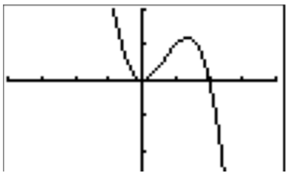
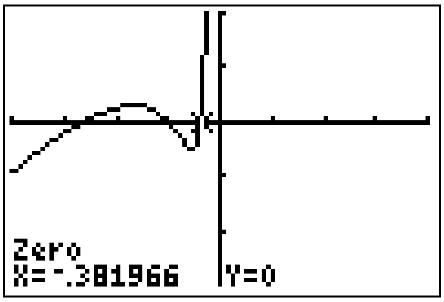
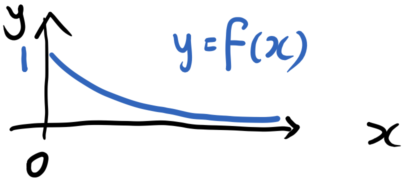
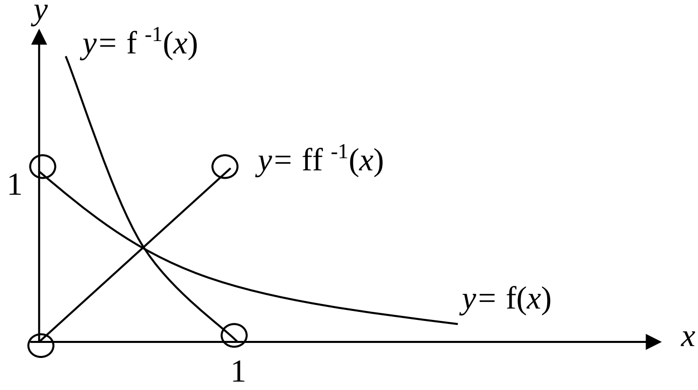
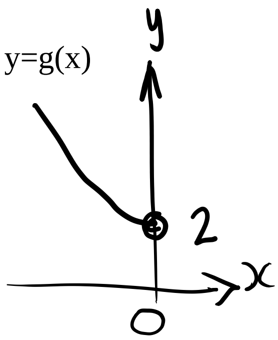
3. Functions (solutions)

1(i)	Let $y = 2 + \frac{1}{x} \Rightarrow xy - 2x = 1 \Rightarrow x = \frac{1}{y-2}$ Hence $\mathrm{g}^{-1} : x \to \frac{1}{x-2}, \quad x \neq 2$
(ii)	$\mathrm{R_f} = \mathbb{R}$ $\mathrm{D_g} = \mathbb{R} \setminus \{0\}$ $\mathrm{R_f} \not\subset \mathrm{D_g}$ Hence gf does not exist $\mathrm{f}(x) = x^2(2-x)$ 
(iii)	$\mathrm{f}\left(2 + \frac{1}{x}\right) = 1$ $\left(2 + \frac{1}{x}\right)^2 \left(-\frac{1}{x}\right) = 1$ $\left(2 + \frac{1}{x}\right)^2 + x = 0$ using GC, $x = -2.62, -1,$ or -0.382 

2(a)	f: $x \mapsto e^{-x}$, $x \in \mathbb{R}^+$, Since the graph of f(x) is strictly decreasing, f is one-to-one Hence f^{-1} exists. 
	Let y = f(x) $y = e^{-x} \Rightarrow x = -\ln y$ $\Rightarrow f^{-1}(x) = -\ln x$ $f^{-1} : x \mapsto -\ln x$, $0 < x < 1$ $(D_{f^{-1}} = R_f = (0,1))$ Range of $f^{-1} = \mathbb{R}^+$ $ff^{-1}(x) = x$, where $0 < x < 1$ since $D_{ff^{-1}} = D_{f^{-1}}$. 
2(b)	g: $x \mapsto 3x^2 + 2$, $x \in \mathbb{R}^-$, $R_g = (2, \infty) \subset (0, \infty) = D_f$ Hence fg exists. $fg(x) = e^{-(3x^2+2)}$, $D_{fg} = D_g = (-\infty, 0)$ or $\mathbb{R}^-$ fg: $x \mapsto e^{-(3x^2+2)}$, $x < 0$. $(-\infty, 0) \xrightarrow{g} (2, \infty) \xrightarrow{f} (0, e^{-2})$ $R_{fg} = (0, e^{-2})$ 

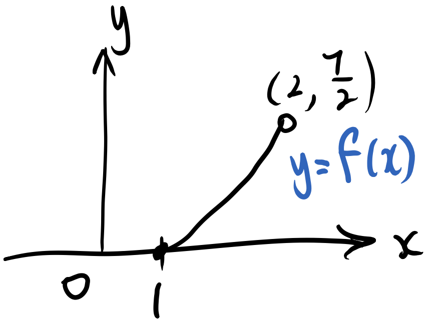
3(i)

$\mathrm{f}'(x) = 2x + \dfrac{1}{x^2} > 0$ for $1 \le x < 2$ $\Rightarrow$ f is strictly increasing.

(ii)

Since f is strictly increasing, its minimum and maximum values correspond to the minimum and maximum x values. Thus

$\mathrm{R_f} = \left[1-1, 4-\dfrac{1}{2}\right) = \left[0, \dfrac{7}{2}\right)$.



(iii)

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{f}(x) = x$$

$$\Rightarrow x^2 - \frac{1}{x} = x$$

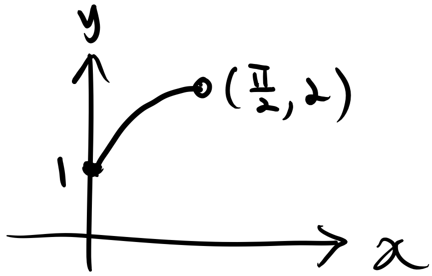
$$\Rightarrow x^3 - x^2 - 1 = 0$$

$$\Rightarrow x = 1.47.$$

(iv)

Since $\mathrm{R_g} = [1, 2) = \mathrm{D_f}$, fg exists.

Since $\mathrm{R_f} = \left[0, \dfrac{7}{2}\right) \not\subset \left[0, \dfrac{\pi}{2}\right) = \mathrm{D_g}$, gf does not exist.



$\mathrm{fg}(x) = \mathrm{f}(\sin x + 1) = (\sin x + 1)^2 - \dfrac{1}{\sin x + 1}$.

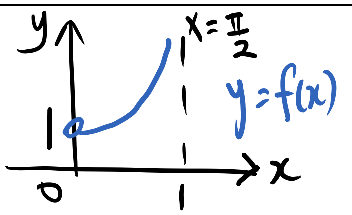
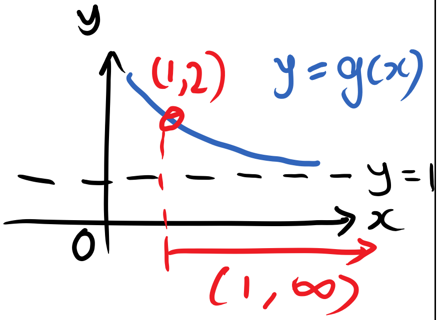
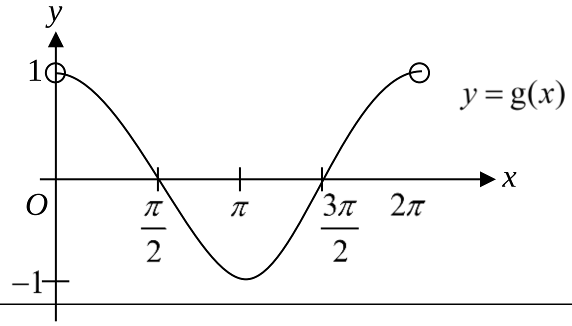
$\mathrm{D_{fg}} = \mathrm{D_g} = \left[0, \dfrac{\pi}{2}\right)$.

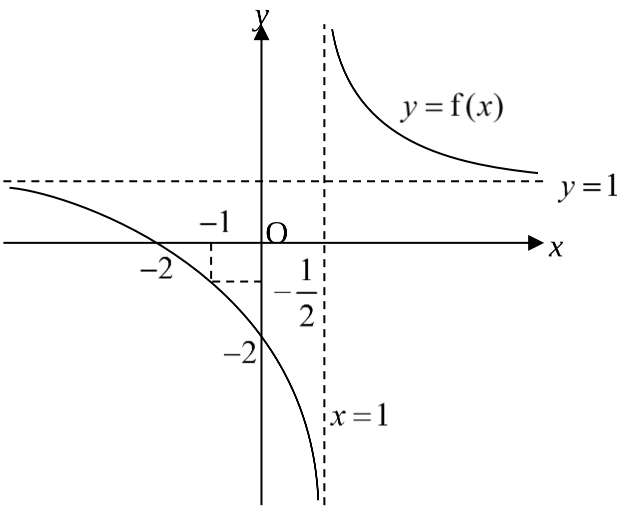
$\mathrm{fg} : x \to (\sin x + 1)^2 - \dfrac{1}{\sin x + 1}$, $x \in \mathbb{R}$, $0 \le x < \dfrac{\pi}{2}$.

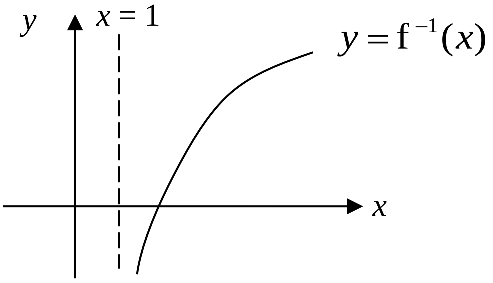
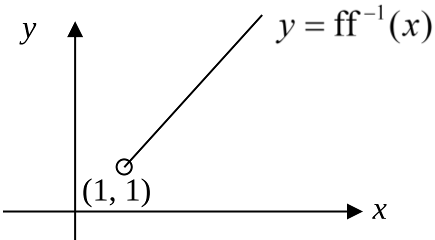
(v)

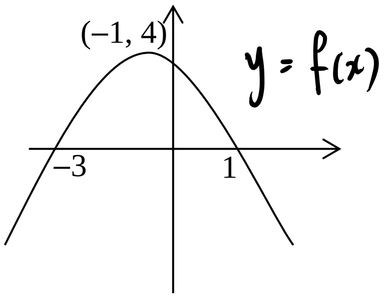
$$\left[0, \frac{\pi}{2}\right) \xrightarrow{\mathrm{g}} [1, 2) \xrightarrow{\mathrm{f}} \left[0, \frac{7}{2}\right)$$

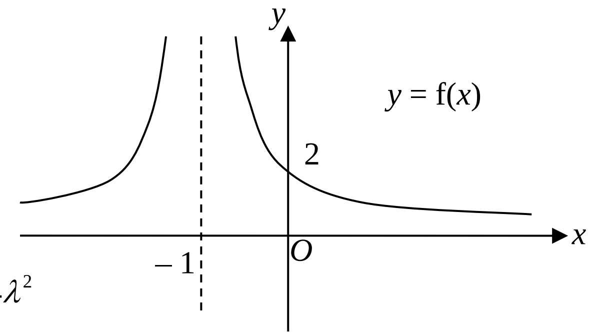
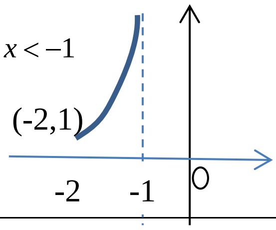
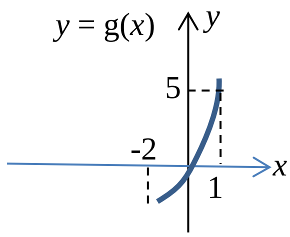
$\mathrm{R_{fg}} = \left[0, \dfrac{7}{2}\right)$.

4(i)	$\mathrm{h}(x)=\dfrac{1}{\mathrm{f}(x)} \quad =\dfrac{1}{\tan x+1}$ $\mathrm{h}'(x)=-\dfrac{\sec^2 x}{(\tan x+1)^2}$ since $\sec^2 x>0,\ (\tan x+1)^2>0$ for $0<x<\dfrac{\pi}{2}$ $\mathrm{h}'(x)=-\dfrac{\sec^2 x}{(\tan x+1)^2}<0$ for $0<x<\dfrac{\pi}{2}$ $\Rightarrow \mathrm{h}(x)$ is a decreasing function. $\therefore$ h is a one-one function.
(ii)	Let $y=\dfrac{1}{\tan x+1} \Rightarrow \quad x=\tan^{-1}\left(\dfrac{1}{y}-1\right)$ $\mathrm{h}^{-1}(x)=\tan^{-1}\left(\dfrac{1}{x}-1\right)$
(iii)	$\mathrm{R_f}=(1,\infty),\ \mathrm{D_g}=(0,\infty) \quad \therefore \mathrm{R_f}\subseteq \mathrm{D_g}$ and gf exists. $\mathrm{gf}(x)=\mathrm{g}(\tan x+1)\ =1+\dfrac{1}{\tan x+1}$ or $=\dfrac{\tan x+2}{\tan x+1}$ Thus $\mathrm{gf}: x\mapsto 1+\dfrac{1}{\tan x+1}, 0<x<\dfrac{\pi}{2}$.   $\left(0,\dfrac{\pi}{2}\right)\xrightarrow{\mathrm{f}}(1,\infty)\xrightarrow{\mathrm{g}}(1,2)$ $\mathrm{R_{gf}}=(1,2)$
5(i)	$\mathrm{f}^2(x)=\dfrac{\left(\dfrac{x+2}{x-1}\right)+2}{\left(\dfrac{x+2}{x-1}\right)-1}=\dfrac{x+2+2x-2}{x+2-x+1}=x$. $\mathrm{f}^3(x)=\mathrm{f}\left(\mathrm{f}^2(x)\right)=\mathrm{f}(x)$, and $\mathrm{f}^4(x)=\mathrm{f}\left(\mathrm{f}^3(x)\right)=\mathrm{f}^2(x)=x$ $\therefore \quad \mathrm{f}^{2012}(x)=x$.
5(ii)	$R_{\mathrm{g}}=[-1,\ 1)$ and $D_{\mathrm{f}}=\mathbb{R}\setminus\{1\}$ Since $R_{\mathrm{g}}\subseteq D_{\mathrm{f}}$, fg exists. 

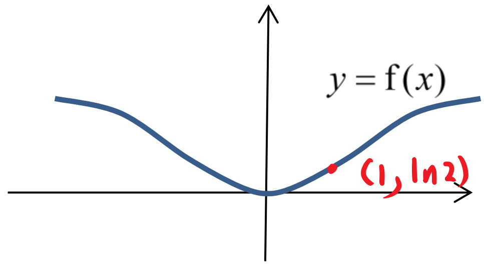
5(iii)	$\text{fg}: x \mapsto \dfrac{\cos x + 2}{\cos x - 1},\ 0 < x < 2\pi$.
5(iv)	$R_{\text{g}} = [-1,\ 1) \xrightarrow{\text{f}} R_{\text{fg}}$ From the graphs of g and f, $R_{\text{fg}} = \left(-\infty,\ -\dfrac{1}{2}\right]$. 

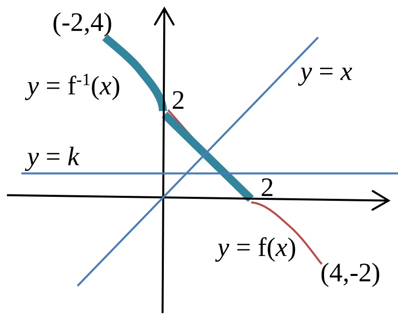
6(i)	Let $y = \mathrm{f}^{-1}(x)$ $y = \ln(x^2 - 1)$ $x^2 = \mathrm{e}^y + 1$ $x = \pm\sqrt{\mathrm{e}^y + 1}$ since $x > 1$ $\therefore \mathrm{f}(x) = \sqrt{\mathrm{e}^x + 1},\ x \in \mathbb{R}$ 
(ii)	$R_{\mathrm{f}^{-1}} = (-\infty, \infty)$ $D_{\mathrm{f}} = (-\infty, \infty)$ $\therefore R_{\mathrm{f}^{-1}} = D_{\mathrm{f}}$, ff^{-1} exists. $\mathrm{ff}^{-1}(x) = x, x > 1$
(iii)	 $R_{\mathrm{ff}^{-1}} = (1, \infty)$

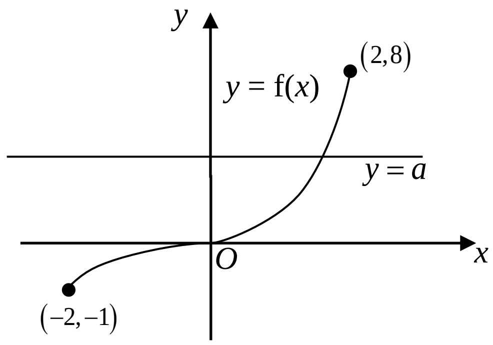
7	$f(x) = 3 - 2x - x^2$ $= 4 - (x+1)^2$ For f^{-1} to exist, f must be one–one. $\therefore k = -1$ To find f^{-1} : Let $y = 4 - (x+1)^2$ $\Rightarrow x = -1 \pm \sqrt{4-y}$ Since $x \le -1$, $x = -1 - \sqrt{4-y}$ Thus, $f^{-1} : x \mapsto -1 - \sqrt{4-x}$, $x \in (-\infty, 4]$ 
(i)	Range of f = $(-\infty, 4]$, domain of g = $[0, 4]$ Since range of f $\not\subseteq$ domain of g, gf does not exist.
(iii)	Let $X = x+1$ $g^{-1}g(x+1) = x+1$ replace X with x+1 $\Rightarrow g^{-1}g(X) = X$ $D_{g^{-1}g(X)} = D_{g(X)} \Rightarrow 0 \le X \le 4 \Rightarrow 0 \le x+1 \le 4 \Rightarrow -1 \le x \le 3$ $D_{g^{-1}g(x+1)} = [-1, 3]$ ----- (1) $D_{gg^{-1}(X)} = D_{g^{-1}(X)} = R_{g(X)} \Rightarrow 1 \le X \le e^2 \Rightarrow 1 \le x+1 \le e^2 \Rightarrow 0 \le x \le e^2 - 1$ $D_{gg^{-1}(x+1)} = [0, e^2 - 1]$ ----- (2) Taking the intersection of (1) & (2), set of values of x = $[0, 3]$

8(i)	$y = x^2 - 4\lambda x$ $\Rightarrow y = (x-2\lambda)^2 - 4\lambda^2$ $\Rightarrow x - 2\lambda = \pm\sqrt{y+4\lambda^2}$ $\Rightarrow x = 2\lambda + \sqrt{y+4\lambda^2}$ since $x > 2\lambda$ $\therefore g^{-1} : x \mapsto 2\lambda + \sqrt{x+4\lambda^2}, \quad x > -4\lambda^2$ 
8(ii)	For gf to exist, $(1, \infty) = R_f \subseteq D_g = (2\lambda, \infty)$ $\Rightarrow 2\lambda \le 1 \Rightarrow \lambda \le \dfrac{1}{2}$ $f(x) = \dfrac{1}{\|x+1\|}, \quad -2 < x < -1$ 
8(iii)	When $\lambda = -1$, $g(x) = x^2 + 4x, \quad x > -2$ $R_f = (1, \infty) \xrightarrow{g} (5, \infty) = R_{gf}$ 

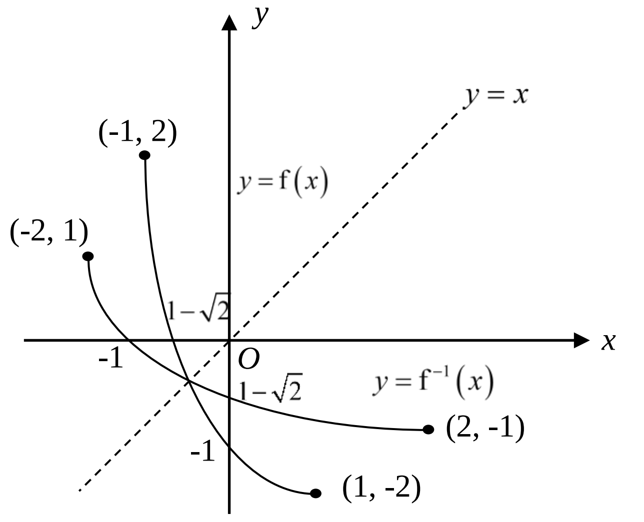
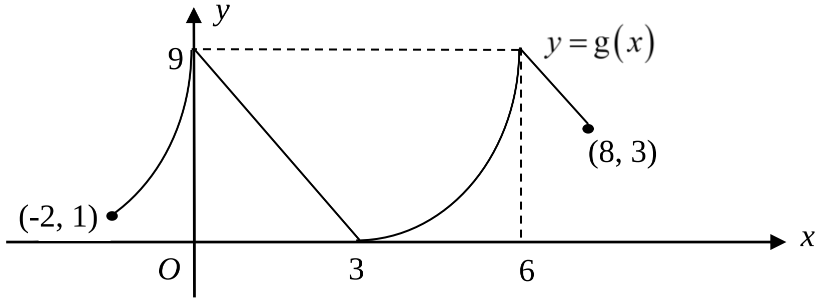
9(i)	$y = f(x)$ is concave upwards when $f''(x) > 0$ $f(x) = \ln(x^2+1) \Rightarrow f'(x) = \dfrac{2x}{x^2+1}$ and $f''(x) = \dfrac{2(x^2+1) - 2x(2x)}{(x^2+1)^2} = \dfrac{2-2x^2}{(x^2+1)^2}$ $\dfrac{2-2x^2}{(x^2+1)^2} > 0$ Since $(x^2+1)^2$ is always positive, $2(1-x^2) > 0$ $(1-x)(1+x) > 0$ $-1 < x < 1$

9 (ii)	$y = \mathrm{f}(x) = \ln(x^2+1)$, $x \in \mathbb{R}$ 
(iii)	f^{-1} exist if $x \geq k$, where $k \geq 0$.
(iv)	$\mathrm{g} : x \mapsto \ln\left(x^2+1\right), x \geq 1.$ $y = \ln(x^2+1)$ Let $x^2+1 = \mathrm{e}^y \quad \Rightarrow$ $x = \pm\sqrt{\mathrm{e}^y - 1}$ But $x \geq 1$, therefore $x = \sqrt{\mathrm{e}^y - 1}$ $\mathrm{g}(1) = \ln 2$ $\mathrm{R_g} = [\ln 2, \infty)$ $\mathrm{g}^{-1}(x) = \sqrt{\mathrm{e}^x - 1}$, $\quad x \in [\ln 2, \infty)$ or $\mathrm{g}^{-1} : x \mapsto \sqrt{\mathrm{e}^x - 1}$, $\quad x \in [\ln 2, \infty)$

10(i)	Every horizontal line $y = k$, $-2 \leq k \leq 2$, cuts the graph of f at exactly 1 point. Therefore f is one-one and f^{-1} exists. 
(ii)	set of values of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is [0,2].
(iii)	$\mathrm{f}^{-1}(x) = 3 \Rightarrow x = \mathrm{f}(3) = -3/4$
(iv)	$\int_2^3 \mathrm{f}(x)\,\mathrm{d}x = \frac{1}{4}\int_2^3 \left(2x - x^2\right)\mathrm{d}x = -\frac{1}{3}$ $\int_{-\frac{3}{4}}^{2} \mathrm{f}^{-1}(x)\,\mathrm{d}x = \left(3 \times \frac{3}{4} - \frac{1}{3}\right) + \frac{1}{2}(2 \times 2) = \frac{47}{12}$

11(i)	The horizontal line $y = a$ cuts the graph $y = \mathrm{f}(x)$ at most once. f is an one-to-one function and thus f^{-1} exists. 

(ii)	For $-2 \le x < 0$, $y = -\dfrac{x^2}{4} \Rightarrow x = \pm\sqrt{-4y}$ $x = -\sqrt{-4y}$ (since $-2 \le x < 0$) For $0 \le x \le 2$, $y = x^3 \Rightarrow x = \sqrt[3]{y}$ $\mathrm{f}^{-1} : x \mapsto \begin{cases} -\sqrt{-4x}, & -1 \le x < 0, \\ \sqrt[3]{x}, & 0 \le x \le 8. \end{cases}$
(iii)	$R_{\mathrm{g}} = [-1, 0)$ and $D_{\mathrm{h}} = [-2, 0)$. Since $R_{\mathrm{g}} \subseteq D_{\mathrm{h}}$, hg exists.
(iv)	$\mathrm{gh}(x) = \mathrm{hg}(x) = -\dfrac{x^2}{4}$ Since $D_{\mathrm{gh}} = D_{\mathrm{h}} = [-2, 0)$ and $D_{\mathrm{hg}} = D_{\mathrm{g}} = [-1, 0)$, solution for $\mathrm{gh}(x) = \mathrm{hg}(x)$ is the intersection of the two domains, i.e. $-1 \le x < 0$.

12i)	Let $y = x^2 - 2x - 1$ $y = (x-1)^2 - 2$ $y + 2 = (x-1)^2$ $(x-1) = \pm\sqrt{y+2}$ $x = 1 \pm \sqrt{y+2}$ Since $-1 \le x \le 1$, $x = 1 - \sqrt{y+2}$ $\mathrm{f}^{-1} : x \mapsto 1 - \sqrt{x+2}, -2 \le x \le 2$ 
(ii)	$\mathrm{g} : x \mapsto \begin{cases} 9 - 3x, & 0 \le x < 3, \\ (x-3)^2, & 3 \le x < 6, \end{cases}$ and that $\mathrm{g}(x) = \mathrm{g}(x+6)$ for all real values of *x*. 

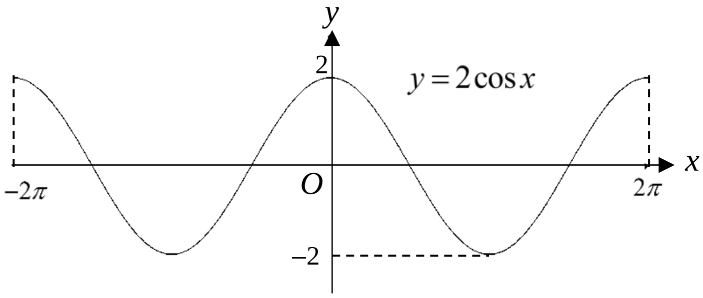
(iii)	$R_f = [-2,2]$ $D_g = (-\infty,\infty)$ Since $R_f \subseteq D_g$, gf exists. $D_f = [-1,1] \xrightarrow{f} R_f = [-2,2] \xrightarrow{g} R_{gf} = [1,9]$

13(i)	$R_g = (-\infty,\infty)$ $D_f = \mathbb{R} \setminus \{1\}$ Since $R_g \not\subset D_f$, function fg does not exist. $R_f = (-\infty,0) \cup (0,\infty)$ $D_g = \mathbb{R}$ Since $R_f \subset D_g$, function gf exists.
	$\text{gf}(x) = \text{g}\left(\frac{-2}{x-1}\right)$ $= 1 - 2\left(\frac{-2}{x-1}\right)$ $= 1 + \frac{4}{x-1}$ $= \frac{x+3}{x-1}$ $D_{gf} = D_f = \mathbb{R} \setminus \{1\}$ $R_{gf} = \mathbb{R} \setminus \{1\}$

(ii)	Let $y = \frac{x+3}{x-1}$ $y(x-1) = x+3$ $x(y-1) = y+3$ $x = \frac{y+3}{y-1}$ $(\text{gf})^{-1}(x) = \frac{x+3}{x-1}$ Since $D_{(gf)^{-1}} = R_{gf} = D_{gf}$, gf($x$) is a self-inverse function.

<table>
<tr><td>14
(i)</td><td>

<u>Method 1</u>



The horizontal line $y = k,\ -2 \le k \le 2$ cuts the graph at more than one point, so f is not 1-1 and f^{-1} does not exist.

<u>Method 2</u>

Since $f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right) = 0$, f is not 1-1 and f^{-1} does not exist.

</td></tr>
<tr><td>(ii)</td><td>

Max $b = \pi$

$y = 2\cos x \quad \Rightarrow \quad x = \cos^{-1}\frac{y}{2}$

$f^{-1} : x \mapsto \cos^{-1}\frac{x}{2},\ \ -2 \le x \le 0$

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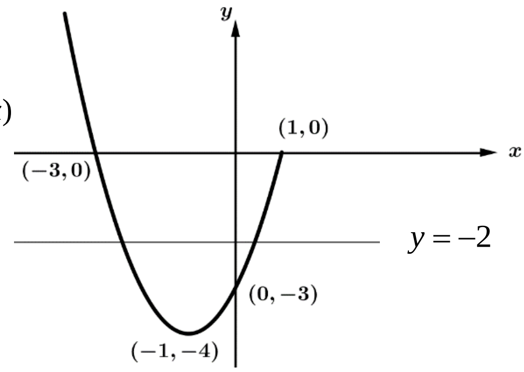
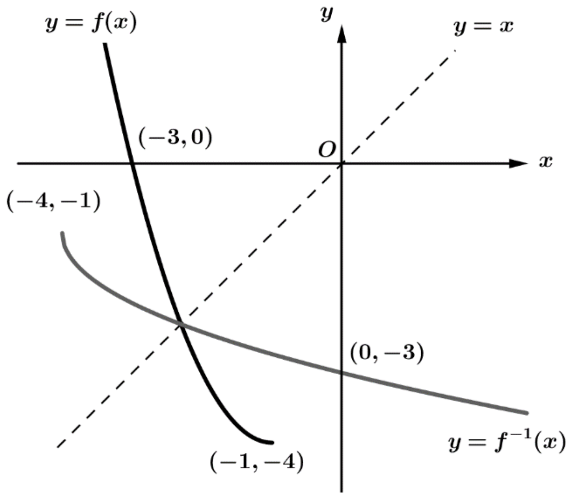
</table>

15	**(i)** $\mathrm{f} : x \mapsto x^2 + 2x - 3, \quad x \le 1.$ The horizontal line $y = -2$ cuts the graph of $y = \mathrm{f}(x)$ twice, thus f is not one-to-one. $\therefore$ f^{-1} does not exist.  **(ii)** For f^{-1} to exist, the largest domain is $(-\infty, -1]$. Largest value of $k = -1$. **(iii)** Let $y = \mathrm{f}(x) = x^2 + 2x - 3.$ $x^2 + 2x - 3 - y = 0$ $\Rightarrow x = \dfrac{-2 \pm \sqrt{4 - 4(-3 - y)}}{2}$ $= \dfrac{-2 \pm \sqrt{4y + 16}}{2} \quad = -1 \pm \sqrt{y + 4}$ Since $x \le -1$, $x = -1 - \sqrt{y + 4}$ $\therefore$ $\mathrm{f}^{-1} : x \mapsto -1 - \sqrt{x + 4}$, $x \ge -4$
	(iv) 

	(v) The graph of $y = \mathrm{f}(x)$ must be reflected in the line $y = x$ in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$. $\therefore \ \mathrm{f}(x) = \mathrm{f}^{-1}(x) \Rightarrow \mathrm{f}(x) = x$ $x^2 + 2x - 3 = x$ $x^2 + x - 3 = 0 \Rightarrow \quad x = \dfrac{-1 \pm \sqrt{13}}{2}$ Since $-4 \le x \le -1$, $x = \dfrac{-1-\sqrt{13}}{2}$. **(vi)** $\mathrm{R_g} = (0, 2)$ and $\mathrm{D_f} = (-\infty, 1]$ $\mathrm{R_g} \not\subseteq \mathrm{D_f}$ Thus, fg does not exist. **(vii)** $\mathrm{D_h} = [1.2, 3)$ $[1.2, 3) \xrightarrow{\mathrm{h}} (0, 1] \xrightarrow{\mathrm{f}} (-3, 0]$ $\mathrm{R_{fh}} = (-3, 0]$

16(i)	$\mathrm{f}(-4) = \mathrm{f}(-4+3) = \mathrm{f}(-1+3) = \mathrm{f}(2) = (3-2)^3 = 1$ $\mathrm{f}(22) = \mathrm{f}(19+3) = \mathrm{f}(19) = \mathrm{f}(16+3) = ... = \mathrm{f}(1) = 8$ Hence, $\mathrm{f}(-4) + \mathrm{f}(22) = 9$
((ii)