O	JURONG SECONDARY SCHOOL 2023 GRADUATION EXAMINATION SECONDARY 4 EXPRESS/ SECONDARY 5 NORMAL (ACADEMIC)

CANDIDATE NAME	

CLASS			INDEX NUMBER	

ADDITIONAL MATHEMATICS **4049/01**

PAPER 1
Candidates answer on the Question Paper.
Additional Materials: Writing Paper (1 sheet)

22 August 2023
2 hours 15 minutes

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

For Examiner's Use
90

This document consists of **19** printed pages including this page.

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer, and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 A curve has an equation $y = 2x^2 - x + 5$.

(a) Express $y = 2x^2 - x + 5$ in the form of $a(x-b)^2 + c$. Hence state the coordinates of the turning point. [3]

$y = 2x^2 - x + 5$

$= 2\left(x^2 - \dfrac{x}{2}\right) + 5$ **M1**: Factor out the 2

$= 2\left[x^2 - \dfrac{x}{2} + \left(\dfrac{1}{4}\right)^2 - \left(\dfrac{1}{4}\right)^2\right] + 5$

$= 2\left[x^2 - \dfrac{x}{2} + \left(\dfrac{1}{4}\right)^2\right] - 2\left(\dfrac{1}{4}\right)^2 + 5$

$= 2\left(x - \dfrac{1}{4}\right)^2 + \dfrac{39}{8}$ **A1**

Turning point: $\left(\dfrac{1}{4}, \dfrac{39}{8}\right)$ **B1**: FT from wrong completed square form

(b) The line $y = 2x + 7$ intersects the curve at points *A* and *B*. Find the distance *AB*. [3]

$y = 2x^2 - x + 5 --- (1)$

$y = 2x + 7 --- (2)$

$\therefore 2x^2 - x + 5 = 2x + 7$

$2x^2 - 3x - 2 = 0$

$(2x+1)(x-2) = 0$ **M1**: Solve quadratic equation (FT)

$\therefore x = -\dfrac{1}{2}$ OR $x = 2$

When $x = -\dfrac{1}{2},\ y = 2\left(-\dfrac{1}{2}\right) + 7 = 6$

When $x = 2,\ y = 2(2) + 7 = 11$

$\therefore$ Coordinates of intersection are $\left(-\dfrac{1}{2}, 6\right)$ or $(2,11)$ **A1**

Required distance $= \sqrt{\left(2 + \dfrac{1}{2}\right)^2 + (11-6)^2} = \dfrac{5\sqrt{5}}{2}$ units **A1**: Accept 5.59

2 Express $\dfrac{3x^3+10x^2+x+1}{x^3+3x^2}$ in partial fractions. [6]

By long division, $\dfrac{3x^3+10x^2+x+1}{x^3+3x^2}=3+\dfrac{x^2+x+1}{x^3+3x^2}$ — **B1**: With long division working

$\dfrac{x^2+x+1}{x^3+3x^2}=\dfrac{x^2+x+1}{x^2(x+3)}=\dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x+3}$ — **M1**: Correct form (FT)

$\therefore x^2+x+1=Ax(x+3)+B(x+3)+Cx^2$ — **M3**: Substitution or comparing coefficients correctly per unknown

When $x=0$:

$1=B(3)$

$\therefore B=\dfrac{1}{3}$

$x^2+x+1=Ax(x+3)+\dfrac{1}{3}(x+3)+Cx^2$

When $x=-3$:

$9-3+1=C(-3)^2$

$7=9C$

$\therefore C=\dfrac{7}{9}$

$x^2+x+1=Ax(x+3)+\dfrac{1}{3}(x+3)+\dfrac{7}{9}x^2$

When $x=1$:

$1+1+1=A(1+3)+\dfrac{1}{3}(1+3)+\dfrac{7}{9}(1)^2$

$3=4A+\dfrac{4}{3}+\dfrac{7}{9}$

$\therefore A=\dfrac{2}{9}$

$\therefore \dfrac{x^2+x+1}{x^2(x+3)}=\dfrac{2}{9x}+\dfrac{1}{3x^2}+\dfrac{7}{9(x+3)}$

$\therefore \dfrac{3x^3+10x^2+x+1}{x^3+3x^2}=3+\dfrac{2}{9x}+\dfrac{1}{3x^2}+\dfrac{7}{9(x+3)}$ — **A1**

3 A curve has equation $y=\frac{1}{3}x^3+x^2+kx$, where k is a constant and $k>1$. Explain why the curve does not have a stationary point. [4]

$y=\frac{1}{3}x^3+x^2+kx$

$\frac{dy}{dx}=x^2+2x+k$

M1: Find $\frac{dy}{dx}$

Method 1:

To find stationary point, $\frac{dy}{dx}=0$:

$x^2+2x+k=0$

M1: Find discriminant of $\frac{dy}{dx}=0$

Assume on the contrary that curve has at least one stationary point.

$\therefore$ There are real roots to $x^2+2x+k=0$.

$\therefore (2)^2-4(1)(k)\geq 0$

$4-4k\geq 0$

$\therefore k\leq 1$

A1: $k\leq 1$

However, $k>1$.

A1: Conclusion

$\therefore$ Curve does not have a stationary point

Method 2:

$x^2+2x+k=(x+1)^2+(k-1)$

M1: Completing the square

$(x+1)^2+(k-1)\geq k-1>0 \ (\because k>1)$

M1: Establishing the inequality

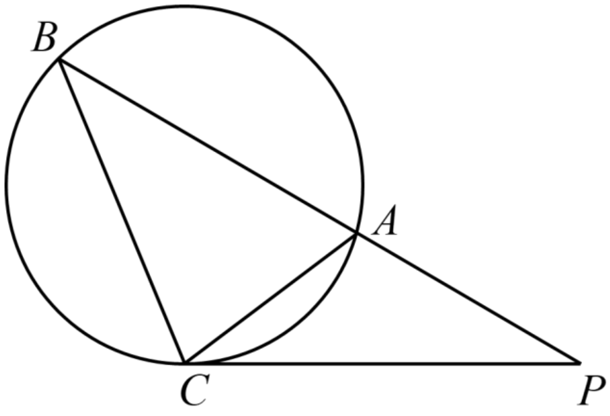
$\therefore \frac{dy}{dx}>0$ for all values of x

$\therefore$ Graph is strictly increasing for all values of x

$\therefore$ Curve does not have a stationary point

A1: Conclusion

4 The diagram shows a circle. The line PC is the tangent to the circle at P. A and B are points on the circle such that PAB is a straight line.



Prove that

(a) triangle BPC is similar to triangle CPA, [3]

$\angle BPC = \angle CPA$ (Common angle) **B1**

$\angle CBP = \angle ACP$ (Angles in alternate segment) **B1**

$\therefore$ By AA similarity test, ΔBPC is similar to ΔCPA. **B1**

(b) $PA \times PB = PC^2$. [2]

$\frac{BPC}{CPA} : \frac{BP}{CP} = \frac{PC}{PA} = \frac{BC}{CA}$ **M1**: Ratio of corresponding sides of similar triangles

$\therefore \frac{BP}{CP} = \frac{PC}{PA}$

$\therefore BP \times PA = CP \times PC$

$\therefore PA \times PB = PC^2$ **A1**

5 The equation of a curve is $y=-\frac{4}{3}x^3-(k+1)x^2-k^2x$, where k is a constant.

(a) Find the range of values of k for which y is always decreasing. [4]

$y=-\frac{4}{3}x^3-(k+1)x^2-k^2x$

$\frac{dy}{dx}=-4x^2-2(k+1)x-k^2$

M1: Differentiation and set $\frac{dy}{dx}<0$

For y to be strictly decreasing,

$\frac{dy}{dx}<0$

$\therefore -4x^2-2(k+1)x-k^2<0$

$4x^2+2(k+1)x+k^2>0$

For $4x^2+2(k+1)x+k^2$ to be always positive,

$b^2-4ac<0$

B1: $b^2-4ac<0$

$[2(k+1)]^2-4(4)(k^2)<0$

$(k+1)^2-4k^2<0$

$(k+1)^2-(2k)^2<0$

$(k+1+2k)(k+1-2k)<0$

$(3k+1)(-k+1)<0$

M1: Solve quadratic inequality

$(3k+1)(k-1)>0$

$\therefore k<-\frac{1}{3}$ OR $k>1$

A1

(b) Given that y has three distinct roots, find the range of values of k. [2]

$\because$ y to have three distinct roots,

y has two turning points.

$\frac{dy}{dx}=0$ has two real roots.

$-4x^2-2(k+1)x-k^2=0$ has two real roots.

$\therefore 4x^2+2(k+1)x+k^2=0$ has two real roots.

$\therefore b^2-4ac>0$

M1: $b^2-4ac>0$

$[2(k+1)]^2-4(4)(k^2)>0$

$(k+1)^2-4k^2>0$

$(3k+1)(k-1)<0$

$\therefore -\frac{1}{3}<k<1$

A1

6 A curve is such that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3\sin x - 4\cos 2x$. The curve passes through $A(0,1)$ and $B(\pi,3)$. Find the equation of the curve. [7]

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 3\sin x - 4\cos 2x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \int(3\sin x - 4\cos 2x)\,\mathrm{d}x$ **M1**: Integration

$= -3\cos x - \frac{4\sin 2x}{2} + c$

$= -3\cos x - 2\sin 2x + c$ **M1**: Integration

$y = \int(-3\cos x - 2\sin 2x + c)\,\mathrm{d}x$

$= -3\sin x - \frac{(-2\cos 2x)}{2} + cx + d$

$= -3\sin x + \cos 2x + cx + d$ **A2**: Minus one mark per mistake

M2: Substitute the two conditions

When $x = 0,\ y = 1$:

$1 = -3\sin 0 + \cos 0 + c(0) + d$

$1 = 1 + d$

$\therefore d = 0$

$y = -3\sin x + \cos 2x + cx$

When $x = \pi,\ y = 3$:

$3 = -3\sin\pi + \cos 2\pi + c\pi$

$3 = 1 + c\pi$

$c = \frac{2}{\pi}$

$\therefore y = -3\sin x + \cos 2x + \frac{2}{\pi}x$ **A1**

7 For the curve $y = 2x^2$, the tangent at point P where $x = a$, intersect the y-axis at A. The normal to the curve at point P intersects the y-axis at B.

Given that $a > 0$, show that the area of triangle ABP is $\dfrac{a\left(16a^2+1\right)}{8}$. [7]

$P = \left(a, 2a^2\right)$	**B1**: Coordinate of P
$y = 2x^2$ $\dfrac{dy}{dx} = 4x$ When $x = a$, $\dfrac{dy}{dx} = 4a$	**M1**: Differentiate to find gradient of tangent at $x = a$
$\therefore$ Gradient of tangent at $P = 4a$ $\therefore$ Gradient of normal at $P = -\dfrac{1}{4a}$	**B1**: Don't give if one of the gradients is wrong
Finding equation of tangent at P: $y = 4ax + c$ When $x = a$, $y = 2a^2$: $2a^2 = 4a(a) + c$ $\therefore c = -2a^2$ $\therefore$ Equation of tangent at P: $y = 4ax - 2a^2$	**M1**: Finding equation of tangent
$\therefore A = \left(0, -2a^2\right)$	**A1**: SOI
Finding equation of tangent at P: $y = -\dfrac{1}{4a}x + c$ When $x = a$, $y = 2a^2$: $2a^2 = -\dfrac{1}{4a}(a) + c$ $\therefore c = 2a^2 + \dfrac{1}{4}$ $\therefore$ Equation of normal at P: $y = -\dfrac{1}{4a}x + 2a^2 + \dfrac{1}{4}$	
$\therefore B = \left(0, 2a^2 + \dfrac{1}{4}\right)$	**A1**: SOI
$\therefore$ Area of $\Delta ABP = \dfrac{1}{2}\left[2a^2 + \dfrac{1}{4} - \left(-2a^2\right)\right](a) = \dfrac{a\left(16a^2+1\right)}{8}$ units^2	**A1**

8 **(a)** The equation of a curve is $y = a\sin bx + c$, $a > 0$. The curve attains maximum and minimum values of 4 and 2 respectively, and the period is π radians. Show that $a = 1$, $b = 2$ and show that $c = 3$. [3]

$-1 \le \sin bx \le 1$

$-a \le a\sin bx \le a$

$-a + c \le a\sin bx + c \le a + c$

$\therefore a + c = 4, \ -a + c = 2$

$\therefore a = 1$ AND $c = 3$

$\dfrac{2\pi}{b} = \pi$

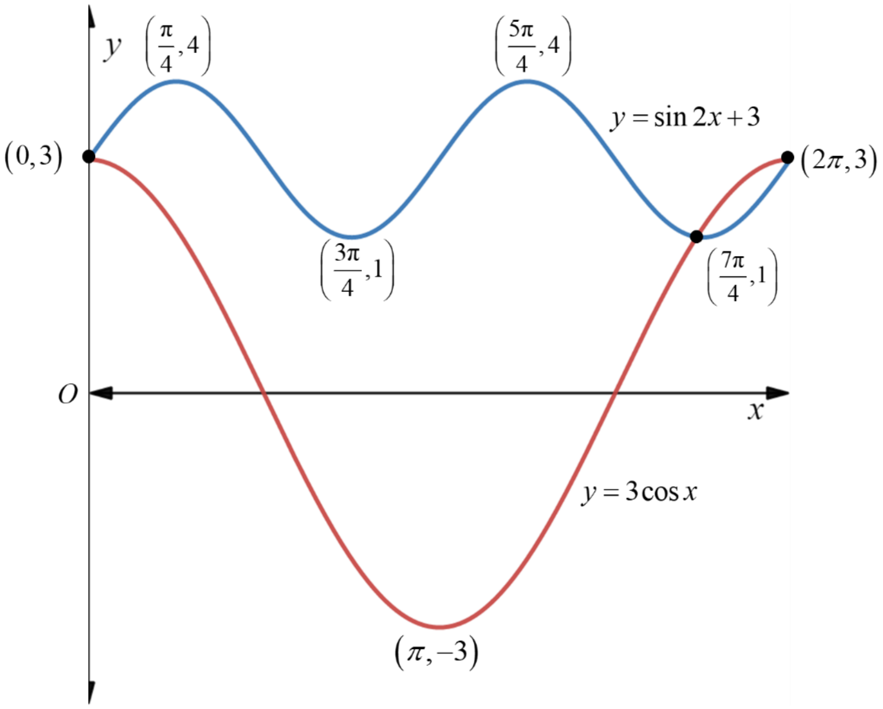
$\therefore b = 1$

M1: Constructing simultaneous equations with the max and min value.

A1

B1: Use period formula

(b) **(i)** Sketch, on the same diagram, the curves $y = \sin 2x + 3$ and $y = 3\cos x$ for $0 \le x \le 2\pi$ radians. [4]



B1: Correct shape for $y = \sin 2x + 3$

B1: Correct shape for $y = 3\cos x$

B2: Correct turning points and end points labelled, minus one for any missing two points.

(ii) Find the number of solutions to the equation $\sin 2x + 3 - 3\cos x = 0$ for $0 \le x \le 2\pi$ radians. [1]

$\sin 2x + 3 - 3\cos x = 0$

$\sin 2x + 3 = 3\cos x$

Number of solutions corresponds to the number of intersections between the curves $y = \sin 2x + 3$ and $y = 3\cos x$.

From the graphs in part **(i)**, there are three intersections.
Hence, there will be three solutions to the given equation.

B1: Three solutions

9 A container of liquid was heated to a temperature of $90°C$. It was then left to cool in a chiller such that its temperature, $T°C$, t minutes after the heat was removed, is given by $T = Ae^{-qt}$, where A and q are constants.
Measured values of t and T are given in the following table.

t (minutes)	2	4	6	8
T°C	66.674	49.393	36.591	27.107

(a) Explain why $A = 90$. [1]

When $t = 0$, $T = 90$:

$90 = Ae^0$

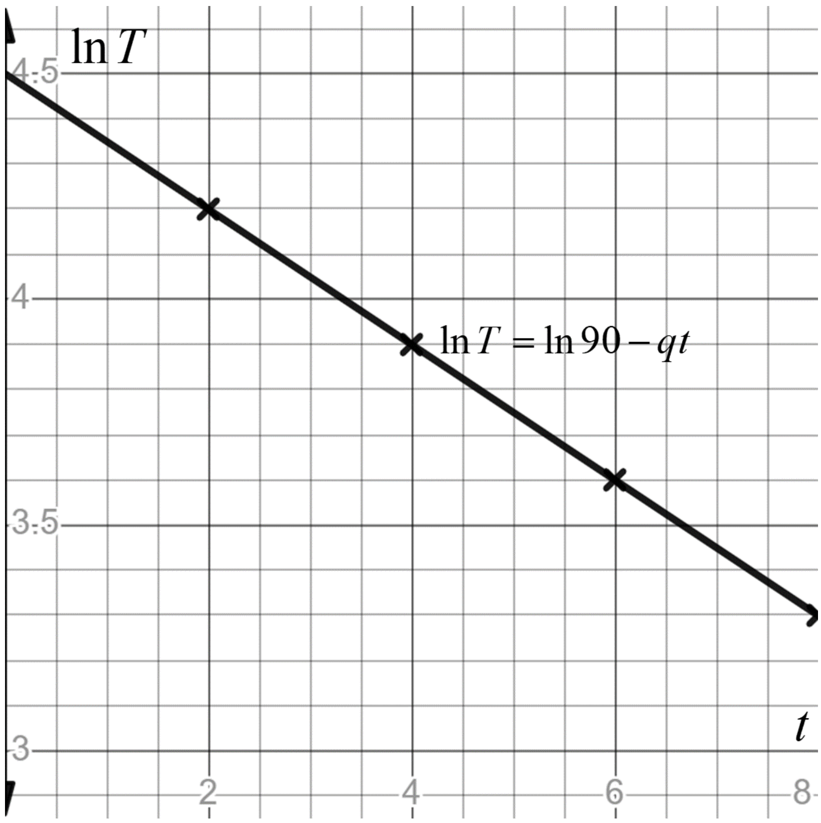
$\therefore A = 90$

B1

(b) Plot $\ln T$ against t and draw a straight line to illustrate the information. [3]

t	2	4	6	8
$\ln T$	4.20	3.90	3.60	3.30

B1



B1: Straight line drawn passing through the $\ln T$ axis (not exceeding)

B1: Correct axes labelled and scale

(c) Use the graph to estimate the value of q. [3]

$T = 90e^{-qt}$

$\ln T = \ln 90e^{-qt}$

$\ln T = \ln 90 + \ln e^{-qt}$

$\ln T = \ln 90 - qt$

$-q$ is the gradient of the straight line

$\therefore -q = -0.150$

$q = 0.150$

M1: Take ln both sides

M1: Find gradient of straight line

A1

(d) Use your graph to estimate the temperature of the liquid 5 minutes after it was left to cool. [2]

From the graph,

when $t = 5$, $\ln T = 3.75$

$\therefore T = e^{3.75} = 42.5$

$\therefore$ Required temperature $= 42.5°C$

M1: Locate $t = 5$ to find the $\ln T$ coordinate

A1

10 The length, breadth and height of a cuboid is $3p$ cm, p cm and $(1-p)$ cm respectively. The volume of the cuboid is $\frac{4}{9}$ cm^3.

(a) Show that $27p^3 - 27p^2 + 4 = 0$. [2]

$(3p)(p)(1-p) = \frac{4}{9}$

$27p^2(1-p) = 4$ **M1**: $(p)(p)(1-p)$

$-27p^3 - 27p^2 - 4 = 0$

$\therefore 27p^3 - 27p^2 + 4 = 0$ (Shown)

A1

(b) Show that $3p-2$ is a factor to $27p^3 - 27p^2 + 4$. [2]

<u>Method 1</u> **M1**: Long division

$$
\begin{array}{r}
9p^2 - 3p - 2 \\
3p-2 \overline{\smash{\big)}\, 27p^3 - 27p^2 + 4} \\
\underline{-(27p^3 - 18p^2)} \\
-9p^2 + 4 \\
\underline{-(-9p^2 + 6p)} \\
-6p + 4 \\
\underline{-(-6p + 4)} \\
0
\end{array}
$$

$\therefore 3p-2$ is a factor to $27p^3 - 27p^2 + 4$ **A1**

<u>Method 2</u>

Note that $27\left(\frac{2}{3}\right)^3 - 27\left(\frac{2}{3}\right)^2 + 4 = 0$. **M1**: Apply factor theorem

Hence, by Factor Theorem, $3p-2$ is a factor of $27p^3 - 27p^2 + 4$. **A1**: Must see "Factor Theorem"

(c) Hence, find p and compute the surface area of the cuboid. [5]

$27p^3 - 27p^2 + 4 = 0$

$(3p-2)(9p^2-3p-2)=0$

M1: $(3p-2)(9p^2-3p-2)$ (Can be shown without working from long division working in the previous part)

$\therefore p = \frac{2}{3}$ OR $9p^2 - 3p - 2 = 0$

$\therefore p = \frac{2}{3}$ OR $p = \frac{-(-3)+\sqrt{(-3)^2-4(9)(-2)}}{2(9)}$

M1: Factorisation or quadratic formula

$\therefore p = \frac{2}{3}$ OR $p = \frac{2}{3}$ OR $p = -\frac{1}{3}$

$\because p > 0$

$\therefore p = \frac{2}{3}$

A1: With reject (No reason required)

Required surface area

$= 2\left[(2)\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)+(2)\left(\frac{1}{3}\right)\right] = \frac{40}{9}\text{cm}^2$

B1

11 **(a)** **(i)** Find the first 4 terms, in ascending powers of x, of the expansion of $(2-kx)^6$ where k is a non-zero constant. [2]

$(2-kx)^6 = 2^6 + \binom{6}{1}(2)^5(-kx)^1 + \binom{6}{2}(2)^4(-kx)^2 + \binom{6}{3}(2)^3(-kx)^3 + \dots$ **M1**: Correct expansion

$= 64 - 192kx + 240k^2x^2 - 160k^3x^3 + \dots$ **A1**

(ii) Given that the coefficient of x^3 is 30 times the coefficient of x, find the possible value(s) of k. [2]

$\dfrac{-160k^3}{-192k} = 30$ **M1**: Relevant ratios (FT)

$\dfrac{5k^2}{6} = 30$

$k^2 = 36$

$k = \pm 6$ **A1**

(iii) Hence, show that there is no term in x^2 in the expansion of $(1-135x^2)(2-kx)^6$. [2]

$(1-135x^2)(2-kx)^6 = (1-135x^2)(64-192kx+240k^2x^2+\dots)$ **M1**: Selective expansion to get coefficient of x^2 (FT)

$= \dots + 240k^2x^2 - 8640x^2 + \dots$

$= \dots + (240k^2 - 8640)x^2 + \dots$

When $k = \pm 6$, coefficient of $x^2 = 240(36) - 8640 = 0$

$\therefore$ Required coefficient of $x^2 = 0$

$\therefore$ Expansion has no term in x^2. **A1**: (With conclusion)

(b) Explain why there is no odd powers of *x* in the expansion of $\left(x+\frac{1}{x}\right)^{2n}$ for $n \in \mathbb{N}$. [4]

General term $=\binom{2n}{r}(x)^{2n-r}\left(\frac{1}{x}\right)^{r}$	**M1**: Consider general term
$=\binom{2n}{r}(x)^{2n-r}\left(x^{-1}\right)^{r}$	
$=\binom{2n}{r}(x)^{2n-2r}$	**A1**: Powers combined (must see simplification)
Note that $2n-2r=2(n-r)$ is always even since *n* and *r* are whole numbers. Hence the general terms in the given expansion always have even powers.	**M1**: Argue that powers are even **since *n* and *r* are whole numbers** (Must see)
Therefore, there is no odd powers of *x*.	**A1**: With conclusion

12 A particle travels in a straight line so that, at time t seconds after leaving a fixed point O, its displacement from O is s metres and its velocity is v ms^{-1}, where $v = 3e^{t} - 60e^{-3t}$.

Find

(a) the initial velocity of the particle, [1]

For $t = 0$, **B1**

$v = 3e^{0} - 60e^{0} = -57 \text{ ms}^{-1}$

$\therefore$ Initial velocity $= -57 \text{ ms}^{-1}$

(b) the value of t when the particle is instantaneously at rest, [3]

For $v = 0$, **M1**: $v = 0$

$0 = 3e^{t} - 60e^{-3t}$

$3e^{t} = 60e^{-3t}$

$e^{4t} = 20$

$\ln e^{4t} = \ln 20$ **M1**: Solve exponential equations

$4t = \ln 20$

$t = \dfrac{1}{4}\ln 20 = 0.749 \text{ s}$ **A1**: Accept $\dfrac{1}{4}\ln 20$

(c) the acceleration of the particle when $t = \ln 8$, [2]

$v = 3e^{t} - 60e^{-3t}$ **M1**: $a = \dfrac{dv}{dt}$

$a = \dfrac{dv}{dt} = 3e^{t} - 60(-3)e^{-3t}$

$a = 3e^{t} + 180e^{-3t}$

When $t = \ln 8$,

$a = 3e^{\ln 8} + 180e^{-3\ln 8} = 24.4 \text{ ms}^{-2}$ **A1**

(d) an expression for s in terms of t, [4]

$v = 3e^{t} - 60e^{-3t}$

M1: Integrate v to get s

$s = \int \left(3e^{t} - 60e^{-3t}\right) dt$

$= 3e^{t} - \frac{60}{(-3)} e^{-3t} + c$

$= 3e^{t} + 20e^{-3t} + c$

A1

When $t = 0,\ s = 0$:

$0 = 3e^{0} + 20e^{0} + c$

M1: Substitute initial conditions

$c = -23$

$\therefore s = 3e^{t} + 20e^{-3t} - 23$

A1

(e) the total distance travelled in the first 5 seconds. [3]

Total distance travelled in the first 5 seconds

$= 2(-1)\left(3e^{\frac{1}{4}\ln 20} + 20e^{-\frac{3}{4}\ln 20} - 23\right) + 3e^{5} + 20e^{-15} - 23$

$= 451$ m

M1: Substitute $t = \frac{1}{4}\ln 20$ (FT from (b)) or $t = 5$ into part (d) (FT from (d))

M1: $\times 2$

A1