RAFFLES INSTITUTION
H2 Mathematics 9758
2023 Year 6 Term 3 Revision 10 (Summary and Tutorial)

Topic: Functions

Summary for Functions

Relations and Functions

- The set of inputs of a function is known as the **domain** while the corresponding set of outputs is known as the **range**.
- To define a function completely, both the rule and domain must be specified.
- The range of a function can be obtaining by looking at its graph (y-values).
- If a function is such that no two elements in the given domain have the same output, it is a **one-one function**.
- To check if a function is one-one, we use horizontal lines to check.

  If function is one-one, such as $\mathrm{f}: x \mapsto x^2, x > 2$, we write since every horizontal line $y = k$, $k \in [4, \infty)$, cuts the graph of f at one and only one point, f is a one-one function. Note that $[4, \infty)$ is the range of f.

  If function is NOT one-one, such as $\mathrm{f}: x \mapsto x^2, x \in \mathbb{R}$, we choose a specific line $y = k$ where k is a value in range of f. In this case, we can write since the horizontal line $y = 1$ cuts the graph of f at more than one point, f is not a one-one function

Inverse Functions

- Given a function f, the **inverse function** f^{-1} exists if and only if f is one-one.
- To define f^{-1}, you must find the rule and domain.
- To find the rule of f^{-1}, let $y = \mathrm{f}(x)$ and **make *x* the subject**, i.e $x = \mathrm{f}^{-1}(y)$.

  Example to find the rule of inverse of $\mathrm{g}: x \mapsto 1 + \mathrm{e}^{-x}, \quad x \in \mathbb{R}$

  Let $y = 1 + \mathrm{e}^{-x}$ and make x the subject to get $x = -\ln(y-1)$.
- To find domain of f^{-1}, use the result: $\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_\mathrm{f}$

  Note also that $\mathrm{R}_{\mathrm{f}^{-1}} = \mathrm{D}_\mathrm{f}$.
- The graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ are reflections of each other in the line $y = x$.

Composite Functions

- Given 2 functions f and g, the **composite function** gf exists if and only if $R_f \subseteq D_g$.
- To define gf, you must find the rule and domain.
- To find the rule of gf, find $f(x)$ first, then find $g(f(x))$.
- To find domain of gf, use the result: $D_{gf} = D_f$
- In general, $fg \neq gf$. i.e. composition of functions is not commutative.
- The composite function ff, if it exists, is written as f^2, i.e. $f^2(x) = ff(x) = f(f(x))$.
    - Useful results: (i) $ff^{-1}(x) = x$, where $x \in D_{f^{-1}}$.
      (ii) $f^{-1}f(x) = x$, where $x \in D_f$.
- To find the range of gf, use either

(1) $R_{gf} = R_g$ restricted to R_f

or (2) sketch the graph of $y = gf(x)$ over D_{gf} and find R_{gf}.

Example

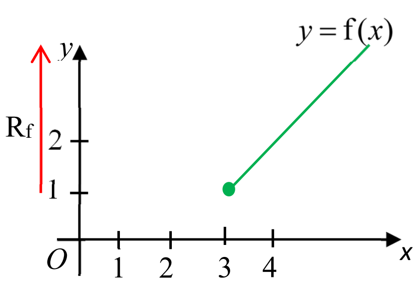
Find the range of the composite gf given that the functions f and g are defined as follows:

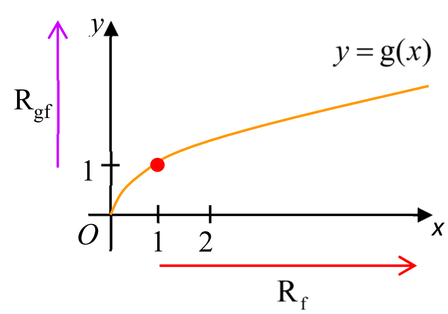
$$f : x \mapsto x - 2, \quad x \geq 3,$$
$$g : x \mapsto \sqrt{x}, \quad x \geq 0.$$

<u>Method 1</u>

Sketch the graphs of the functions f and g on separate diagrams.

For the composite function gf, f is first applied to each $x \in D_f$ to find the image $f(x)$, and then the rule for g is applied to $f(x)$ to obtain the image $g(f(x))$. i.e. $gf(x) = g(f(x))$.





Consequently, we can obtain the range of gf as the range of g whose domain is restricted range of f, i.e $[3, \infty) \xrightarrow{f} [1, \infty) \xrightarrow{g} [1, \infty)$.

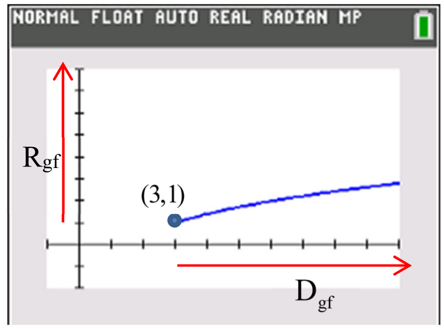
Thus $R_{gf} = [1, \infty)$.

Method 2

Find the function gf followed by sketching its graph to find the range.

$\mathrm{gf}(x) = \mathrm{g}[\mathrm{f}(x)] = \sqrt{x-2}, \quad \mathrm{D}_{\mathrm{gf}} = \mathrm{D}_{\mathrm{f}} = [3, \infty)$

From sketch, $\mathrm{R}_{\mathrm{gf}} = [1, \infty)$.



Revision Tutorial Questions

Source of Question: EJC/Y5 Promo/2018/Q3

1 The function f is defined by $\mathrm{f}: x \mapsto \ln|1-2x|,\ x \in \mathbb{R},\ x \neq \frac{1}{2}$.

(i) Explain why f does not have an inverse. **[1]**

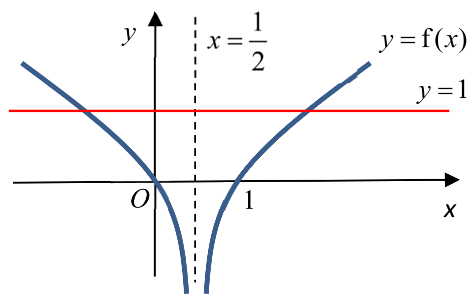
The function g is now defined by $\mathrm{g}: x \mapsto \ln|1-2x|,\ x > a$ where a is a real constant.

(ii) State the minimum value of a such that g^{-1} exists. **[1]**

(iii) Using the value of a stated in **(ii)**, define g^{-1} in a similar form and state the range of g^{-1}. **[4]**

[**(ii)** $a = \frac{1}{2}$ **(iii)** $\mathrm{g}^{-1}: x \mapsto \frac{1}{2}\left(\mathrm{e}^{x}+1\right),\ x \in \mathbb{R}$; $\mathrm{R}_{\mathrm{g}^{-1}} = \left(\frac{1}{2}, \infty\right)$
]

Solution:

1(i) **[1]**	 From sketch, the line $y=1$ intersects the graph of $y=\mathrm{f}(x)$ twice. Hence, f is not one-one and does not have an inverse. Alternatively, since $\mathrm{f}(0)=0=\mathrm{f}(1)$, f is not one-one and does not have an inverse.
1(ii) **[1]**	Minimum value of $a=\frac{1}{2}$
1(iii) **[4]**	Let $y=\ln\|1-2x\|$ For $x>\frac{1}{2}$, $y=\ln(2x-1)$ $2x-1=\mathrm{e}^{y}$ $x=\frac{1}{2}\left(\mathrm{e}^{y}+1\right)$ $\mathrm{g}^{-1}(y)=\frac{1}{2}\left(\mathrm{e}^{y}+1\right)$, $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=\mathbb{R}$ $\therefore \mathrm{g}^{-1}: x \mapsto \frac{1}{2}\left(\mathrm{e}^{x}+1\right),\ x \in \mathbb{R}$ and $\mathrm{R}_{\mathrm{g}^{-1}}=\mathrm{D}_{\mathrm{g}}=\left(\frac{1}{2}, \infty\right)$

Source of Question: NJC/Y5 Promo/2018/Q10

2 The functions g and h are defined by

$$g : x \mapsto \frac{8x}{x^2+2}, \qquad x \geq k,$$
$$h : x \mapsto \frac{1}{4}(x-2)^2, \qquad 0 \leq x \leq 4.$$

(i) Find the smallest exact value of k such that g^{-1} exists. **[2]**

Use $k=2$ for the remainder of this question.

(ii) Find the exact range of g. **[1]**

(iii) Sketch the graphs of

$$y = g(x), \; y = g^{-1}(x) \text{ and } y = gg^{-1}(x)$$

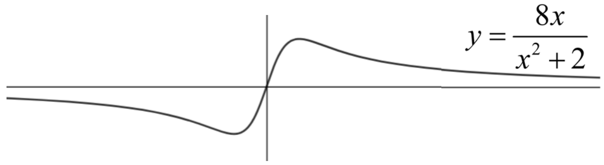
on the same diagram.

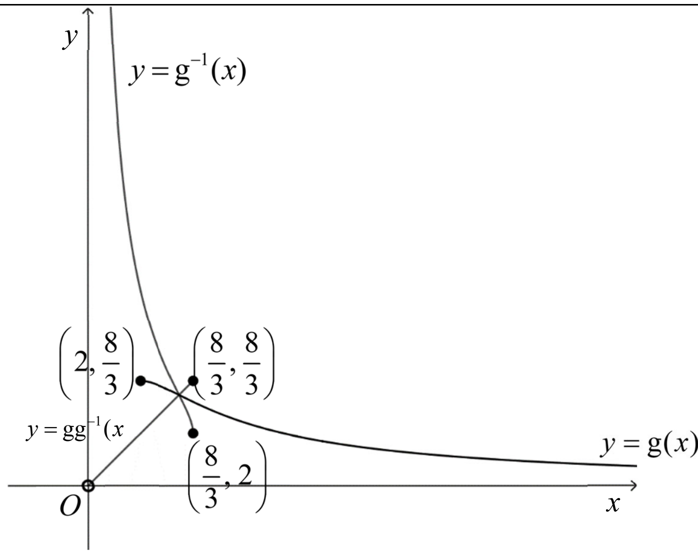
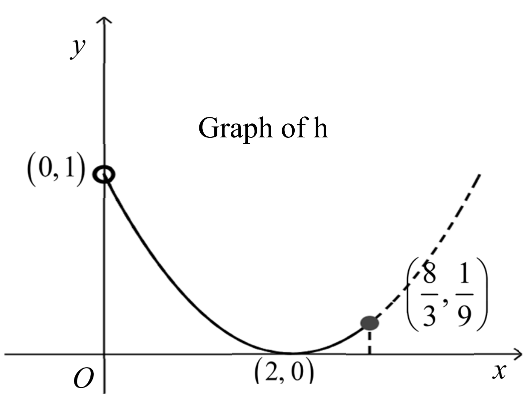
Hence find the exact solution of $g^{-1}(x) = g(x)$. **[6]**

(iv) Show that hg exists and determine the range of hg. **[3]**

[**(i)** $k=\sqrt{2}$ **(ii)** $\left(0, \frac{8}{3}\right]$ **(iii)** $x=\sqrt{6}$ **(iv)** $[0,1)$]

Solution:

2(i) **[2]**	 $g(x)=\frac{8x}{x^2+2}$ $g'(x)=\frac{8(x^2+2)-8x(2x)}{(x^2+2)^2}=\frac{16-8x^2}{(x^2+2)^2}$ For maximum point of $y=g(x)$, $g'(x)=0$. $16-8x^2=0$ $x=\pm\sqrt{2}$ From the graph, it is a one-one function if $x \geq \sqrt{2}$, so the smallest exact value of k is $\sqrt{2}$.
2(ii) **[1]**	$g(2)=\frac{16}{2^2+2}=\frac{8}{3}$, so the range of g is $\left(0, \frac{8}{3}\right]$ as observed from the graph in (i).

2(iii) **[6]**	 From the diagram, we may solve $g(x) = x$ instead of $g^{-1}(x) = g(x)$. $\frac{8x}{x^2+2} = x$ $8x = x^3 + 2x$ $x^3 - 6x = 0$ $x(x^2 - 6) = 0$ $x(x+\sqrt{6})(x-\sqrt{6}) = 0$ $x = 0$ or $x = -\sqrt{6}$ or $x = \sqrt{6}$ From the diagram in **(ii)**, $x = \sqrt{6}$.
2(iv) **[3]**	Since $R_g = \left(0, \frac{8}{3}\right] \subseteq [0, 4] = D_h$, hg exists.  $\left(0, \frac{8}{3}\right] \xrightarrow{h} [0, 1)$

Source of Question: MI/Promo/PU2/2017/Q6

3 A function f is defined by $\mathrm{f}(x)=4-(3-x)^2$ for $x\in\mathbb{R}$.

(i) Show that f^{-1} does not exist. **[1]**

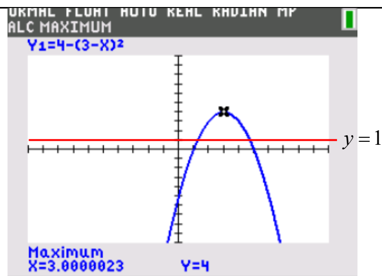
(ii) If the domain of f is restricted to $x\le k$, state the largest exact value of k for which the function f^{-1} exists. **[1]**

(iii) Using the domain defined in part **(ii)**, find f^{-1} and its range. **[3]**

(iv) Solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ exactly. **[3]**

[**(ii)** $k=3$ **(iii)** $\mathrm{f}^{-1}(x)=3-\sqrt{4-x}, x\in(-\infty,4]$; $\mathrm{R}_{\mathrm{f}^{-1}}=(-\infty,3]$ **(iv)** $x=\dfrac{5-\sqrt{5}}{2}$]

Solution:

3(i) **[1]**	 Since the horizontal line $y=1$ intersects the graph of $y=\mathrm{f}(x)$ more than once, f is not one-one. Hence, f^{-1} does not exist. Alternatively, since $\mathrm{f}(1)=0=\mathrm{f}(4)$, f is not one-one and does not have an inverse.
3(ii) **[1]**	$k=3$
3(iii) **[3]**	Let $y=4-(3-x)^2$ $3-x=\pm\sqrt{4-y}$ $x=3\mp\sqrt{4-y}$ Since $x\le 3$, $x=3-\sqrt{4-y}$ Hence, $\mathrm{f}^{-1}(x)=3-\sqrt{4-x}, x\in(-\infty,4]$. So $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=(-\infty,3]$
3(iv) **[3]**	We may solve $\mathrm{f}(x)=x$ instead of $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. $4-(3-x)^2=x$ $x^2-5x+5=0$ $x=\dfrac{5\pm\sqrt{25-4(5)}}{2}=\dfrac{5-\sqrt{5}}{2}$ or $\dfrac{5+\sqrt{5}}{2}$ (rejected $\because$ $x\le 3$)

Source of Question: NYJC/Prelim/2017/01/Q7

4 The functions f and g are defined by

$$\mathrm{f}: x \mapsto \mathrm{e}^{-x^2}, \quad x \in \mathbb{R},\ x < 0,$$

$$\mathrm{g}: x \mapsto \frac{1}{x+3}, \quad x \in \mathbb{R},\ x \neq -3.$$

(i) Show that g^{-1} exists, and define g^{-1} in a similar form. **[2]**

(ii) State the solution set for $\mathrm{g}\mathrm{g}^{-1}(x) = x$. **[1]**

(iii) Explain why fg^{-1} does not exist. **[1]**

Let the function h be defined by

$$\mathrm{h}: x \mapsto \mathrm{g}(x), \quad x \in \mathbb{R},\ x < k,$$

where k is a real constant.

(iv) Given that $\mathrm{f}\,\mathrm{h}^{-1}$ exists, state the maximum value of k. **[1]**

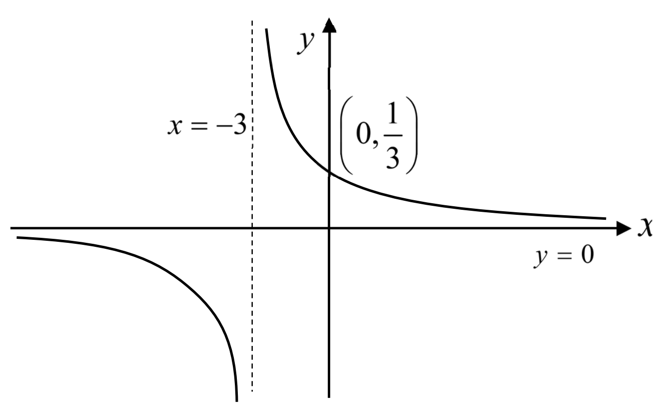
(v) For the value of k found in **(iv)**,

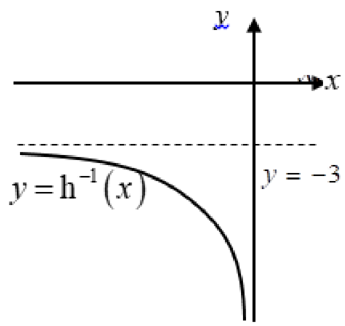
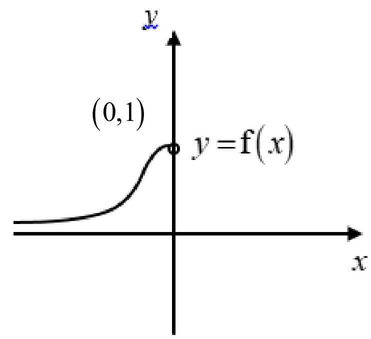
(a) find the exact range of $\mathrm{f}\,\mathrm{h}^{-1}$, **[2]**

(b) solve $\mathrm{h}(x) = \mathrm{h}^{-1}(x)$. **[2]**

[**(i)** $\mathrm{g}^{-1}: x \mapsto \dfrac{1}{x} - 3, x \in \mathbb{R},\ x \neq 0$ **(ii)** $\{x \in \mathbb{R} \mid x \neq 0\}$ **(iv)** $k = -3$
(v)(a) $\mathrm{R}_{\mathrm{fh}^{-1}} = \left(0, \mathrm{e}^{-9}\right)$ **(v)(b)** $x = -3.30$]

Solution:

4(i) **[2]**	(i)  Every horizontal line $y = k$ cuts the graph at most once. This implies g is one-one. Therefore g^{-1} exists

	$g^{-1}: x \mapsto \dfrac{1}{x} - 3, x \in \mathbb{R},\ x \neq 0$
4(ii) **[1]**	$\{x \in \mathbb{R} \mid x \neq 0\}$
4(iii) **[1]**	$R_{g^{-1}} = D_g = \mathbb{R} \setminus \{-3\}$, $D_f = \mathbb{R}^-$. Since $R_{g^{-1}} \not\subset D_f$, fg^{-1} does not exist.
4(iv) **[1]**	$k = -3$
4(v)(a) **[2]**	  From sketch, we have $(-\infty, 0) \xrightarrow{h^{-1}} (-\infty, -3) \xrightarrow{f} (0, e^{-9})$ $R_{fh^{-1}} = (0, e^{-9})$
4(v)(b) **[2]**	$h(x) = h^{-1}(x)$ $h(x) = x$ $\dfrac{1}{x+3} = x$ $x^2 + 3x - 1 = 0$ Since $x < -3$, $x = -3.30$ (3sf)

Source of Question: ACJC/Y5 Promo/2017/Q11

5 **(a)** The function f is defined by

$$\mathrm{f}: x \mapsto \left|\mathrm{e}^{x^2} - 2\right|, \quad x \in \mathbb{R}.$$

If the domain of f is restricted to $k \le x \le 0$,

(i) state the least value of k for which the function f^{-1} exists, **[1]**

(ii) find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. **[3]**

(b) The functions g and h are defined by

$$\mathrm{g}: x \mapsto \begin{cases} \sin^{-1} x & \text{for } -1 < x \le 1, \\ \pi\left(1 - \dfrac{x}{2}\right) & \text{for } 1 < x \le 3, \end{cases}$$

$$\mathrm{h}: x \mapsto 2x, \text{ for } -\frac{\sqrt{3}}{4} \le x \le 1.$$

(i) Show that the composite function gh exists and find its range, giving all values in terms of π. **[3]**

It is given that $\mathrm{g}(x+4) = \mathrm{g}(x)$ for all real values of x.

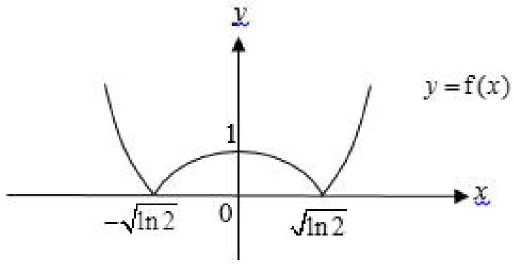
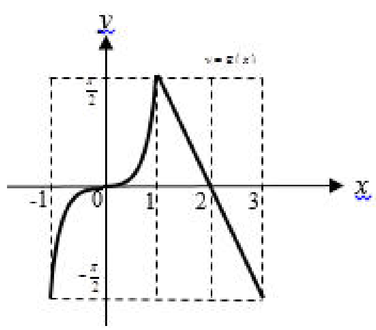
(ii) Evaluate $\mathrm{g}(17)$. **[2]**

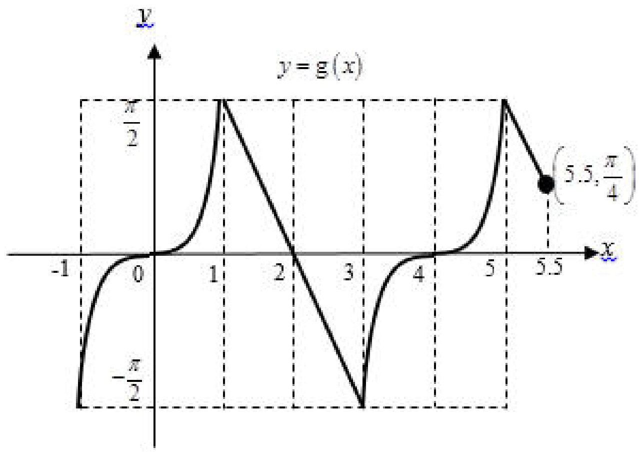
(iii) Sketch the graph of $y = \mathrm{g}(x)$ for $-1 \le x \le 5.5$. **[3]**

[**(a)(i)** $k = -\sqrt{\ln 2}$ **(a)(ii)** $\mathrm{f}^{-1}(x) = -\sqrt{\ln(2-x)}$; $\mathrm{D}_{\mathrm{f}^{-1}} = [0,1]$

(b)(i) $\mathrm{R}_{\mathrm{gh}} = \left[-\frac{\pi}{3}, \frac{\pi}{2}\right]$ **(b)(ii)** $\frac{\pi}{2}$]

Solution:

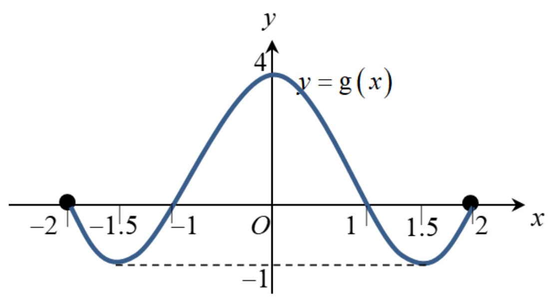
5(a)(i) **[1]**	 f^{-1} exists if least value of k is $-\sqrt{\ln 2}$.
5(a)(ii) **[3]**	If the domain of f is restricted to $-\sqrt{\ln 2} \le x \le 0$, let $y = -\left(e^{x^2} - 2\right)$ $e^{x^2} = 2 - y$ $x^2 = \ln(2 - y)$ $x = \pm\sqrt{\ln(2-y)}$ Since $-\sqrt{\ln 2} \le x \le 0$, $\therefore x = -\sqrt{\ln(2-y)}$ $f^{-1}(x) = -\sqrt{\ln(2-x)},\ 0 \le x \le 1.$ Note: $D_{f^{-1}} = R_f = [0,1]$
5(b)(i) **[3]**	 $R_h = \left[-\frac{\sqrt{3}}{2}, 2\right]$, $D_g = (-1, 3]$ Since $R_h \subseteq D_g$, $\therefore$ gh exist. So $R_{gh} = \left[-\frac{\pi}{3}, \frac{\pi}{2}\right]$.
5(b)(ii) **[2]**	Since $g(x+4) = g(x)$,

	$g(17) = g(13)$ i.e. when $x = 13$ $= g(9)$ $= g(5)$ $= g(1) = \sin^{-1}(1) = \dfrac{\pi}{2}$
5(b)(iii) [3]	

Source of Question: TPJC/Prelim/2017/01/Q3

6 A function f is defined by $\mathrm{f}: x \mapsto (x-k)^2$, $x<k$ where $k>5$.

(i) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**



The diagram above shows the curve with equation $y=\mathrm{g}(x)$, where $-2 \le x \le 2$. The curve crosses the x–axis at $x=-2$, $x=-1$, $x=1$ and $x=2$, and has turning points at $(-1.5,-1)$, $(0,4)$ and $(1.5,-1)$.

(ii) Explain why the composite function fg exists. **[2]**

(iii) Find in terms of k,

(a) the value of $\mathrm{fg}(-1)$, **[1]**

(b) range of fg. **[2]**

[**(i)** $\mathrm{f}^{-1}(x)=-\sqrt{x}+k$, $D_{\mathrm{f}^{-1}}=[0,1]$ **(iii)(a)** k^2 **(iii)(b)** $\left[(4-k)^2,\ (1+k)^2\right]$]

Solution:

6(i) **[3]**	Let $y=(x-k)^2$ $x-k=\pm\sqrt{y}$ $x=-\sqrt{y}+k \quad (\because x<k)$ $\mathrm{f}^{-1}(x)=-\sqrt{x}+k$ $D_{\mathrm{f}^{-1}}=(0,\ \infty)$
6(ii) **[2]**	$R_{\mathrm{g}}=[-1,\ 4]$, $D_{\mathrm{f}}=(-\infty,\ k)$ Since $k>5$, $R_{\mathrm{g}} \subseteq D_{\mathrm{f}}$. Thus fg exists.
6(iii)(a) **[1]**	$\mathrm{fg}(-1)=\mathrm{f}(0)=k^2$

6(iii)(b) **[2]**	Using $R_g = [-1,\ 4]$, and the fact that f is a strictly decreasing function in the given domain, $$R_{fg} = \left[(4-k)^2,\ (-1-k)^2\right]$$ $$= \left[(4-k)^2,\ (1+k)^2\right]$$

Source of Question: HCI/Promo/2018/Q6

7 The function f is defined by

$$\mathrm{f}: x \mapsto \frac{x}{ax-1}, \quad \text{for } x \in \mathbb{R},\ -\frac{1}{a} \le x < \frac{1}{a},$$

where a is a positive constant.

(i) Sketch the curve $y = \mathrm{f}(x)$, stating the equation of the asymptote and the coordinates of the end-point. **[2]**

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. **[3]**

(iii) Sketch the curve $y = \mathrm{f}^{-1}(x)$ on the same diagram as the curve $y = \mathrm{f}(x)$ in part **(i)**. Hence obtain the range of values of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ in terms of a. **[2]**

The function g is defined by

$$\mathrm{g}: x \mapsto \left(x - \frac{1}{3a}\right)^2 - \frac{1}{9a^2} \quad \text{for } x \in \mathbb{R},\ x < \frac{1}{a}.$$

(iv) Show that the composite function gf exists, and find the range of gf in terms of a.

[2]

[**(ii)** $\mathrm{f}^{-1}(x) = \dfrac{x}{ax-1}$; $D_{\mathrm{f}^{-1}} = \left(-\infty, \dfrac{1}{2a}\right]$ **(iii)** $-\dfrac{1}{a} \le x \le \dfrac{1}{2a}$ **(iv)** $\left[-\dfrac{1}{9a^2},\ \infty\right)$]

Solution:

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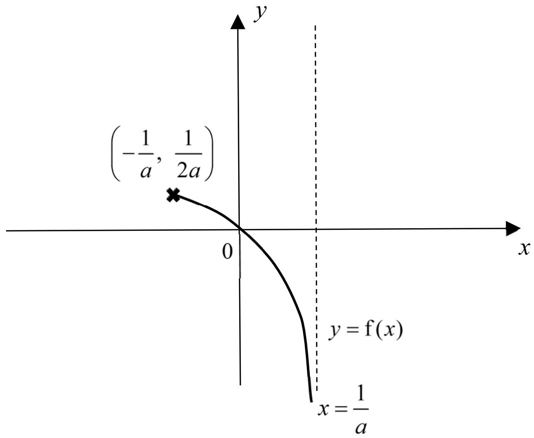
<td>7(i)
[2]</td>
<td>


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<tr>
<td>7(ii)
[3]</td>
<td>
Let $y = \dfrac{x}{ax-1}$
$\Rightarrow axy - y = x$
$\Rightarrow x(ay-1) = y$
$\Rightarrow x = \dfrac{y}{ay-1}$

Therefore $\mathrm{f}^{-1}(x) = \dfrac{x}{ax-1}$, $\quad \mathrm{D}_{\mathrm{f}^{-1}} = \left(-\infty, \dfrac{1}{2a}\right]$
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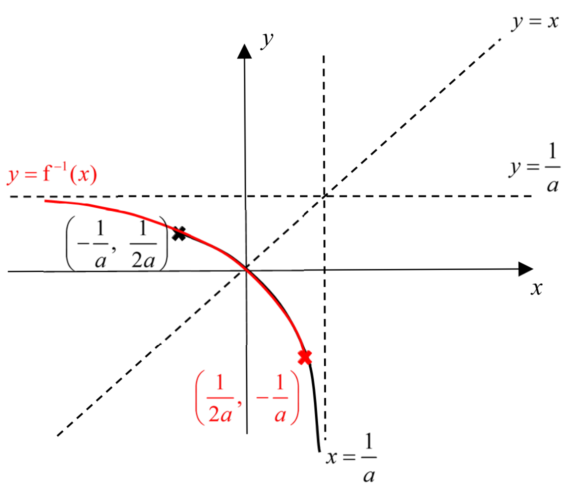
<tr>
<td>7(iii)
[2]</td>
<td>


From graph, the range of values of x for which $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ is
$-\dfrac{1}{a} \le x \le \dfrac{1}{2a}$.
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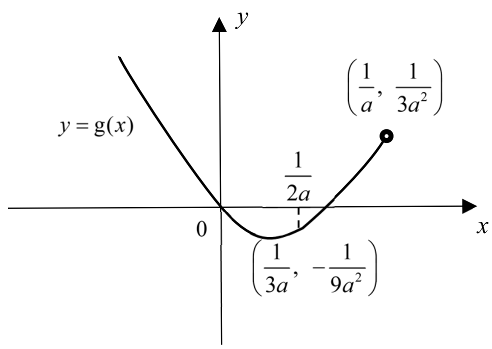
7(iv) **[2]**	Since $\frac{1}{2a} < \frac{1}{a}$, $R_{\mathrm{f}} = \left(-\infty,\ \frac{1}{2a}\right]$ is a subset of $D_{\mathrm{g}} = \left(-\infty,\ \frac{1}{a}\right)$. Therefore gf exists. From the given expression of $\mathrm{g}(x)$, the graph of $y = \mathrm{g}(x)$ is a quadratic graph with minimum point $\left(\frac{1}{3a},\ -\frac{1}{9a^2}\right)$  $\mathrm{R_f} = \left(-\infty,\ \frac{1}{2a}\right] \xrightarrow{\mathrm{g}} \left[-\frac{1}{9a^2},\ \infty\right)$ $\quad \therefore \mathrm{R_{gf}} = \left[-\frac{1}{9a^2},\ \infty\right)$	

Source of Question: SRJC JC2 Mid-Year CT 9758/2018/01/Q2

8 The functions f and g are defined by

$$\mathrm{f} : x \mapsto 2\cos\left(x + \frac{\pi}{4}\right),\ -\pi \le x \le \pi,$$

$$\mathrm{g} : x \mapsto (x-1)^2 + a,\ -3 < x \le 2 \text{ and } -1 < a < 0.$$

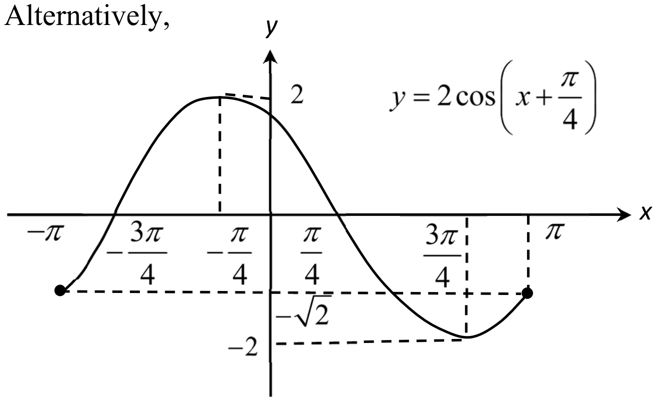
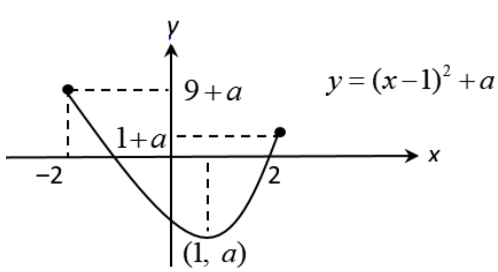
(i) Explain why f does not have an inverse. **[1]**

(ii) Explain why the composite function fg does not exist. Given that the composite function gf exists, find gf in similar form and find its range in terms of a. **[4]**

(iii) The function f has an inverse if the domain is restricted to $-\pi \le x \le p$. State the largest possible exact value of p. **[1]**

[**(ii)** $\mathrm{gf} : x \mapsto \left[2\cos\left(x + \frac{\pi}{4}\right) - 1\right]^2 + a,\ -\pi \le x \le \pi$; $\mathrm{R_{gf}} = [a, 9+a]$ **(iii)** $p = -\frac{\pi}{4}$]

Solution:

8(i) **[1]**	Since $\text{f}\left(-\frac{3\pi}{4}\right)=\text{f}\left(\frac{\pi}{4}\right)=0$, f is not 1–1 and hence f does not have an inverse. Alternatively,  Since the line $y = 0$ cuts the curve twice, so f is not a 1-1 function and so f^{-1} does not exist.	
8(ii) **[4]**	Since $\text{R}_\text{g} = [a,16+a)\not\subset[-\pi,\pi] = \text{D}_\text{f}$, fg does not exist. ($\because -1<a<0$, $15<16+a<16$) $\text{D}_{\text{gf}} = \text{D}_\text{f} = [-\pi,\pi]$ $\text{gf}: x \mapsto \left[2\cos\left(x+\frac{\pi}{4}\right)-1\right]^2+a,\ -\pi\le x\le\pi$  Range of gf $= [a, 9+a]$	
8(iii) **[1]**	$p=-\frac{\pi}{4}$	

Source of Question: SAJC/Prelim/2017/01/Q3

9 It is given that

$$f(x)=\begin{cases} b\sqrt{1-\dfrac{x^2}{a^2}} & \text{for } -a<x\le a \\ -a\sqrt{1-\dfrac{(x-2a)^2}{a^2}} & \text{for } a<x\le 3a \end{cases}$$

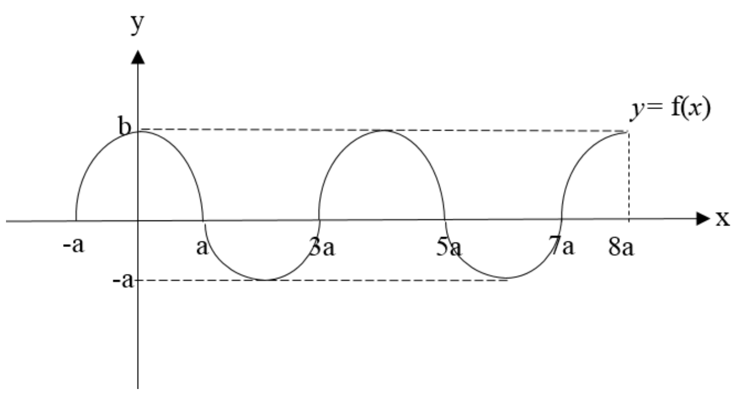
and that $f(x+4a)=f(x)$ for all real values of x, where a and b are real constants and $0<a<b$.

(i) Sketch the graph of $y=f(x)$ for $-a\le x\le 8a$. **[3]**

(ii) Use the substitution $x=a\cos\theta$ to find the exact value of $\int_{3a}^{4a} f(x)\,dx$ in terms of a, b and π. **[5]**

[**(ii)** $\dfrac{\pi}{4}ab$]

Solution:

9(i) **[3]**	

9(ii) **[5]**	$\int_{3a}^{4a} \mathrm{f}(x)\ \mathrm{d}x$ $= \int_{-a}^{0} b\sqrt{1-\frac{x^2}{a^2}}\ \mathrm{d}x$ $=b\int_{\pi}^{\frac{\pi}{2}} \sqrt{1-\frac{a^2\cos^2\theta}{a^2}}\ (-a\sin\theta)\ \mathrm{d}\theta$ $=ab\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\ \mathrm{d}\theta$ $= ab\int_{\frac{\pi}{2}}^{\pi} \frac{1-\cos 2\theta}{2}\ \mathrm{d}\theta$ $= \frac{ab}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{\pi}$ $= \frac{ab}{2}\left[\pi - \frac{\pi}{2}\right]$ $= \frac{\pi}{4}ab$

Source of Question: HCI/Prelim/2017/01/Q1

10 The *floor function*, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x. For example, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 3.5 \rfloor = 3$.

The function f is defined by

$$\mathrm{f}(x) = \begin{cases} \lfloor x \rfloor & \text{for } x \in \mathbb{R}, \quad -1 \le x < 2, \\ 0 & \text{for } x \in \mathbb{R}, \quad 2 \le x < 3, \end{cases}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

It is given that $\mathrm{f}(x) = \mathrm{f}(x+4)$.

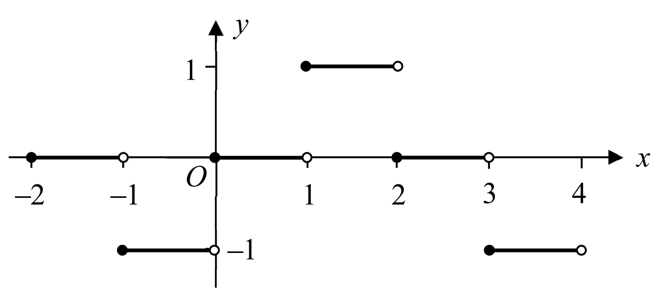
(i) Find the values of $\mathrm{f}(-1.2)$ and $\mathrm{f}(3.6)$. **[2]**

(ii) Sketch the graph of $y = \mathrm{f}(x)$ for $-2 \le x < 4$. **[2]**

(iii) Hence evaluate $\int_{-2}^{4} \mathrm{f}(x)\, \mathrm{d}x$. **[1]**

[**(i)** $\mathrm{f}(-1.2) = 0$, $\mathrm{f}(3.6) = -1$ **(iii)** -1]

Solution:

10(i) **[2]**	$\mathrm{f}(-1.2) = \mathrm{f}(2.8) = 0$ $\mathrm{f}(3.6) = \mathrm{f}(-0.4) = -1$	
10(ii) **[2]**		
10(iii) **[1]**	$\int_{-2}^{4} \mathrm{f}(x)\mathrm{d}x = -1+1-1 = -1$	