as it orbits the planet.

$$F_g = F_c$$

$$G\frac{Mm}{r^2} = m\frac{v^2}{r}$$

$$v^2 = \frac{GM}{r}$$

$$E_K = \frac{1}{2}mv^2$$

$$E_K = \frac{1}{2}\frac{GMm}{r}$$

$$E_T = E_P + E_K$$

$$E_T = -\frac{GMm}{r} + \frac{1}{2}\frac{GMm}{r}$$

$$E_T = -\frac{GMm}{2r}$$

The total energy of a satellite that is bound is equal to the sum of its kinetic energy and gravitational potential energy and is negative (since a total energy ≥ 0 means that the object will be able to escape to infinity).

S6 A geostationary satellite is a satellite that orbits directly above the equator (sharing the same axis of rotation as the Earth and is in the same plane as the equator) in the same direction as Earth's rotation (West to East). Its orbital period is the same as the period of the Earth's rotation (24 hrs).

The orbital radius is unique since it must follow the same period as the Earth's rotation and it is also independent of the mass of the satellite (i.e. all geostationary satellites have the same orbital radius.)

S7 For object to escape Earth:

$$E_T \geq 0$$

$$E_P + E_K \geq 0$$

$$-\frac{GMm}{R_{Earth}} + \frac{1}{2}mv^2 \geq 0$$

$$v \geq \sqrt{\frac{2GM}{R_{Earth}}}$$

$$v \geq \sqrt{2gR_{Earth}}$$

$$v_{escape} = \sqrt{\frac{2GM}{R_{Earth}}}$$

For object orbiting Earth, the gravitational force on it provides for the centripetal force:

$$\frac{GMm}{r^2} = m\frac{v_{orbital}^{\ 2}}{r}$$

$$v_{orbital} = \sqrt{\frac{GM}{r}}$$

Escape velocity is derived from the principle of conservation of energy, whereas the orbital velocity is derived from Newton's second law of motion.

The radius in the escape velocity refers to the radius of the Earth, whereas the radius in the orbital velocity refers to the radius of the orbit.

There is a factor of 2 in the expression for escape velocity.

SP1

$$F = \frac{GMm}{r^2}$$

$$G = \frac{Fr^2}{Mm}$$

$$\text{units of G} = \frac{\text{kg m s}^{-2}\text{ m}^2}{\text{kg}^2} = \text{m}^3\text{ kg}^{-1}\text{ s}^{-2}$$

Answer: C

SP2

$$g = \frac{GM}{r^2} \Rightarrow GM = gr^2$$

$$g_{r+h} = \frac{GM}{(r+h)^2} = \frac{gr^2}{(r+h)^2}$$

Answer: A

SP3

$$U_f = 0 \quad \text{and} \quad U_i = -\frac{GMm}{r}$$

Energy needed = gain in gravitational potential energy

$$= U_f - U_i$$

$$= 0 - \left(-\frac{GMm}{r}\right) = \frac{GMm}{r} = \frac{(6.67\times10^{-11})(5.0\times10^{24})(2.0)}{6.1\times10^6} = 1.1\times10^8\text{ J}$$

Answer: C

SP4

$$\phi_X = -\frac{GM}{R} = -800 \text{ kJ kg}^{-1} \Rightarrow U_X = m\phi_X = -800 \text{ kJ}$$

$$\phi_Y = -\frac{GM}{2R} = -400 \text{ kJ kg}^{-1} \Rightarrow U_Y = m\phi_Y = -400 \text{ kJ}$$

Work done $= \Delta U = U_f - U_i = U_Y - U_X = -400 - (-800) = 400$ kJ

Answer: D

SP5

$$U = -\frac{GMm}{r}$$

When $r\uparrow$, $U\uparrow$ (i.e. less negative).

A: $F = \frac{GMm}{r^2} \Rightarrow$ when $r\uparrow$, $F\downarrow$.

B: $a_c = \frac{F}{m} = \frac{GM}{r^2} \Rightarrow$ when $r\uparrow$, $a_c\downarrow$.

C: $v_{orbital} = \sqrt{\frac{GM}{r}} \Rightarrow$ when $r\uparrow$, $v_{orbital}\downarrow$

D: $\omega = \frac{v_{orbital}}{r} = \sqrt{\frac{GM}{r^3}} \Rightarrow$ when $r\uparrow$, $\omega\downarrow$.

Answer: E

SP6 The gravitational force on the planet provides for the centripetal force as it orbits the sun.

$$\frac{GM_SM_P}{R^2} = M_PR\omega^2$$

$$\frac{GSP}{R^2} = PR\left(\frac{2\pi}{T}\right)^2$$

$$T^2 \propto \frac{R^3}{S}$$

$$\therefore T \propto R^{3/2} \text{ or } T \propto S^{-1/2}$$

Answer: B

SP7 Minimum energy to escape Earth's attraction: $E_{Earth} = E_{infinity}$

$$-\frac{GMm}{R} + \frac{1}{2}mv^2 = 0$$

$$v^2 = \frac{2GM}{R}$$

$$= 2\frac{GM}{R^2}R \quad \text{, since } g = \frac{GM}{R^2}$$

$$= 2gR$$

$$v = \sqrt{2gR}$$

Answer: C

SP8 For geostationary satellites, the following are the same:

- Magnitude of centripetal acceleration
- Linear speed, angular speed
- Radius, height
- Period, frequency

As force, kinetic energy and potential energy depends on the mass of the satellite, they may not be the same for all geostationary satellites.

Answer: C

SP9 **(a)**

$$F = \frac{GMm}{r^2}$$

(b) The gravitational force on the planet provides for the centripetal force.

$$F_g = F_c$$

$$\frac{GMm}{r^2} = mr\omega^2$$

$$\omega = \sqrt{\frac{GM}{r^3}}$$

(c)

$$\omega = \sqrt{\frac{GM}{r^3}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{GM}{r^3}}$$

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

(d)

$$\frac{T_J^2}{T_E^2} = \frac{r_J^3}{r_E^3}$$

$$r_J = \left(\frac{T_J}{T_E}\right)^{2/3} r_E$$

$$= \left(\frac{11.9}{1}\right)^{2/3} \left(1.50 \times 10^{11}\right)$$

$$= 7.82 \times 10^{11} \text{ m}$$

SP10 **(a)**

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 60 \times 60} = 7.27 \times 10^{-5} \text{ rad s}^{-1}$$

(b)

$$G\frac{Mm}{r^2} = mr\omega^2$$

$$mr\left(\frac{2\pi}{T}\right)^2 = G\frac{Mm}{r^2}$$

$$r^3 = \frac{GM}{4\pi^2}T^2$$

$$r = \sqrt[3]{\frac{6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{4\pi^2} \times \left(24 \times 60 \times 60\right)^2}$$

$$= 4.23 \times 10^7 \text{ m}$$

Altitude = $(4.23 \times 10^7) - (6.37 \times 10^6) = 3.59 \times 10^7$ m

(c) It has the same orbital period of 24 hours as the Earth about its axis of rotation.
It rotates from West to East in the same direction as the Earth about its axis of rotation.
Its plane of orbit lies in the same plane as the equator.