2024 TJC H2 FM Prelim Paper 1 (Marking Scheme)

1 Polar equations of the form $r = a + a\cos n\theta,\ a,\ n \in \mathbb{Z},\ n \geq 2$ are commonly called petal curves because of the shapes of their graphs. The number and sizes of the petals are dependent on the values of a and n.

(a) State the relationship between the number of petals and n. [1]

(b) Show that the area bounded by the curve is independent of n. [3]

Solution
(a) number of petals = n (b) Area bounded by the graph $\frac{1}{2}\int_0^{2\pi} r^2 \mathrm{d}\theta$ $= \frac{1}{2}\int_0^{2\pi} (a + a\cos n\theta)^2 \mathrm{d}\theta$ $= \frac{a^2}{2}\int_0^{2\pi} (1 + 2\cos n\theta + \cos^2 n\theta)\ \mathrm{d}\theta$ $= \frac{a^2}{2}\int_0^{2\pi} \left(1 + 2\cos n\theta + \frac{1}{2}\cos 2n\theta + \frac{1}{2}\right) \mathrm{d}\theta$ = $= \frac{a^2}{2}\int_0^{2\pi} \left(\frac{3}{2} + 2\cos n\theta + \frac{1}{2}\cos 2n\theta\right) \mathrm{d}\theta$ $= \frac{a^2}{2}\left[\frac{3}{2}\theta + \frac{2}{n}\sin n\theta + \frac{1}{4n}\sin 2n\theta\right]_0^{2\pi}$ $= \frac{a^2}{2}\frac{3}{2}(2\pi)$ $= \frac{3\pi a^2}{2}$, independent of n

2 Show that $(1+\mathrm{i}\tan\theta)^k = \sec^k\theta(\cos k\theta + \mathrm{i}\sin k\theta)$. [1]

Hence or otherwise, show that $\sum_{k=0}^{n-1} \cos k\theta \ \sec^k\theta = \cot\theta \ \sin n\theta \ \sec^n\theta,$ provided θ is not an integer multiple of $\frac{\pi}{2}$. [5]

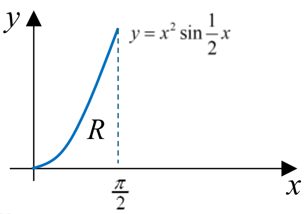
Solution
$(1+\mathrm{i}\tan\theta)^k = \left(1+\mathrm{i}\frac{\sin\theta}{\cos\theta}\right)^k$ $= \sec^k\theta(\cos\theta + \mathrm{i}\sin\theta)^k$ $= \sec^k\theta(\cos k\theta + \mathrm{i}\sin k\theta)$ $\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k = \frac{(1+\mathrm{i}\tan\theta)^n - 1}{\mathrm{i}\tan\theta}$ $= \frac{\sec^n\theta(\cos n\theta + \mathrm{i}\sin n\theta) - 1}{\mathrm{i}\tan\theta}$ $= \frac{\sec^n\theta\cos n\theta + \mathrm{i}\sec^n\theta\sin n\theta - 1}{\mathrm{i}\tan\theta}$ $= \frac{\cot\theta\sec^n\theta\cos n\theta - \cot\theta + \mathrm{i}\sec^n\theta\sin n\theta\cot\theta}{\mathrm{i}}$ $= -\mathrm{i}\cot\theta\sec^n\theta\cos n\theta + \mathrm{i}\cot\theta + \sec^n\theta\sin n\theta\cot\theta$ $= \sec^n\theta\sin n\theta\cot\theta + \mathrm{i}\left(\cot\theta - \cot\theta\sec^n\theta\cos n\theta\right)$ $\sum_{k=0}^{n-1}\cos k\theta\sec^k\theta = \mathbb{R}\left(\sum_{k=0}^{n-1}(1+\mathrm{i}\tan\theta)^k\right)$ $= \sec^n\theta\sin n\theta\cot\theta$

3 Let $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, \mathrm{d}x$.

(a) Prove that for $n \geq 2$,

$$I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}.$$ [3]

(b) R is the region enclosed by the graph of $y = x^2 \sin\frac{1}{2}x$, the line $x = \frac{\pi}{2}$ and the x-axis. By using the result in **(a)**, find the exact volume of the solid generated when R is rotated 2π radians about the x-axis. [5]

Solution	**Remarks**
(a) $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x \, \mathrm{d}x$ $= \left[x^n \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} nx^{n-1} \sin x \, \mathrm{d}x$ $= \left(\frac{\pi}{2}\right)^n - \left[nx^{n-1}(-\cos x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} n(n-1)x^{n-2} \cos x \, \mathrm{d}x$ $= \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$	Use by parts to show the desired result. Not recommended to use MI to prove.
(b) Volume $= \pi \int_0^{\frac{\pi}{2}} x^4 \sin^2 \frac{1}{2}x \, \mathrm{d}x$  $= \pi \int_0^{\frac{\pi}{2}} x^4 \left(\frac{1-\cos x}{2}\right) \mathrm{d}x$ $= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} x^4 \, \mathrm{d}x - \frac{\pi}{2} \int_0^{\frac{\pi}{2}} x^4 \cos x \, \mathrm{d}x$ $= \frac{\pi}{2}\left[\frac{x^5}{5}\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2} I_4$ $= \frac{\pi^6}{320} - \frac{\pi}{2} I_4$ $I_0 = \int_0^{\frac{\pi}{2}} \cos x \, \mathrm{d}x = 1$ From result in (a), $I_2 = \left(\frac{\pi}{2}\right)^2 - 2(2-1)I_0 = \frac{\pi^2}{4} - 2$ $I_4 = \left(\frac{\pi}{2}\right)^4 - 4(4-1)I_2$ $= \frac{\pi^4}{16} - 12\left(\frac{\pi^2}{4} - 2\right)$	Use disc method to find volume since region R is just between curve and x axis.

$=\dfrac{\pi^4}{16}-3\pi^2+24$ Required Volume $=\dfrac{\pi^6}{320}-\dfrac{\pi}{2}\left(\dfrac{\pi^4}{16}-3\pi^2+24\right)$ $=\dfrac{\pi^6}{320}-\dfrac{\pi^5}{32}+\dfrac{3\pi^3}{2}-12\pi$	

4 Let P_n be the vector space of real polynomials of degree n or less.

The transformation $T: P_4 \to P_3$ is defined by,

$$T(\mathrm{f}(x))=\mathrm{f}(0)+\mathrm{f}(1)x+\mathrm{f}(-1)x^2+\mathrm{f}(2)x^3.$$

Show that T is a linear transformation. [2]

Find a basis for the null space of T. [7]

Solution

For $\mathrm{f}(x),\mathrm{g}(x)\in P_4$, $\alpha\in\mathbb{R}$ the "vectors" here are the polynomials and not x!

$T(\mathrm{f}+\mathrm{g})(x)=(\mathrm{f}+\mathrm{g})(0)+(\mathrm{f}+\mathrm{g})(1)x+(\mathrm{f}+\mathrm{g})(-1)x^2+(\mathrm{f}+\mathrm{g})(2)x^3$

$=\mathrm{f}(0)+\mathrm{f}(1)x+\mathrm{f}(-1)x^2+\mathrm{f}(2)x^3+\mathrm{g}(0)+\mathrm{g}(1)x+\mathrm{g}(-1)x^2+\mathrm{g}(2)x^3$

$=T(\mathrm{f})(x)+T(\mathrm{g})(x)$

$T(\alpha\mathrm{f})(x)=(\alpha\mathrm{f})(0)+(\alpha\mathrm{f})(1)x+(\alpha\mathrm{f})(-1)x^2+(\alpha\mathrm{f})(2)x^3$

$=\alpha\left[\mathrm{f}(0)+\mathrm{f}(1)x+\mathrm{f}(-1)x^2+\mathrm{f}(2)x^3\right]$

$=\alpha T(\mathrm{f})(x)$

$\therefore T$ is a linear transformation.

To show that it is a linear transformation, we need to show:
1. $\mathrm{T}(\mathrm{f}(x)+\mathrm{g}(x))=\mathrm{T}(\mathrm{f}(x))+\mathrm{T}(\mathrm{g}(x))$
2. $\mathrm{T}(\alpha\mathrm{f}(x))=\alpha\mathrm{T}(\mathrm{f}(x))$

All workings regardless how simple it is must be explicitly shown

For the null space of T:

$T(\mathrm{f}(x))=\mathrm{f}(0)+\mathrm{f}(1)x+\mathrm{f}(-1)x^2+\mathrm{f}(2)x^3=0$

$\Rightarrow \mathrm{f}(0)=\mathrm{f}(1)=\mathrm{f}(-1)=\mathrm{f}(2)=0$

$\mathrm{f}(x)=ax^4+bx^3+cx^2+dx+e$ since $\mathrm{f}(x)\in \mathrm{P}_4$

$\mathrm{f}(0)=e=0$

$\mathrm{f}(1)=a+b+c+d+e=0\Rightarrow a+b+c+d=0$

$\mathrm{f}(-1)=a-b+c-d+e=0\Rightarrow a-b+c-d=0$

$\mathrm{f}(2)=16a+8b+4c+2d+e=0\Rightarrow 16a+8b+4c+2d=0$

For null space,

We need to find the set of polynomials of degree 4 such that $\mathrm{T}(\mathrm{f}(x))=0$

i.e.

$\mathrm{T}(ax^4+bx^3+cx^2+dx+e)=0$

Augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 \\ 16 & 8 & 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0.5 & 0 \end{array}\right) \text{ using GC}$$

$$\Rightarrow \begin{cases} a = 0.5\lambda \\ b = -\lambda \\ c = -0.5\lambda \\ d = \lambda \end{cases}$$

$$\mathrm{f}(x) = 0.5\lambda x^4 - \lambda x^3 - 0.5\lambda x^2 + \lambda x = \lambda\left(x^4 - 2x^3 - x^2 + 2x\right)$$

$\therefore$ Basis for null space of $T = \left\{x^4 - 2x^3 - x^2 + 2x\right\}$ (express answer in polynomial)

5 The half-line L with equation

$$y = mx + 8,\ x > 0,\ m < 0,$$

is tangential to the locus $|z| = 4$ at point Q represented by complex number z_4.

(a) Sketch the locus $|z| = 4$ and L on a single Argand diagram. [2]

(b) Find the exact value of m. [2]

(c) Sketch the locus of $\arg(z - z_4) = -\frac{5\pi}{6}$ on the same Argand diagram in part **(a)**. [2]

(d) The complex number $z = x + \mathrm{i}y$ satisfies the following relations:

$$y \le mx + 8,\ x > 0,$$

$$\mathrm{Im}(z - z^*) > 0,$$

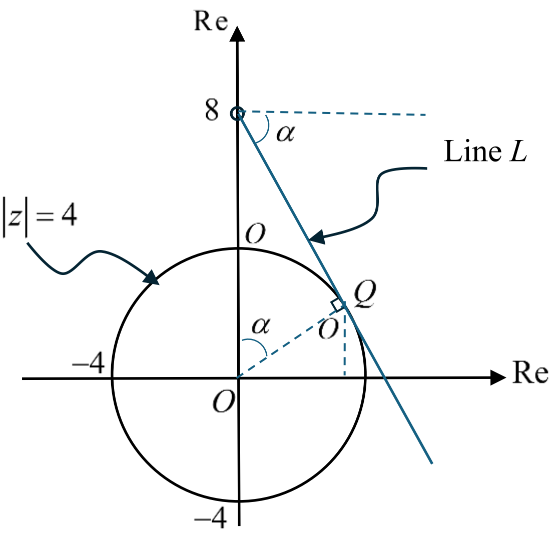
$$|z| \ge 4,$$

$$-\frac{5\pi}{6} \le \arg(z - z_4) \le -\frac{\pi}{3}.$$

Shade the region R that contains z. Hence find the least $\arg(z + 4\mathrm{i})$, leaving your answer in radians, correct to 4 decimal places. [3]

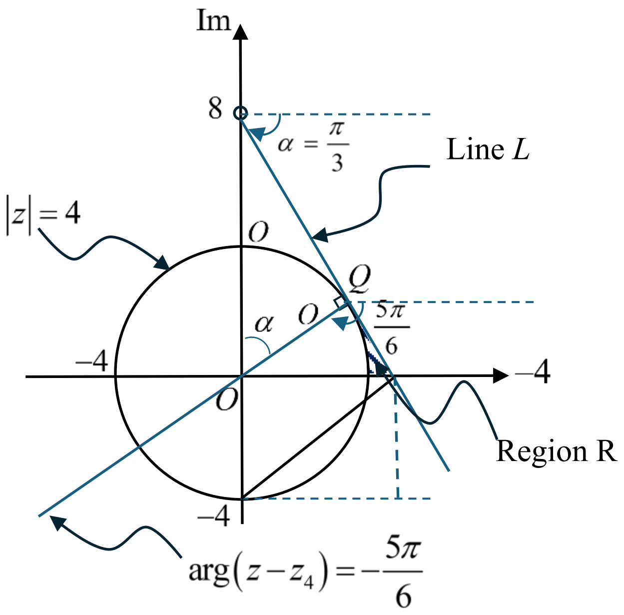
Solution

(a)



(b) Method 1

$$\cos\alpha = \frac{4}{8} \Rightarrow \alpha = \frac{\pi}{3}$$

	Therefore $m=-\tan\frac{\pi}{3}=-\sqrt{3}$ <u>Method 2</u> Solve $x^2+y^2=4^2$ -------(1) and $y=mx+8$ ----------(2) Sub (2) into (1): $x^2+(mx+8)^2=16$ $x^2+m^2x^2+16mx+64=16$ $(m^2+1)x^2+16mx+48=0$ For tangent, $b^2-4ac=0$ $(16m)^2-4(m^2+1)(48)=0$ $64m^2-192=0$ $m=\pm\sqrt{3}$ Since $m<0$, $m=-\sqrt{3}$ (c)  (d) Least $\arg(z+4\text{i})=\tan^{-1}\left(4\div\frac{8}{\sqrt{3}}\right)$ $=0.7137$

6 The polar equations of two curves are given by

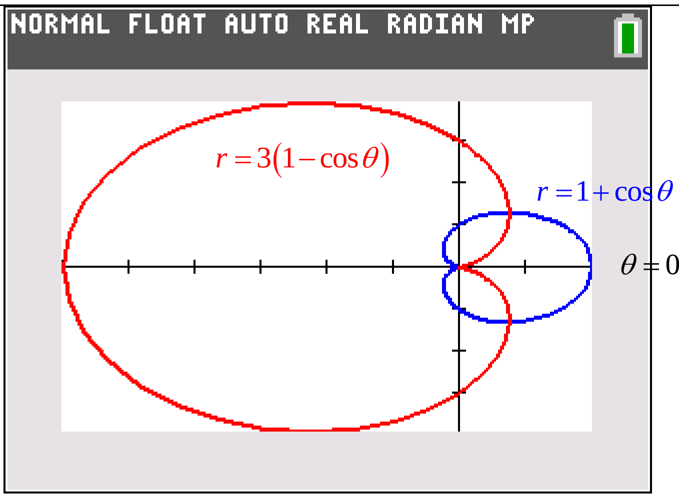
$$C_1:\ r = 1+\cos\theta \quad \text{and} \quad C_2:\ r = 3(1-\cos\theta) \ \text{ for } 0 \le \theta < 2\pi .$$

(a) Sketch C_1 and C_2 on the same diagram, indicating clearly the symmetries and the exact polar coordinates of the point(s) of intersection of the curves. [4]

(b) Find the exact polar coordinates of the P on C_2, where $0 \le \theta < \pi$, and P is the furthest away from the x-axis. [4]

(c) Find in exact form, the shortest distance between the points P and A, where A is the point of intersection between C_1 and C_2 in the first quadrant. [2]

(d) Express the equation of C_1 in parametric form using the given θ as the parameter. Hence find the area of the curved surface generated when the segment OA on C_1 is rotated 2π radians about the x-axis. [4]

6	Solution
(a)	 $1+\cos\theta = 3(1-\cos\theta)$ $\cos\theta = \frac{1}{2}$ $\theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$ At $\theta = \frac{\pi}{3}$, $r = 1+\cos\left(\frac{\pi}{3}\right) = \frac{3}{2}$ At $\theta = \frac{5\pi}{3}$, $r = 1+\cos\left(\frac{5\pi}{3}\right) = \frac{3}{2}$ The polar coordinates of the points of intersections are

	$\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$
(b)	$r = 3(1-\cos\theta)$ $y = r\sin\theta = 3(1-\cos\theta)\sin\theta$ $\frac{dy}{d\theta} = 3\left[(\sin\theta)(\sin\theta) + \cos\theta(1-\cos\theta)\right]$ $= 3(\sin^2\theta + \cos\theta - \cos^2\theta)$ $= 3(1 - 2\cos^2\theta + \cos\theta)$ $= -3(2\cos\theta + 1)(\cos\theta - 1)$ At $\frac{dy}{d\theta} = 0 \Rightarrow -3(2\cos\theta+1)(\cos\theta-1) = 0$ $\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$ $\theta = \frac{2\pi}{3}$ or $\theta = \frac{4\pi}{3}$ (rejected) $\theta = 0$ At $\theta = \frac{2\pi}{3}$, $r = 3\left(1-\cos\frac{2\pi}{3}\right) = 4.5$ Hence, polar coordinates of P is $\left(4.5, \frac{2\pi}{3}\right)$.
(c)	Using cosine rule, $d^2 = \left(\frac{9}{2}\right)^2 + \left(\frac{3}{2}\right)^2 - 2\left(\frac{9}{2}\right)\left(\frac{3}{2}\right)\cos\left(\frac{\pi}{3}\right)$ $d^2 = \frac{63}{4}$ $d = \frac{\sqrt{63}}{2} = \frac{3\sqrt{7}}{2}$
(d)	Since $r = 1+\cos\theta$, the parametric equation of C_1 is $\begin{cases} x = r\cos\theta = \cos\theta + \cos^2\theta, \\ y = r\sin\theta = \sin\theta + \sin\theta\cos\theta, \end{cases}$ for $0 \le \theta < 2\pi$. Hence $\frac{dx}{d\theta} = -\sin\theta - 2\cos\theta\sin\theta = -\sin\theta - \sin 2\theta$ $\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta = \cos\theta + \cos 2\theta$ Therefore surface area generated is $\int_{\frac{\pi}{3}}^{\pi} 2\pi\cos\theta(1+\cos\theta)\sqrt{(\sin\theta+\sin 2\theta)^2 + (\cos\theta+\cos 2\theta)^2}\ d\theta$

	=9.79 (3sf)
	Extension of question: *Show that the perimeter of the outer border formed by the two curves is given by* $k\left(\int_{a}^{\pi} 3\sqrt{1-\cos\theta}\ \mathrm{d}\theta + \int_{0}^{a} \sqrt{1+\cos\theta}\ \mathrm{d}\theta\right)$, *where a and k are constants to be determined.* *[4]* **[Solution]** $r = 1+\cos\theta$ $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = (1+\cos)^2 + \sin^2\theta$ $= 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta$ $= 2 + 2\cos\theta$ $= 2(1+\cos\theta)$ $r = 3(1-\cos\theta)$ $r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \left[3(1-\cos\theta)\right]^2 + (-3\sin\theta)^2$ $= 9(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)$ $= 9(2 - 2\cos\theta)$ $= 18(1-\cos\theta)$ Required length $= 2 \times \left[\int_{\frac{\pi}{3}}^{\pi} \sqrt{18(1-\cos\theta)}\ \mathrm{d}\theta + \int_{0}^{\frac{\pi}{3}} \sqrt{2(1+\cos\theta)}\ \mathrm{d}\theta\right]$ $= 2\sqrt{2}\left(\int_{\frac{\pi}{3}}^{\pi} 3\sqrt{1-\cos\theta}\ \mathrm{d}\theta + \int_{0}^{\frac{\pi}{3}} \sqrt{1+\cos\theta}\ \mathrm{d}\theta\right)$ (shown)

7 **(a)** Let **A** be a square matrix with eigenvalue λ. Show that λ is also an eigenvalue for $\mathbf{A}^T$. [2]

A square matrix is called a *stochastic matrix* if all the entries are non-negative and the sum of entries of each column is 1.

Let $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$ be a stochastic matrix.

(b) By considering $\mathbf{B}^T$ and a suitable column vector, show that an eigenvalue of **B** is 1. [2]

Let $\mathbf{x} = (x_1 \quad x_2 \quad \cdots \quad x_n)^T$ be an eigenvector of $\mathbf{B}^T$ with corresponding eigenvalue λ.

(c) By considering $\mathbf{B}^T\mathbf{x}$, find an expression for λx_k, $1 \le k \le n$. [2]

It is further given that x_k is the entry in $\mathbf{x}$ with the largest absolute value.

(d) By considering $|\lambda x_k|$, show that $|\lambda| \le 1$. [4]

(You may apply without proof the result $|a_1 + a_2 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n|$ for any real numbers $a_1, \cdots, a_n$.)

	Solution
(a)	$\lvert\mathbf{A} - \lambda\mathbf{I}\rvert = 0$ $\lvert(\mathbf{A} - \lambda\mathbf{I})^T\rvert = 0$ $\lvert\mathbf{A}^T - \lambda\mathbf{I}\rvert = 0$ Thus λ is also an eigenvalue for $\mathbf{A}^T$.
(b)	$\mathbf{B}^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} b_{11} + b_{21} + \cdots + b_{n1} \\ b_{12} + b_{22} + \cdots b_{n2} \\ \vdots \\ b_{1n} + b_{2n} + \cdots b_{nn} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ Since 1 is an eigenvalue of $\mathbf{B}^T$, 1 is thus an eigenvalue of **B** (from (a)).
(c)	$\mathbf{B}^T\mathbf{x} = \lambda\mathbf{x}$ Consider the *k*-th row of $\mathbf{B}^T\mathbf{x}$, i.e. $b_{1k}x_1 + b_{2k}x_2 + b_{3k}x_3 + \cdots b_{nk}x_n = \lambda x_k$

(d)	$\left\|\lambda x_k\right\| = \left\|b_{1k}x_1 + b_{2k}x_2 + b_{3k}x_3 + \cdots b_{nk}x_n\right\|$ $\leq \left\|b_{1k}x_1\right\| + \left\|b_{2k}x_2\right\| + \left\|b_{3k}x_3\right\| + \cdots + \left\|b_{nk}x_n\right\|$ $= b_{1k}\left\|x_1\right\| + b_{2k}\left\|x_2\right\| + b_{3k}\left\|x_3\right\| + \cdots + b_{nk}\left\|x_n\right\|$ $\leq b_{1k}\left\|x_k\right\| + b_{2k}\left\|x_k\right\| + b_{3k}\left\|x_k\right\| + \cdots + b_{nk}\left\|x_k\right\|$ $\leq \left\|x_k\right\|(b_{1k} + b_{2k} + b_{3k} + \cdots + b_{nk})$ $= \left\|x_k\right\|$ $\left\|\lambda\right\| \leq 1$

8 Diseases such as smallpox is generally considered to impart immunity for life. To assess the effect of smallpox, an experimental group of individuals born on the same day in one specific year are closely monitored until they reach t years of age. Let

- $N = N(t)$ denote the number in the group who survived to age t,
- $S = S(t)$ denote the number who have not had the disease but are still susceptible to it at age t,
- p denote the probability of a susceptible individual getting the disease,
- $\frac{1}{m}$ denote the proportion of those who die due to the disease.

To study the effects of smallpox, the Swiss mathematician Daniel Bernoulli proposed in 1760 the following differential equation:

$$\frac{\mathrm{d}S}{\mathrm{d}t} = -pS + \frac{S}{N}\frac{\mathrm{d}N}{\mathrm{d}t} + p\frac{S^2}{mN}. \qquad (1)$$

(a) By multiplying both sides of (1) by $\frac{N}{S^2}$, show that

$$\frac{\mathrm{d}R}{\mathrm{d}t} = pR - \frac{p}{m} \qquad (2)$$

where $R = \frac{N}{S}$, displaying your working clearly. [3]

(b) Regarding p and m as constants, obtain the general solution of equation (2) for R in terms of t. [3]

(c) Assuming that no individual died at birth and no one was born with smallpox, write down a relation between $N(0)$ and $S(0)$ and hence show that

$$S(t) = \frac{mN(t)}{1+(m-1)\mathrm{e}^{pt}}. \qquad [3]$$

(d) Bernoulli's data states that $p = \frac{1}{8}$ and $m = 8$ when $t = 24$. Estimate the proportion of individuals who would not have had smallpox by the time they reached 24 years of age, giving your answer to the nearest form $\frac{1}{n}$ where n is a positive integer to be determined. [3]

Solution		
	(a)	Multiplying both sides of (1) by $\frac{N}{S^2}$ gives $\frac{N}{S^2}\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{N}{S^2}\left(-pS + \frac{S}{N}\frac{\mathrm{d}N}{\mathrm{d}t} + p\frac{S^2}{mN}\right)$

		$\frac{N}{S^2}\frac{\mathrm{d}S}{\mathrm{d}t}=-\frac{pN}{S}+\frac{1}{S}\frac{\mathrm{d}N}{\mathrm{d}t}+\frac{p}{m}$ $\frac{N}{S^2}\frac{\mathrm{d}S}{\mathrm{d}t}=-\frac{pN}{S}+\frac{S}{S^2}\frac{\mathrm{d}N}{\mathrm{d}t}+\frac{p}{m}$ $\frac{1}{S^2}\left(S\frac{\mathrm{d}N}{\mathrm{d}t}-N\frac{\mathrm{d}S}{\mathrm{d}t}\right)=\frac{pN}{S}-\frac{p}{m}$ Now we have $\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N}{S}\right)=\frac{S\frac{\mathrm{d}N}{\mathrm{d}t}-N\frac{\mathrm{d}S}{\mathrm{d}t}}{S^2}$ by the quotient rule. Substitute $R=\frac{N}{S}$ and $\frac{\mathrm{d}R}{\mathrm{d}t}=\frac{S\frac{\mathrm{d}N}{\mathrm{d}t}-N\frac{\mathrm{d}S}{\mathrm{d}t}}{S^2}$ into the above DE gives DE (2) $\frac{\mathrm{d}R}{\mathrm{d}t}=pR-\frac{p}{m}$.
	(b)	$\frac{\mathrm{d}R}{\mathrm{d}t}=pR-\frac{p}{m}=\frac{p}{m}(mR-1)$ $\Rightarrow\int\frac{1}{mR-1}\mathrm{d}R=\frac{p}{m}\int\mathrm{d}t$ $\Rightarrow\frac{1}{m}\ln(mR-1)=\frac{p}{m}t+C\quad(\because m,R>1\Rightarrow mR>1)$ $\Rightarrow mR-1=A\mathrm{e}^{pt}$ where $A=\mathrm{e}^{Cm}$ $\Rightarrow R=\frac{1}{m}\left(1+A\mathrm{e}^{pt}\right)$ which gives the GS to the DE (2).
	(c)	Since $N(0)$ and $S(0)$ are respectively the number of individuals born and the number of individuals without smallpox at birth, the two values are equal, i.e. $N(0)=S(0)\Rightarrow R(0)=1$. Putting $t=0$ into $R=\frac{1}{m}\left(1+A\mathrm{e}^{pt}\right)$ gives $1=\frac{1}{m}(1+A)\Rightarrow A=m-1$. Therefore $R(t)=\frac{1}{m}\left[1+(m-1)\mathrm{e}^{pt}\right]$ $\Rightarrow\frac{N(t)}{S(t)}=\frac{1}{m}\left[1+(m-1)\mathrm{e}^{pt}\right]$ $\Rightarrow S(t)=\frac{mN(t)}{1+(m-1)\mathrm{e}^{pt}}$.
	(d)	Put $t=24,\ p=\frac{1}{8}$ and $m=8$ into above equation gives

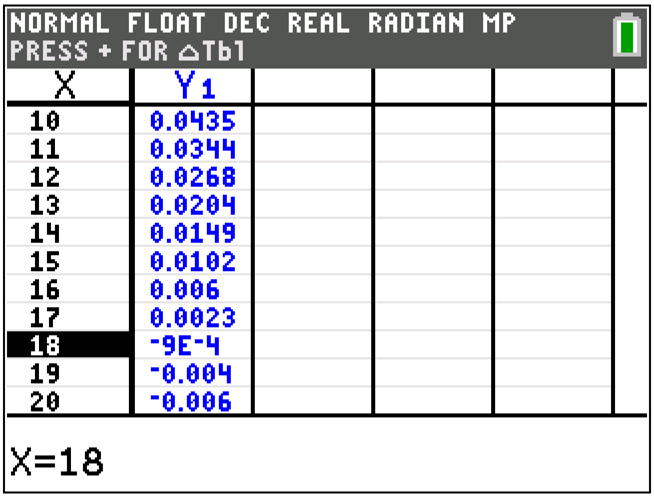
$$S(24) = \frac{8N(24)}{1+(8-1)e^{\frac{1}{8}(24)}} = \frac{8N(24)}{1+7e^{3}}$$

$$\Rightarrow \frac{S(24)}{N(24)} \approx 0.05649767$$

Thus the proportion of individuals who would not have had smallpox by the time they reached 24 years of age is $\frac{1}{18}$.

Key into GC

$Y_1 = \frac{1}{X} - 0.05649767$ and find the value of X such that the value $\frac{1}{X} - 0.05649767$ is least.



NORMAL FLOAT DEC REAL RADIAN MP
PRESS + FOR △Tbl

X	Y1
10	0.0435
11	0.0344
12	0.0268
13	0.0204
14	0.0149
15	0.0102
16	0.006
17	0.0023
18	-9E-4
19	-0.004
20	-0.006

X=18

9 Plankton are a collection of tiny organisms that live at and beneath the surface of lakes, rivers, ponds, and oceans across the planet. A group of plankton biologists are investigating plankton in a lake on a boat. At a depth of h metres, the density of plankton, in millions per cubic metre, is modelled by the function, $\mathrm{p}(h) = 0.15h^2\mathrm{e}^{-0.005h^2}$ for $0 \le h \le 30$ and is modelled by a continuous function, $\mathrm{f}(h)$ for $h \ge 30$ which is not explicitly given.

(a) Find $\mathrm{p}'(20)$. Using correct units, interpret the meaning of $\mathrm{p}'(20)$ in the context of the problem. [2]

b) Consider a vertical column of water in this lake with a uniform horizontal cross sectional area of 2 square metres. Use Simpson's rule with 6 equal intervals to estimate $\int_0^{30} \mathrm{p}(h)\,\mathrm{d}h$. Hence, estimate the number of plankton, to the nearest million, in this column of water from $h = 0$ to $h = 30$. [4]

(c) It is given that there is a function u such that $0 \le \mathrm{f}(h) \le \mathrm{u}(h)$ for all $h \ge 30$ and $\int_{30}^{\infty} \mathrm{u}(h)\,\mathrm{d}h = 105$. The column of water in part **(b)** is H metres deep, where $H > 30$. Write down an expression involving one or more integrals that gives the number of plankton, in millions, in the entire column.

By using your answer in **(b)**, explain why the number of plankton in this entire column is less than or equal to 600 million. [3]

(d) The boat is moving on the surface of the lake. At time $t \ge 0$, the position of the boat is $\left(x(t), y(t)\right)$, where

$$x(t) = 8t^{\frac{1}{2}} - t \text{ and } y(t) = \frac{16}{3}t^{\frac{3}{4}}.$$

Time t is measured in hours, and $x(t)$ and $y(t)$ are measured in kilometres. Find the total distance travelled by the boat over the time interval $0 \le t \le T$, leaving your answer in terms of T. [5]

Solution
(a) Using GC, $\mathrm{p}'(20) = -0.812$. It means that the density of plankton is decreasing at a rate of 0.812 million **per cubic metre per metre** when the depth is 20 metres.
(b) $\int_0^{30} \mathrm{p}(h)\,\mathrm{d}h$ $\approx \left(\frac{5}{3}\right)\left[\mathrm{p}(0) + 4\mathrm{p}(5) + 2\mathrm{p}(10) + 4\mathrm{p}(15) + 2\mathrm{p}(20) + 4\mathrm{p}(25) + \mathrm{p}(30)\right]$ $= 182.465$

Number of plankton $= 2\int_0^{30} \mathrm{p}(h)\,\mathrm{d}h$ ≈ 365 million
(c) Number of plankton in the entire column $= 2\left(\int_0^{30} \mathrm{p}(h)\,\mathrm{d}h + \int_{30}^{H} \mathrm{f}(h)\,\mathrm{d}h\right)$ $\leq 2\left(\int_0^{30} \mathrm{p}(h)\,\mathrm{d}h + \int_{30}^{\infty} \mathrm{u}(h)\,\mathrm{d}h\right)$ $= 365 + 2(105)$ $= 575 < 600$
(d) Distance travelled by the boat $= \int_0^T \sqrt{\left(4t^{-\frac{1}{2}} - 1\right)^2 + \left(4t^{-\frac{1}{4}}\right)^2}\,\mathrm{d}t$ $= \int_0^T \sqrt{16t^{-1} - 8t^{-\frac{1}{2}} + 1 + 16t^{-\frac{1}{2}}}\,\mathrm{d}t$ $= \int_0^T \sqrt{16t^{-1} + 8t^{-\frac{1}{2}} + 1}\,\mathrm{d}t$ $= \int_0^T \sqrt{\left(4t^{-\frac{1}{2}} + 1\right)^2}\,\mathrm{d}t$ $= \int_0^T \left(4t^{-\frac{1}{2}} + 1\right)\mathrm{d}t$ $= \left[8t^{\frac{1}{2}} + t\right]_0^T$ $= 8\sqrt{T} + T$

10 A group of engineers are conducting tests on an experimental cruise missile.

The missile is programmed to be launched from the launchpad, situated at the origin O, to reach cruising altitude along an initial path given by the equation $y = 2\tan^{-1}(x - \sqrt{3}) + \frac{2\pi}{3}, x \in \mathbb{R}, x \geq 0$, where 1 unit denotes 10 metres.

To track the missile, a laser rangefinder is set up to direct a laser beam along the path $y = x$. When the missile crosses the path of the laser beam, the rangefinder will allow the engineers to determine if the missile is moving along its intended path.

The point $A(x_a, y_a)$ denotes the first point (other than O) where the missile crosses the path of the laser beam.

(a) Find the integer n such that $n < x_a < n + 1$. [1]

(b) Use the iterative formula

$$x_{n+1} = 2\tan^{-1}(x_n - \sqrt{3}) + \frac{2\pi}{3} \quad \text{------ (*)}$$

with $x_0 = n$ to obtain 2 successive estimates of x_a, leaving your answers corrected to 4 decimal places. [2]

(c) Using sketches of $y = 2\tan^{-1}(x - \sqrt{3}) + \frac{2\pi}{3}$ and $y = x$ on the same set of axes, explain why the estimates obtained by the iterative formula in **(b)** does not converge to x_a. [2]

(d) Use equation (*) in **(b)** to obtain another iterative formula $x_{n+1} = g(x_n)$ that converges. Taking $x_0 = n$, use this formula to estimate the altitude of the missile at the point (other than O) which the missile first crosses the path of the laser beam, correct to 3 significant figures. Show that your answer has achieved the required level of accuracy. [5]

After cruising for a distance, the missile is programmed to dive towards its target following a path given by the function

$$\mathrm{g}(x) = 5 - (x - 5003)\mathrm{e}^{x-5003}, x \in \mathbb{R}, x \geq 5000.$$

(e) Use linear interpolation on the interval [5000, 5005] to obtain one estimate, α, of the x-coordinate of the point where the missile will impact the ground (which is modelled by the line $y = 0$). Give your answer correct to 2 decimal places. [2]

(f) By using the value of g(α), determine whether α is an overestimate or underestimate of the actual value. [2]

Solution	**[14 marks]**

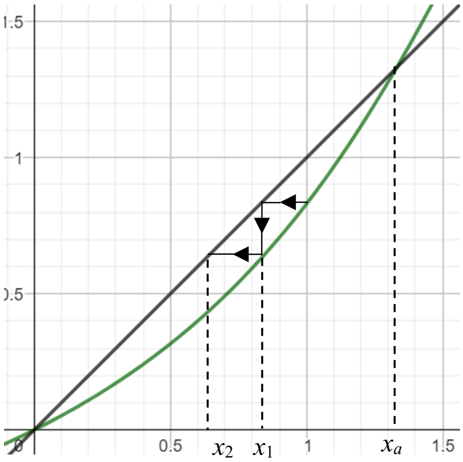
(a) Let $f(x) = 2\tan^{-1}(x - \sqrt{3}) + \frac{2\pi}{3} - x$

f(1) = −0.169 < 0
f(2) = 0.618 > 0
Hence $n = 1$.

(b) Using $x_{n+1} = 2\tan^{-1}(x_n - \sqrt{3}) + \frac{2\pi}{3}$ with $x_0 = 1$:

$x_1 = 0.8306$, $x_2 = 0.6271$

(c)



From the sketch, we see that x_1, x_2 and successive estimates of x_a do not approach x_a; hence the iterative formula does not converge to x_a.

(d) $x = 2\tan^{-1}(x - \sqrt{3}) + \frac{2\pi}{3} \Rightarrow \frac{x}{2} - \frac{\pi}{3} = \tan^{-1}(x - \sqrt{3}) \Rightarrow x = \tan\left(\frac{x}{2} - \frac{\pi}{3}\right) + \sqrt{3}$

Using $x_{n+1} = \tan\left(\frac{x_n}{2} - \frac{\pi}{3}\right) + \sqrt{3}$ with $x_0 = 1$:

$x_1 = 1.123$, $x_2 = 1.204$, $x_3 = 1.255$, $x_4 = 1.286$, $x_5 = 1.304$, $x_6 = 1{,}315$, $x_7 = 1.321$,
$x_8 = 1.327 \approx 1.33$, $x_9 = 1.328 \approx 1.33$
f(1.325) = −0.0037 < 0
f(1.335) = 0.00347 > 0
Hence $x_a = 1.33$ (correct to 3sf)

(e) Let $g(x) = 5 - (x - 5003)e^{x-5003}$

Estimated value = $\frac{5000|g(5005)| + 5005|g(5000)|}{|g(5005)| + |g(5000)|} = 5001.72$

(f) g(5000) = 5.15 > 0
g(5005) = −9.78 < 0
g(5001.72) = 5.36 > 0
Hence the actual value lies between 5001.72 and 5005, and the estimate is an underestimation.