2023 SH1 H2 Further Mathematics Promotional Examination Solutions

Qn	Solution
1	Method 1: Standard check for linear independence. Consider the equation $\alpha\mathbf{u}+\beta\mathbf{Au}=\mathbf{0}$. (1) $\mathbf{A}(\alpha\mathbf{u}+\beta\mathbf{Au})=\mathbf{A0}$ $\alpha\mathbf{Au}+\beta\mathbf{A}^2\mathbf{u}=\mathbf{0}$ $\alpha\mathbf{Au}+\mathbf{0}=\mathbf{0}$ ($\because \mathbf{A}^2=\mathbf{O}$) $\alpha\mathbf{Au}=\mathbf{0}$ $\alpha=0$ ($\because \mathbf{Au}\neq\mathbf{0}$). Sub into (1): $\beta\mathbf{Au}=\mathbf{0}$ $\beta=0$ ($\because \mathbf{Au}\neq\mathbf{0}$). $\therefore$ Equation (1) has only the trivial solution. Therefore, **u** and **Au** are linearly independent.
	Method 2: Proof by contradiction. Suppose **u** and **Au** are linearly dependent. Version 1 Then, $\mathbf{u}=k\mathbf{Au}$ for some $k\in\mathbb{R}$. $\mathbf{Au}=\mathbf{A}(k\mathbf{Au})$ $=k\mathbf{A}^2\mathbf{u}$ $=k\mathbf{Ou}$ $=k\mathbf{0}$ $=\mathbf{0}$, contradicting $\mathbf{Au}\neq\mathbf{0}$. Version 2 Then, $\mathbf{Au}=k\mathbf{u}$ for some $k\in\mathbb{R}$. (1) $\mathbf{Au}=k\mathbf{u}$ $\mathbf{A}^2\mathbf{u}=\mathbf{A}(k\mathbf{u})$ $\mathbf{Ou}=k\mathbf{Au}$ $k\mathbf{Au}=\mathbf{0}$. Since $\mathbf{Au}\neq\mathbf{0}$, it must be the case that $k=0$. Then, from (1), $\mathbf{Au}=0\mathbf{u}=\mathbf{0}$, contradicting $\mathbf{Au}\neq\mathbf{0}$. Therefore, **u** and **Au** are linearly independent.
2(i)	Consider the interval $0\le\theta\le 2\pi$. When $\cos 2\theta=0$, $\theta=\frac{\pi}{4}$ or $\frac{3\pi}{4}$. Total area $=4\times\frac{1}{2}\int_0^{\frac{\pi}{4}} a^2\cos 2\theta\ \mathrm{d}\theta$ (by symmetry) $=2a^2\left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$ $=a^2\left(\sin\frac{\pi}{2}-\sin 0\right)$ $=a^2(1-0)$ $=a^2$.

(ii)	Differentiating both sides of $r^2 = a^2 \cos 2\theta$ w.r.t. θ: $2r\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = -2a^2 \sin 2\theta$ $\dfrac{\mathrm{d}r}{\mathrm{d}\theta} = -\dfrac{a^2 \sin 2\theta}{r}$ $\left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \dfrac{a^4 \sin^2 2\theta}{r^2} = \dfrac{a^4 \sin^2 2\theta}{a^2 \cos 2\theta} = \dfrac{a^2 \sin^2 2\theta}{\cos 2\theta}.$ $\text{Total length} = 4\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{a^2 \cos 2\theta + \frac{a^2 \sin^2 2\theta}{\cos 2\theta}}\ \mathrm{d}\theta$ $= 4a\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}}\ \mathrm{d}\theta$ $= 4a\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\frac{1}{\cos 2\theta}}\ \mathrm{d}\theta$ $= 4a\int_0^{\frac{\pi}{4}} \sqrt{\sec 2\theta}\ \mathrm{d}\theta$ (shown)
3(i) (a)	When $c = \frac{1}{2}$, the recurrence relation becomes $u_{n+2} = u_{n+1}$ which results in a constant sequence.
(i) (b)	When $c = 0$, the recurrence relation becomes $u_{n+2} - u_{n+1} + \frac{1}{4}u_n = 0.$ Auxiliary equation: $m^2 - m + \frac{1}{4} = 0$ $\left(m - \frac{1}{2}\right)^2 = 0$ $m = \frac{1}{2}.$ General solution: $u_n = (A + Bn)\left(\frac{1}{2}\right)^n$. As $n \to \infty$, $\left(\frac{1}{2}\right)^n \to 0$, $u_n = (A + Bn)\left(\frac{1}{2}\right)^n \to 0$. The sequence converges to 0 as *n* becomes very large.
(ii)	$u_{n+2} - u_{n+1} + \left(\frac{1}{4} - c^2\right)u_n = 0$ Auxiliary equation: $m^2 - m + \frac{1}{4} - c^2 = 0$ $m = \dfrac{1 \pm \sqrt{1 - 4\left(\frac{1}{4} - c^2\right)}}{2}$ $= \dfrac{1 \pm \sqrt{4c^2}}{2}$ $= \dfrac{1}{2} \pm \lvert c \rvert.$ Since we have shown the sequence converges when $c = 0$ in part **(i)(b)**, we may consider the case where $c \neq 0$ here. General solution: $u_n = A\left(\frac{1}{2} - \lvert c \rvert\right)^n + B\left(\frac{1}{2} + \lvert c \rvert\right)^n$. <u>Method 1</u>: Magnitude of both roots are at most 1.

	$\left\|\frac{1}{2}-\|c\|\right\| \le 1$ and $\left\|\frac{1}{2}+\|c\|\right\| \le 1$ $-1 \le \frac{1}{2}-\|c\| \le 1$ $\quad$ $-1 \le \frac{1}{2}+\|c\| \le 1$ $-1 \le \|c\|-\frac{1}{2} \le 1$ $\quad$ $-\frac{3}{2} \le \|c\| \le \frac{1}{2}$ $-\frac{1}{2} \le \|c\| \le \frac{3}{2}$ $\quad$ $-\frac{1}{2} \le c \le \frac{1}{2}$ $-\frac{3}{2} \le c \le \frac{3}{2}$ $\therefore -\frac{1}{2} \le c \le \frac{1}{2}$. <u>Method 2</u>: Observe that both roots are equidistant from $\frac{1}{2}$. Observe that the larger root, $\frac{1}{2}+\|c\|$, is at least $\frac{1}{2}$. Thus, for the sequence to converge, we must have $\frac{1}{2}+\|c\| \le 1$ $\|c\| \le \frac{1}{2}$ $-\frac{1}{2} \le c \le \frac{1}{2}.$ Also, check that when $\|c\| \le \frac{1}{2}$, the smaller root $\frac{1}{2}-\|c\|$ satisfies $0 \le \frac{1}{2}-\|c\| \le \frac{1}{2}$ and will thus have a magnitude of at most 1. Hence, the required range of values of c is $-\frac{1}{2} \le c \le \frac{1}{2}$.
4(i)	$S_n = \left\{ \mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R}) \middle\| \mathbf{v} \in \text{null space of } \mathbf{X} \right\}$ Take $\mathbf{O} \in \mathbf{M}_{n\times n}(\mathbb{R})$. Then, $\mathbf{Ov} = \mathbf{0} \Rightarrow \mathbf{v} \in$ null space of $\mathbf{O}$. $\therefore \mathbf{O} \in S_n$. Take $\mathbf{A}, \mathbf{B} \in S_n$ and $k \in \mathbb{R}$. Then, $\mathbf{v} \in$ null space of $\mathbf{A}$ and $\mathbf{v} \in$ null space of $\mathbf{B}$ $\Rightarrow \mathbf{Av} = \mathbf{0}$ and $\mathbf{Bv} = \mathbf{0}$. $(\mathbf{A}+\mathbf{B})\mathbf{v} = \mathbf{Av}+\mathbf{Bv} = \mathbf{0}+\mathbf{0} = \mathbf{0}$. $\Rightarrow \mathbf{v} \in$ null space of $\mathbf{A}+\mathbf{B}$ $\Rightarrow \mathbf{A}+\mathbf{B} \in S_n$. $\therefore S_n$ is closed under addition. $(k\mathbf{A})\mathbf{v} = k(\mathbf{Av}) = k\mathbf{0} = \mathbf{0}$ $\Rightarrow \mathbf{v} \in$ null space of $k\mathbf{A}$ $\Rightarrow k\mathbf{A} \in S_n$. $\therefore S_n$ is closed under scalar multiplication. Therefore, S_n is a subspace of $\mathbf{M}_{n\times n}(\mathbb{R})$.

(ii)	$S_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{R}) \middle\vert \begin{pmatrix} 1 \\ 2 \end{pmatrix} \in \text{null space of} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$ $= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{R}) \middle\vert \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ $= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{R}) \middle\vert a + 2b = 0,\ c + 2d = 0 \right\}$ $= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{M}_{2\times 2}(\mathbb{R}) \middle\vert a = -2b,\ c = -2d \right\}$ $= \left\{ \begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} \middle\vert b, d \in \mathbb{R} \right\}$ $= \left\{ b\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \middle\vert b, d \in \mathbb{R} \right\}$ $= \text{span}\left\{ \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \right\}.$ Since $\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$ are not scalar multiples of each other, they are linearly independent. Therefore, a basis for S_2 is $\left\{ \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} \right\}$.
(iii)	Take $\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in S_n$ and denote $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n$. Then, $\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $a_{11}v_1 + a_{12}v_2 + \ldots + a_{1n}v_n = 0$ $\vdots$ $a_{n1}v_1 + a_{n2}v_2 + \ldots + a_{nn}v_n = 0$ Each equality reduces the degree of freedom of the entries in a matrix by 1. Since **A** has n^2 entries and there are *n* linearly independent equalities, the entries in **A** have a degree of freedom of $\dim\left(\mathbf{M}_{n\times n}(\mathbb{R})\right) - n = n^2 - n$. Thus, the dimension of S_n, which is the degree of freedom of the entries of a matrix from S_n, is $n^2 - n$.
(iv)	Since $\mathbf{O}_{n\times n}\mathbf{x} = \mathbf{0} \neq \mathbf{v}$ for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{v} \notin$ column space of $\mathbf{O}$. $\therefore\ \mathbf{O} \notin \left\{ \mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R}) \middle\vert \mathbf{v} \in \text{column space of } \mathbf{X} \right\}$. Therefore, $\left\{ \mathbf{X} \in \mathbf{M}_{n\times n}(\mathbb{R}) \middle\vert \mathbf{v} \in \text{column space of } \mathbf{X} \right\}$ is not a subspace of $\mathbf{M}_{n\times n}(\mathbb{R})$.

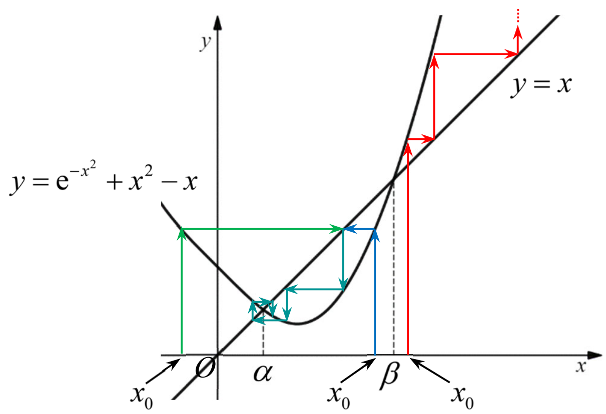
5(i)	$a_n = (a_{n-1}+60)\left(\frac{1}{16}\right)$ $a_n = \frac{1}{16}a_{n-1} + \frac{15}{4}$ General solution: $a_n = A\left(\frac{1}{16}\right)^n + B$, $n \geq 0$. $a_0 = 0 \Rightarrow A + B = 0$. Since $a_1 = (0+60)\left(\frac{1}{16}\right) = \frac{15}{4}$, $\frac{15}{4} = A\left(\frac{1}{16}\right)^1 + B$. By GC, $A = -4,\ B = 4$. $\therefore a_n = -4\left(\frac{1}{16}\right)^n + 4,\ n \geq 0.$
(ii)	$\lim\limits_{n\to\infty} a_n = 4$.
(iii)	Let b_m denote the amount of drug, in mg, at the end of the m^{th} day of taking *d* mg of the drug a day. Then, $b_0 = 4$. $$b_m = (b_{m-1}+d)\left(\frac{1}{16}\right)$$ $$b_m = \frac{1}{16}b_{m-1} + \frac{1}{16}d$$ General solution: $b_m = C\left(\frac{1}{16}\right)^m + D$, $m \geq 0$. $b_0 = 4 \Rightarrow C + D = 4$. Since $b_1 = (4+d)\left(\frac{1}{16}\right) = \frac{4+d}{16}$, $$\frac{4+d}{16} = C\left(\frac{1}{16}\right) + D$$ $\Rightarrow C = 4 - \frac{d}{15},\ D = \frac{d}{15}$. $\therefore b_m = \left(4-\frac{d}{15}\right)\left(\frac{1}{16}\right)^m + \frac{d}{15},\ n \geq 0.$ Since $\left\lvert\frac{1}{16}\right\rvert < 1$ and $d > 60 \Rightarrow \frac{d}{15} > 4 \Rightarrow 4 - \frac{d}{15} < 0$, $\{b_n\}$ is an increasing sequence converging to $\frac{d}{15}$. So, for an overdose to never occur, $\frac{d}{15} + d \leq 80$. Thus, we have $$\frac{16d}{15} \leq 80$$ $$d \leq 75.$$ Therefore, the maximum value of *d* is 75.
6(i)	When $y = 0$, $t\cos 2t = 0$, $t = 0$ or $\cos 2t = 0$, $t = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$. $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{2\sqrt{t}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}t} = \cos 2t - 2t\sin 2t$.

	Surface area $= 2\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\ \mathrm{d}t+\pi\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)$ $= 2\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{t}\sqrt{\left(\frac{1}{2\sqrt{t}}\right)^2+\left(\cos 2t-2t\sin 2t\right)^2}\ \mathrm{d}t+\frac{\pi^2}{2}$ $= 31.958 \text{ units}^2$ (to 5 s.f.) $= 32.0 \text{ units}^2$ (to 3 s.f.)
(ii)	Volume $= 2\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x(-y)\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$ $= 2\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sqrt{t}\,(-t\cos 2t)\left(\frac{1}{2\sqrt{t}}\right)\mathrm{d}t$ $= -\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} t\cos 2t\ \mathrm{d}t$ $= -\pi\left(\left[\frac{t\sin 2t}{2}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}-\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{\sin 2t}{2}\ \mathrm{d}t\right)$ $= -\pi\left[\left(-\frac{3\pi}{8}-\frac{\pi}{8}\right)+\left[\frac{\cos 2t}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\right]$ $= \frac{\pi^2}{2}-\pi(0-0)$ $= \frac{\pi^2}{2}$ units^3.
(iii)	$-\pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} t\cos 2t\ \mathrm{d}t$ $\approx -\pi\left(\frac{1}{3}\right)\left(\frac{\pi}{8}\right)\left[\mathrm{f}\left(\frac{\pi}{4}\right)+4\mathrm{f}\left(\frac{3\pi}{8}\right)+2\mathrm{f}\left(\frac{\pi}{2}\right)+4\mathrm{f}\left(\frac{5\pi}{8}\right)+\mathrm{f}\left(\frac{3\pi}{4}\right)\right]$ $= -\frac{\pi^2}{24}\left(0-\frac{3\pi}{2\sqrt{2}}-\pi-\frac{5\pi}{2\sqrt{2}}+0\right)$ $= 4.94605$ (to 5 d.p.) Percentage error $= \left\lvert\frac{4.94605-\frac{\pi^2}{2}}{\frac{\pi^2}{2}}\right\rvert = 0.228\%$ which is very small. Thus, the approximation is very accurate.

7(i)	$\mathbf{A}=\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$. $\mathbf{v}_n=\mathbf{A}\mathbf{v}_{n-1}=\mathbf{A}^2\mathbf{v}_{n-2}=\ldots=\mathbf{A}^n\mathbf{v}_0$. $\mathbf{B}=\mathbf{A}^n$.
(ii)	$\det\left(\lambda\mathbf{I}-\mathbf{A}\right)=0$ $\begin{vmatrix}\lambda-1 & -2\\ -1 & \lambda-1\end{vmatrix}=0$ $\left(\lambda-1\right)^2-2=0$ $\lambda-1=\pm\sqrt{2}$ $\lambda=1\pm\sqrt{2}$ For $\lambda=1-\sqrt{2}$, we solve the system $\left(\left(1-\sqrt{2}\right)\mathbf{I}-\mathbf{A}\right)\mathbf{x}=\mathbf{0}$. $\begin{pmatrix}-\sqrt{2} & -2\\ -1 & -\sqrt{2}\end{pmatrix}\xrightarrow{R_1-\sqrt{2}R_2}\begin{pmatrix}0 & 0\\ -1 & -\sqrt{2}\end{pmatrix}\xrightarrow[R_1\leftrightarrow\sqrt{2}R_2]{-R_2}\begin{pmatrix}1 & \sqrt{2}\\ 0 & 0\end{pmatrix}$. $\therefore$ A corresponding eigenvector is $\begin{pmatrix}\sqrt{2}\\ -1\end{pmatrix}$. For $\lambda=1+\sqrt{2}$, we solve the system $\left(\left(1+\sqrt{2}\right)\mathbf{I}-\mathbf{A}\right)\mathbf{x}=\mathbf{0}$. $\begin{pmatrix}\sqrt{2} & -2\\ -1 & \sqrt{2}\end{pmatrix}\xrightarrow{R_1+\sqrt{2}R_2}\begin{pmatrix}0 & 0\\ -1 & \sqrt{2}\end{pmatrix}\xrightarrow[R_1\leftrightarrow\sqrt{2}R_2]{-R_2}\begin{pmatrix}1 & -\sqrt{2}\\ 0 & 0\end{pmatrix}$. $\therefore$ A corresponding eigenvector is $\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix}$. $\therefore\mathbf{A}=\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}\begin{pmatrix}1-\sqrt{2} & 0\\ 0 & 1+\sqrt{2}\end{pmatrix}\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}^{-1}$. $\mathbf{A}^n=\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}\begin{pmatrix}1-\sqrt{2} & 0\\ 0 & 1+\sqrt{2}\end{pmatrix}^n\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}^{-1}$ $=\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}\begin{pmatrix}\left(1-\sqrt{2}\right)^n & 0\\ 0 & \left(1+\sqrt{2}\right)^n\end{pmatrix}\begin{pmatrix}\sqrt{2} & \sqrt{2}\\ -1 & 1\end{pmatrix}^{-1}$.

(iii)	$\mathbf{v}_n = \mathbf{A}^n \mathbf{v}_0$ $= \mathbf{PDP}^{-1}\mathbf{v}_0$ $= \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \left(1-\sqrt{2}\right)^n & 0 \\ 0 & \left(1+\sqrt{2}\right)^n \end{pmatrix} \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} \sqrt{2}\left(1-\sqrt{2}\right)^n & \sqrt{2}\left(1+\sqrt{2}\right)^n \\ -\left(1-\sqrt{2}\right)^n & \left(1+\sqrt{2}\right)^n \end{pmatrix} \dfrac{1}{\sqrt{2}+\sqrt{2}} \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $= \dfrac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{2}\left(1-\sqrt{2}\right)^n & \sqrt{2}\left(1+\sqrt{2}\right)^n \\ -\left(1-\sqrt{2}\right)^n & \left(1+\sqrt{2}\right)^n \end{pmatrix} \begin{pmatrix} 1-\sqrt{2} \\ 1+\sqrt{2} \end{pmatrix}$ $= \dfrac{1}{2\sqrt{2}} \begin{pmatrix} \sqrt{2}\left(1-\sqrt{2}\right)^{n+1} + \sqrt{2}\left(1+\sqrt{2}\right)^{n+1} \\ -\left(1-\sqrt{2}\right)^{n+1} + \left(1+\sqrt{2}\right)^{n+1} \end{pmatrix}.$ Since $\left\lvert 1-\sqrt{2} \right\rvert < 1$, $\left(1-\sqrt{2}\right)^n \to 0$ as $n \to \infty$. Thus, as $n \to \infty$, $\dfrac{a_n}{b_n} = \dfrac{\sqrt{2}\left(1-\sqrt{2}\right)^{n+1} + \sqrt{2}\left(1+\sqrt{2}\right)^{n+1}}{-\left(1-\sqrt{2}\right)^{n+1} + \left(1+\sqrt{2}\right)^{n+1}} \to \dfrac{\sqrt{2}\left(1+\sqrt{2}\right)^{n+1}}{\left(1+\sqrt{2}\right)^{n+1}} = \sqrt{2}.$
8 **(i)**	Take $\mathrm{f}(x), \mathrm{g}(x) \in \mathbf{P}_2$, $k \in \mathbb{R}$. $\mathrm{T}\left(\mathrm{f}(x)+\mathrm{g}(x)\right) = \mathrm{f}(x)+\mathrm{g}(x)+\left(\mathrm{f}(x)+\mathrm{g}(x)\right)'$ $= \mathrm{f}(x)+\mathrm{g}(x)+\mathrm{f}'(x)+\mathrm{g}'(x)$ $= \mathrm{f}(x)+\mathrm{f}'(x)+\mathrm{g}(x)+\mathrm{g}'(x)$ $= \mathrm{T}\left(\mathrm{f}(x)\right)+\mathrm{T}\left(\mathrm{g}(x)\right).$ $\mathrm{T}\left(k\mathrm{f}(x)\right) = k\mathrm{f}(x)+\left(k\mathrm{f}(x)\right)'$ $= k\mathrm{f}(x)+k\mathrm{f}'(x)$ $= k\left(\mathrm{f}(x)+\mathrm{f}'(x)\right)$ $= k\mathrm{T}\left(\mathrm{f}(x)\right)$ Therefore, T is a linear transformation.
(ii)	$\mathrm{T}\left(ax^2+bx+c\right) = 0x^2+0x+0$ $\left(ax^2+bx+c\right)+\left(2ax+b\right) = 0x^2+0x+0$ $ax^2+\left(2a+b\right)x+\left(b+c\right) = 0x^2+0x+0$ $\left.\begin{aligned} a &= 0 \\ 2a+b &= 0 \\ b+c &= 0 \end{aligned}\right\} \Rightarrow a=b=c=0$ $\Rightarrow$ Null space of $\mathrm{T} = \left\{0x^2+0x+0\right\}$. $\therefore$ A basis for the null space of T is $\varnothing$.

	$\text{rank}(\mathrm{T}) = \dim(\mathbf{P}_2) - \text{nullity}(\mathrm{T})$ $= 3 - 0$ $= 3.$
(iii)	$\mathrm{T}^2\left(ax^2+bx+c\right) = \mathrm{T}\left(ax^2+(2a+b)x+(b+c)\right)$ $= ax^2+(2a+b)x+(b+c)+2ax+2a+b$ $= ax^2+(4a+b)x+2a+2b+c$ (shown). $\mathrm{T}^3\left(ax^2+bx+c\right) = \mathrm{T}\left(ax^2+(4a+b)x+2a+2b+c\right)$ $= ax^2+(4a+b)x+2a+2b+c+2ax+4a+b$ $= ax^2+(6a+b)x+6a+3b+c.$ Conjecture: $\mathrm{T}^n\left(ax^2+bx+c\right) = ax^2+(2na+b)x+n(n-1)a+nb+c$ for all $n \in \mathbb{Z}^+$. Let P_n be the statement $\mathrm{T}^n\left(ax^2+bx+c\right) = ax^2+(2na+b)x+n(n-1)a+nb+c$ for all $n \in \mathbb{Z}^+$. When $n=1$, $\text{RHS} = ax^2+(2a+b)x+(b+c)$ $= \text{LHS}$ (by part **(ii)**). $\therefore P_1$ is true. Assume that P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\mathrm{T}^k\left(ax^2+bx+c\right) = ax^2+(2ka+b)x+k(k-1)a+kb+c$. We want to prove that P_{k+1} is true, i.e., $\mathrm{T}^{k+1}\left(ax^2+bx+c\right)$ $= ax^2+\left(2(k+1)a+b\right)x+(k+1)(k)a+(k+1)b+c.$ $\mathrm{T}^{k+1}\left(ax^2+bx+c\right)$ $= \mathrm{T}\left(ax^2+(2ka+b)x+k(k-1)a+kb+c\right)$ $= ax^2+(2ka+b)x+k(k-1)a+kb+c+2ax+2ka+b$ $= ax^2+(2ka+b+2a)x+\left(k^2-k+2k\right)a+(k+1)b+c$ $= ax^2+\left(2(k+1)a+b\right)x+\left(k^2+k\right)a+(k+1)b+c$ $= ax^2+\left(2(k+1)a+b\right)x+(k+1)(k)a+(k+1)b+c.$ $\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true. Since P_1 is true, and P_k is true $\Rightarrow$ P_{k+1} is true, by mathematical induction, P_n is true for all $n \in \mathbb{Z}^+$.

(iv)	$ax^2+bx+c=0$ has real roots $\Rightarrow b^2-4ac\ge 0$. For the equation $$\mathrm{T}^n\left(ax^2+bx+c\right)=0,$$ $$ax^2+(2na+b)x+n(n-1)a+nb+c=0,$$ discriminant $=(2na+b)^2-4a\left[n(n-1)a+nb+c\right]$ $=4n^2a^2+4nab+b^2-4a\left[n^2a-na+nb+c\right]$ $=4n^2a^2+4nab+b^2-4n^2a^2+4na^2-4nab-4ac$ $=b^2-4ac+4na^2$. Now, $b^2-4ac\ge 0$ and $4na^2>0$ since $n\ge 1$ and $a\ne 0\Rightarrow a^2>0$. $\therefore$ Discriminant $=b^2-4ac+4na^2>0$. Therefore, the equation $\mathrm{T}^n\left(ax^2+bx+c\right)=0$ has real and distinct roots for any positive integer *n*.
9(i)	Let $\mathrm{f}(x)=\mathrm{e}^{-x^2}+x^2-2x$. Check: $\mathrm{f}(0)=1>0$ $\mathrm{f}(1)=-0.63212<0$ $\mathrm{f}(2)=0.018316>0$ Since f is continuous for all $x\in\mathbb{R}$, $0<\alpha<1<\beta<2$. $N=0$.
(ii)	 From the sketch, it can be observed that if the initial estimate x_0 is not too far below α or between α and β, successive estimates converge to α and not to β. Also, an initial estimate x_0 larger than β will diverge away from β. Hence, the iterative formula is suitable for finding an approximation only for α, but not for β.

(iii) Let $x_0 = 0.5$.

i	x_i
0	0.5000
1	0.5288
2	0.5069
3	0.5235
4	0.5109
5	0.5204
6	0.5132
7	0.5187
8	0.5145
9	0.5176
10	0.5153

Check:

$\mathrm{f}(0.515) = -0.0153 < 0$

$\mathrm{f}(0.525) = 0.00226 > 0$

$\therefore \alpha = 0.52$ (to 2 d.p.).

(iv) Using the Newton-Raphson method,

$$x_1 = 1.5 - \frac{\mathrm{f}(1.5)}{\mathrm{f}'(1.5)}$$

$= 2.442671$ (to 6 d.p.)

$= 2.4427$ (to 4 d.p.)

(v)

$$\mathrm{f}(x) = \mathrm{e}^{-x^2} + x^2 - 2x$$

$$\mathrm{f}'(x) = -2x\mathrm{e}^{-x^2} + 2x - 2$$

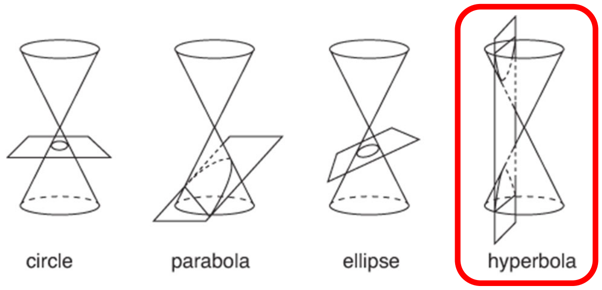
$$\begin{aligned}\mathrm{f}''(x) &= -2\mathrm{e}^{-x^2} + 4x^2\mathrm{e}^{-x^2} + 2 \\ &= 4x^2\mathrm{e}^{-x^2} + 2\left(1 - \mathrm{e}^{-x^2}\right)\end{aligned}$$

Now, $x \neq 0 \Rightarrow x^2 > 0 \Rightarrow \mathrm{e}^{x^2} > 1 \Rightarrow \mathrm{e}^{-x^2} < 1 \Rightarrow 1 - \mathrm{e}^{-x^2} > 0$ and $4x^2\mathrm{e}^{-x^2} > 0$. Thus,

$\mathrm{f}''(x) = 4x^2\mathrm{e}^{-x^2} + 2\left(1 - \mathrm{e}^{-x^2}\right) > 0$. □

By GC, the coordinates of the minimum point are $(1.2583, -0.72799)$. Since $\mathrm{f}''(x) > 0$ for all $x \neq 0$, $\mathrm{f}'(x) > 0$ for all $x > 1.2583$.

Therefore, $k > 1.26$ (3 s.f.).

10 (i)	The wall is a plane that cuts through both cones of the light that is emitted from the wall light. Such an intersection will produce the conic section known as a hyperbola. 
(ii)	Let the equation of the hyperbola be $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$, with $a,b>0$. Since the distance between the vertices is 12 cm, $b=\frac{12}{2}=6$. Gradient of asymptotes are $\pm\frac{12}{8}=\pm\frac{3}{2}=\pm\frac{b}{a}$ $\Rightarrow a=\frac{2b}{3}=\frac{2(6)}{3}=4$. Therefore, equation of hyperbola is $\frac{y^2}{6^2}-\frac{x^2}{4^2}=1$ $\frac{y^2}{36}-\frac{x^2}{16}=1$ (shown)
(iii)	No. Even with the installation of the supporting rod, the ratio $\frac{b}{a}$ asymptotes remain the same as this ratio only depends on the ratio of the height to the radius of the cylinder. The eccentricity of the hyperbola is given by $e=\sqrt{1+\frac{b^2}{a^2}}$. Thus, the new eccentricity remains the same as the value of $\frac{b}{a}$ is unaffected.
(iv)	By similar triangles, the vertical distance between each vertex and the light source is now $\frac{3}{2}\times 6=9\text{ cm}$. Therefore, equation of new hyperbola is $\frac{y^2}{9^2}-\frac{x^2}{\left(\frac{2}{3}\times 9\right)^2}=1$ $\frac{y^2}{81}-\frac{x^2}{36}=1.$
(v)	$e=\sqrt{1-\frac{1^2}{3^2}}$ $=\sqrt{\frac{8}{9}}$ $=\frac{2\sqrt{2}}{3}.$

(vi)	Let θ denote the angle of elevation. Then, an approximation for θ can be obtained by $$\begin{aligned} \sin\theta &\approx \frac{1}{3} \\ \theta &\approx \sin^{-1}\left(\frac{1}{3}\right) \quad (\because \theta \text{ is acute}) \\ &= 19.5^\circ \text{ (to 1 d.p.)} \end{aligned}$$