Chapter 1: Binomial Theorem

Expansion of linear algebraic factors	$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ... + \binom{n}{r}a^{n-r}b^r + ... + b^n$ where n is a **positive integer** and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$	
Expansion of quadratic functions	$(ax^2+bx+c)^n = [c+(bx+ax^2)]^n$ $= c^n + \binom{n}{1}c^{n-1}(x+x^2) + \binom{n}{2}c^{n-2}(x+x^2)^2 + ...$	Express the quadratic factor into the form $(a + b)^n$ where a is the constant and b would be x and x^2.
Expansion of (1 + x)ⁿ	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... \frac{n(n-1)...(n-r+1)}{r!}x^r + ...$ $\|x\|<1$	To expand $(1 + kx)$, replace x with kx.

Expansion of (a + bx)ⁿ	***Ascending powers of x*** **(or descending 1/x)**	***Descending powers of x***
	$(a+bx)^n = \left[a\left(1+\frac{b}{a}x\right)\right]^n = a^n\left(1+\frac{b}{a}x\right)^n$ $= a^n\left[1+n\left(\frac{b}{a}x\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}x\right)^2 + ... + \frac{n(n-1)...(n-r+1)}{r!}\left(\frac{b}{a}x\right)^r + ...\right]$ Range of validity: $\left\|\frac{b}{a}x\right\|<1 \Rightarrow \|x\|<\left\|\frac{a}{b}\right\| \Rightarrow -\left\|\frac{a}{b}\right\|<x<\left\|\frac{a}{b}\right\|$ **(small x values)** (closer to 0 gives better estimate)	$(a+bx)^n = \left[bx\left(\frac{a}{bx}+1\right)\right]^n = (bx)^n\left(1+\frac{a}{bx}\right)^n$ $= b^n x^n\left[1+n\left(\frac{a}{bx}\right)+\frac{n(n-1)}{2!}\left(\frac{a}{bx}\right)^2 + ... + \frac{n(n-1)...(n-r+1)}{r!}\left(\frac{a}{bx}\right)^r + ...\right]$ Range of validity: $\left\|\frac{a}{bx}\right\|<1 \Rightarrow \|x\|>\left\|\frac{a}{b}\right\| \Rightarrow x<-\left\|\frac{a}{b}\right\|$ or $x>\left\|\frac{a}{b}\right\|$ **(large x values)**

Chapter 2: Sequences and Series

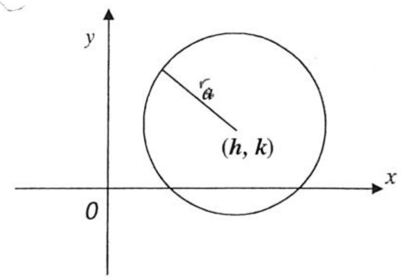
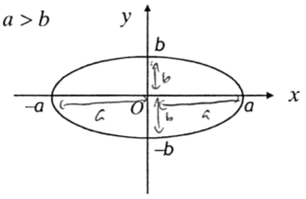
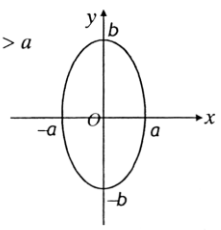
Simple definitions	$u_1 = S_1$ [1st term = sum of 1st term]	$u_n = S_n - S_{n-1}; n \geq 2$ [The difference in 2 sums gives the term]
	Arithmetic progression (A.P.)	***Geometric progression (G.P.)***
Definitions	difference of any two successive members of the sequence is a **constant**	each term after the first is found by multiplying the previous one by a fixed non-zero number called the *common ratio*.
General term	$u_n = a+(n-1)d$ where a = first term, n = no. of terms, d = common difference.	$u_n = ar^{n-1}$ where r = common ratio; $T_n = S_n - S_{n-1}$
Proving	$d = u_n - u_{n-1}$ a **non-zero constant**	$r = \frac{u_n}{u_{n-1}}$ is a **constant independent of *n*.**
Sum of first n terms	$S_n = \frac{n}{2}[2a+(n-1)d]$; $S_n = \frac{n}{2}(a+l)$, l = last term	$S_n = \frac{a(1-r^n)}{1-r}$; $S_n = \frac{a(r^n-1)}{r-1}$ where $r \neq 0,1$; $S_\infty = \frac{a}{1-r}$, S_∞ exists $\Leftrightarrow \|r\|<1$
Sigma Notation	$\sum_{r=a}^{b} u_r = u_a + u_{a+1} + u_{a+2} + ... + u_b$ where a and b are integers. n = **($b - a$ + 1)** i.e. **upper limit – lower limit + 1.**	
Rules of sigma notation	$\sum_{r=1}^{n} a = an$; $\sum_{r=1}^{\infty}(\text{G.P.})^r = \frac{a}{1-r}$; $\sum_{r=1}^{n} au_r = a\sum_{r=1}^{n} u_r$	$\sum_{r=1}^{n}(u_r+v_r) = \sum_{r=1}^{n} u_r + \sum_{r=1}^{n} v_r$; $\sum_{r=m}^{n}(u_r) = \sum_{r=1}^{n} u_r + \sum_{r=1}^{m-1} v_r$
Method of Differences	**Note:** Watch out for method of differences when there is a **small difference in the terms being subtracted**. Also note that the **cancelling is symmetrical**, i.e. cancelling the 2nd term on the LHS will cancel the **last 2nd term** on the RHS.	
Limit of sequence	$\frac{1}{n}, \frac{1}{n!} \to 0$; $a^n \to 0$ (if $0 < a < 1$)	$\frac{1}{a^n} \to 0$ (if $a > 1$); The other terms, e.g. an, a^n where $a > 1$ will **tend to infinity** and are not included in the limit of a sequence.
Convergence of sequences	A sequence is convergent if $\lim_{n\to\infty} u_n = L$ where L is a **finite number**	$u_n \to L$, $u_{n+1} \to L$; $S_\infty = L$ (Use G.C. to generate terms of sequence)

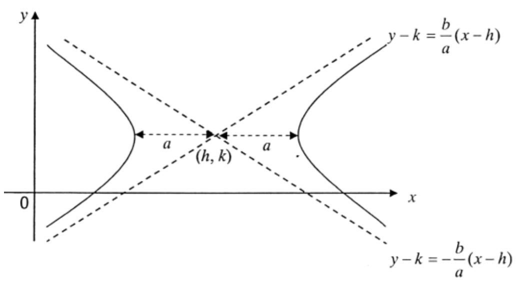
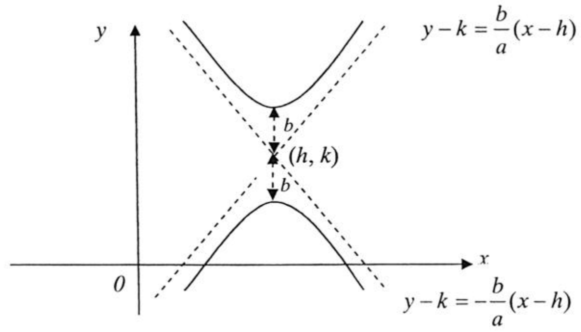
Chapter 3: Mathematical Induction

Format	Let P_n be the statement where [*copy eqn here*] for all $n \in \mathbb{Z}^+$. LHS of $P_1 = n$ RHS of $P_1 = n$ Therefore, P_1 is true. In the process of proving, you may use what you wrote for P_k into that equation. [esp. for sigmas] Assume that P_k is true for **some** $k \in \mathbb{Z}^+$, i.e. [*replace n* by *k*] We want to prove P_{k+1}, i.e. [*replace k* by *k+1*] LHS of P_{k+1} = [Start proving] = RHS (**Proven**) Since P_1 is true and P_k is true $\Rightarrow$ P_n is true for all $n \in \mathbb{Z}^+$.
Conjecture	Some induction questions may involve **conjectures** (guess) of the general formula. They will usually ask you to write the values of a sequence/sigma for n = 1, 2, 3, 4 then **deduce a general equation** which will then be proven by Mathematical Induction.

Chapter 4: Graphing Techniques

Points that must be labeled	Axial intercepts (x = 0, y = 0)	Turning Points (Let $\frac{dy}{dx} = 0$ and solve for x)	Asymptotes (Express in form $f(x) = Q(x) + \frac{R(x)}{D(x)}$) **Vertical** asymptote: Let $D(x) = 0$, solve for x.	**Horizontal/Oblique** asymptote: $Q(x)$.

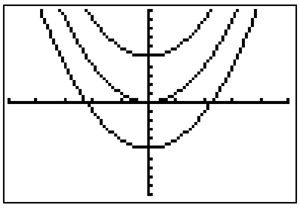
	Circle	***Eclipse (oval)***
Conics	$(x-h)^2 + (y-k)^2 = r^2$ represents a circle with centre **(*h*, *k*)** and radius r. 	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where $a \neq b$ represents an ellipse with centre (h, k) x-intercepts at $(\pm a, 0)$ and y-intercepts at $(0, \pm b)$. Lines of symmetry: $x = h$, $y = k$. $a > b$  $b > a$ 

	Hyperbola	
Conics	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where $a \neq b$ represents an horizontal hyperbola with centre (h, k). Oblique asymptotes: $y-k = \frac{b}{a}(x-h)$ and $y-k = -\frac{b}{a}(x-h)$. Lines of symmetry: $x = h$, $y = k$. 	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ where $a \neq b$ represents a vertical hyperbola with centre (h, k). Oblique asymptotes: $y-k = \frac{b}{a}(x-h)$ and $y-k = -\frac{b}{a}(x-h)$. Lines of symmetry: $x = h$, $y = k$. 

Transformations

$y = \mathbf{f}(x) + \mathbf{a}$

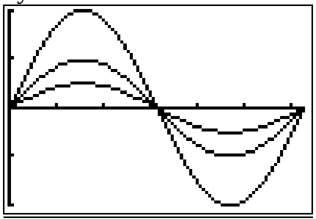
For a > 0), the graph of $y = \mathbf{f}(x) + \mathbf{a}$ is obtained by translating the graph of $y = \mathrm{f}(x)$ by ***a*** units in the positive *y* direction.



This **vertical translation** only affects the ***y-values*** and the **horizontal asymptote** by the constant **a**.

$y = \mathbf{a}\,\mathbf{f}(x)$

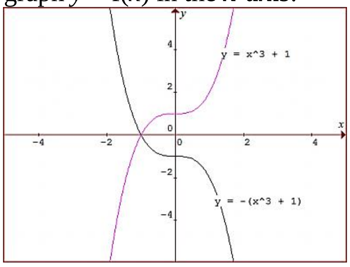
For a > 0), the graph of $y = a\,\mathbf{f}(x)$ is obtained by scaling the graph of $y = \mathrm{f}(x)$ through a **scaling parallel** to the ***y-axis*** by a **scale factor of *a*.**



This **vertical scaling** only affects the ***y-values*** and the **horizontal asymptote** by the constant **a**.

$y = -\mathbf{f}(x)$

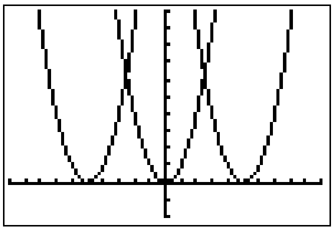
the graph of $y = -\mathbf{f}(x)$ is a **reflection** of the graph $y = \mathrm{f}(x)$ in the ***x-axis***.



This **vertical transformation** only affects the ***y-values*** and the **horizontal asymptote** by a **negative sign**.
(Just **negate** the points/equation)

$y = \mathrm{f}(x + \mathrm{a})$

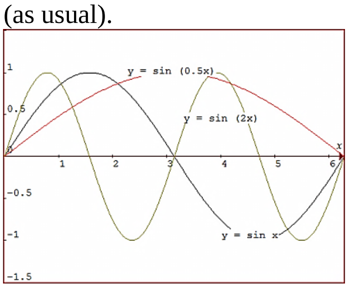
The graph of $y = \mathbf{f}(x - a)$ is obtained by translating the graph of $y = \mathrm{f}(x)$ by ***a*** units in the *x* direction. If $a > 0$, shift towards the **positive *x-direction***, if $a < 0$, shift towards the **negative *x*-direction.**



This **horizontal translation** only affects the ***x-values*** and the **vertical asymptote** by the constant **a**.

$y = \mathrm{f}\left(\dfrac{x}{a}\right)$

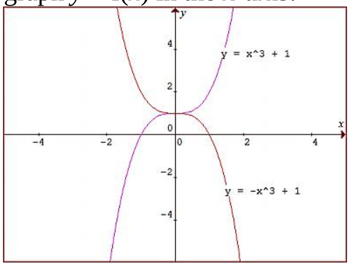
The graph of $y = \mathrm{f}\left(\dfrac{x}{a}\right)$ is obtained by scaling the graph of $y = \mathrm{f}(x)$ by ***a*** units in the *x* direction. If $a > 0$ you expand the graph, if $a < 0$ you compress the graph (as usual).



This **horizontal translation** only affects the ***x-values*** and the **vertical asymptote** by the constant **a**.

$y = \mathrm{f}(-x)$

the graph of $y = -\mathbf{f}(x)$ is a **reflection** of the graph $y = \mathrm{f}(x)$ in the ***x-axis***.



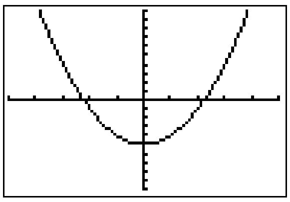
This **horizontal transformation** only affects the ***x-values*** and the **vertical asymptote** by a **negative sign**.
(Just **negate** the points/equation)

All transformations are done in the order $y = d + c\mathrm{f}\left(\dfrac{x}{b} - a\right)$

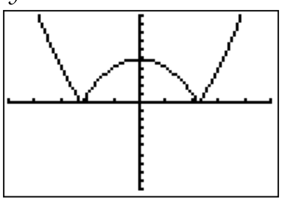
$y = |\mathbf{f}(x)|$

the parts of the graph **below the *x-axis*** (<0) are **reflected upwards**.

1. Keep $y = \mathrm{f}(x)$ where $\mathrm{f}(x) \geq 0$,
2. Reflect $y = \mathrm{f}(x)$ where $\mathrm{f}(x) < 0$ above the ***x-axis***.



$y = x^2 - 5$

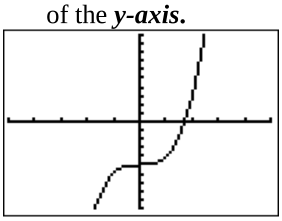


$y = |x^2 - 5|$

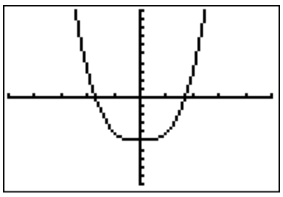
$y = \mathbf{f}(|x|)$

the ***x-values*** in the graph are modulused.

1. Discard $y = \mathrm{f}(x)$ where $x < 0$ [the part of the curve **left of *y-axis***]
2. Keep and reflect $y = \mathrm{f}(x)$ where $\mathrm{f}(x) \geq 0$ to the **left** of the ***y-axis.***



$y = x^3 - 5$



$y = |x^3| - 5$

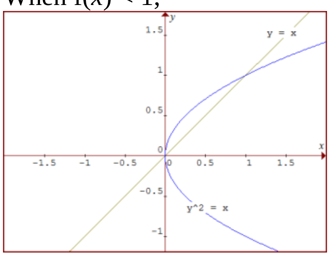
$y^2 = \mathbf{f}(x)$,

The graph of $y = \pm\sqrt{\mathrm{f}(x)}$ consists of 2 parts **symmetrical to the *x-axis***.

1. Only consider the part of the graph for which $\mathbf{f}(x) \geq \mathbf{0}$ (area above ***x-axis***).
2. First sketch $y = \sqrt{\mathrm{f}(x)}$:

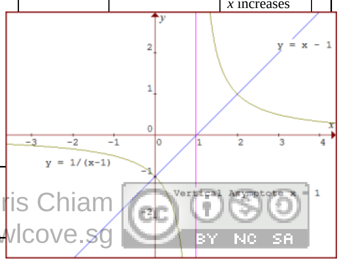
When $\mathrm{f}(x) > 1$, $\mathrm{f}(x) > \sqrt{\mathrm{f}(x)}$

When $\mathrm{f}(x) < 1$, $\sqrt{\mathrm{f}(x)} > \mathrm{f}(x)$



$y = \dfrac{1}{\mathrm{f}(x)}$

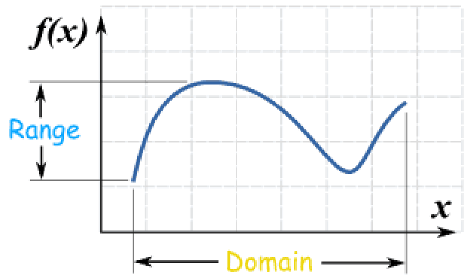
	$y = \mathbf{f}(x)$	$y^2 = \mathbf{f}(x)$
Asymptotes and x-intercepts	*x*-intercept at (*h*, 0)	Vertical asy. $x = h$
	Vertical asymptote $x = h$	*x*-intercept at (*h*, 0)
Horizontal asymptotes	$y = k, k \neq 0$	$y = \frac{1}{k}$
	$y = 0$	No horizontal asymptote
	If function **goes to positive and negative infinity** (i.e. there is no horizontal asymptote)	Horizontal asymptote at ***y = 0***
Maximum and Minimum points	(*h*, *k*)	$\left(h, \frac{1}{k}\right)$
	Maximum pt (*h*, *k*)	Minimum point $\left(h, \frac{1}{k}\right)$
	Minimum pt (*h*, *k*)	Maximum point $\left(h, \frac{1}{k}\right)$
Other trends	f(*x*) is increasing as *x* increases	f(*x*) decreases as *x* increases
	f(*x*) is decreases as *x* increases	f(*x*) increases as *x* increases



Chapter 5: Functions

Definitions	***Definition***	A **function** f, or mapping, is a rule which assigns every element of $x \in X$ to only one element of $y \in Y$.
	Domain	**Domain** is the **set of <u>x-values</u>** (input values) for which a function **f** is defined as the **domain** (D_f) Note: **<u>It must always be stated in the answer.</u>**
	Range	**Range** is the **set of <u>y-values</u>** (output values) corresponding to **every x value** in the domain (R_f)

$x \in (a,b)$	$a < x < b$
$x \in [a,b]$	$a \leq x \leq b$
$x \in (a,b]$	$a < x \leq b$
$x \in [a,b)$	$a \leq x < b$

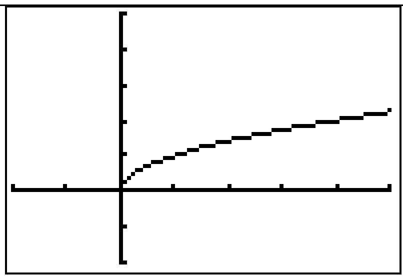


Vertical line test		If $f\!:\!A \to B$ is a function, for each $a \in A$, the vertical line $x = a$ cuts the graph $y = f(x)$ at **only one point**.
	Positive	From the sketch, any **vertical line** $x = a$, where $a \in D_f$ will cut the graph of $y = f(x)$ at **exactly one point**. Hence **f(x)** is a function.
	Negative	From the sketch, the line x = **a** (state a constant)cuts the graph of y = f(x) at **2 different points**. Hence **f(x)** is **not** a function.
Inverse functions	***Definition***	For any **one-to-one** function, $f\!:\!x \mapsto R_f$, there is an inverse function that exists, i.e. $f^{-1}\!:\!x \mapsto R_f$ such that $f^{-1}(y) = x \Rightarrow y = f(x)$ for every $x \in X$.
	Horizontal line test	A function f:A→B is **one-one** if each **horizontal line y = b** where $b \in R_f$ **cuts** the graph of y = f(x) at **only 1 point**.

Example 1

Determine if the function $f\!:\!x \mapsto \sqrt{x}$ is a one-to-one function.

From the diagram, any *horizontal line* ***y* = b**, where $b \in R_f$ cuts the graph of y = f(x) at exactly **one point**. Hence **f** is **one-one**.

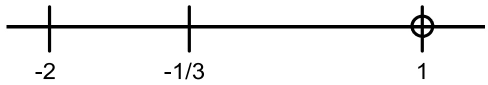
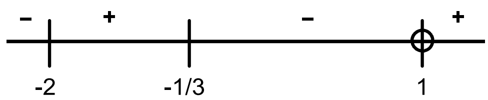


Notes

$D_{f^{-1}} = R_f$ and $R_{f^{-1}} = D_f$	$f(x) = y \Leftrightarrow x = f^{-1}(y)$
To obtain the inverse function f^{-1}, follow these steps: **1.** Let **y = f(x)** and express in terms of y. **2.** Rewrite **x** as $\mathbf{f^{-1}(y)}$ to obtain $f^{-1}(y)$ as a function in y. **3.** Replace all **y** with **x** to obtain $f^{-1}(x)$ as a function in x. **4.** State the domain of f^{-1} using $D_{f^{-1}} = R_f$	The graphs of f and f^{-1} are **reflections** of each other in the line **y = x**. Hence finding the solution of $f(x) = f^{-1}(x)$ is the same as the solution of the equation f(x) = x or $f^{-1}(x) = x$

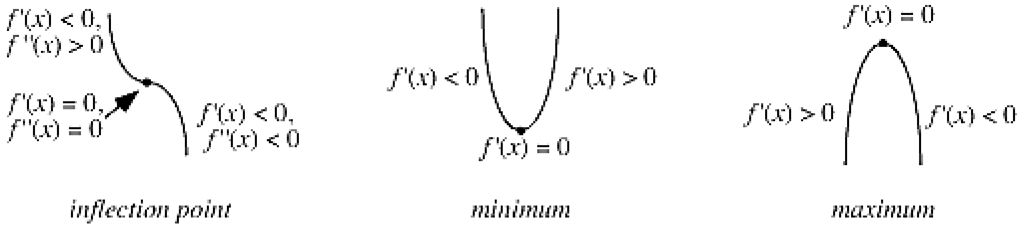
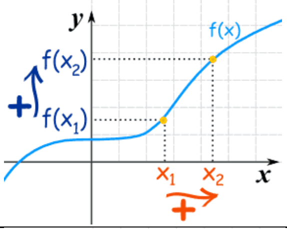
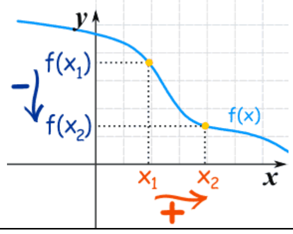
Composite functions	***Definition***	**Function composition** (or composite function) is the application of one function to the results of another function. For instance, the composite function gf as gf(x) = g(f(x)) where $x \in D_f$	
	Conditions	The composite function gf exists $\Leftrightarrow R_f \subseteq D_g$	the composite function fg exists $\Leftrightarrow R_g \subseteq D_f$
	Notes	$D_{gf} = D_f$, $D_{fg} = D_g$	$ff^{-1}(x) = f^{-1}f(x) = x$; $f^2(x) = ff(x)$

Chapter 6: Equations and Inequalities

Properties of Inequalities	***Addition and Subtraction***	If $a > b$, then $a + c > b + c$ and $a - c > b - c$	If $a > b$ and $c > d$, then $a + c > b + d$
	Multiplication and Division	If $c > 0$ and $a > b$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$. (can only cross-multiply positive terms)	If $d < 0$ and $a > b$, then $ad < bd$ and $\frac{a}{d} < \frac{b}{d}$ (reverse the inequality sign – especially when dealing with **logarithmic terms**)
Solving inequalities	***Without G.C.***	**1.** Make one side of the inequality zero. (never cross-multiply terms that can take positive or negative values)	$\frac{3x^2 + 7x + 2}{x - 1} \geq 0$
		2. Factorise the inequality into **linear factors** (by using completing the square, etc.) Positive factors can be **ignored**.	$\frac{(3x+1)(x+2)}{(x-1)} \geq 0$
		3. Equate each factor to zero and solve to get the critical points.	Let $3x + 1 = 0, x + 2 = 0, x - 1 = 0$ $\therefore x = -\frac{1}{3}, x = -2, x = 1$
		4. Draw a no. line with the critical points. Circle the critical points that will make the denominator zero (or not satisfy the inequality)	
		5. Test the sign in any region. The critical points will divide the no. line into alternating positive and negative regions.	Let $x = 0, \frac{(0+1)(0+2)}{(0-1)} = -2(< 0)$ 
		6. The solution will be the region required by the inequality.	Hence, $-2 \leq x \leq -\frac{1}{3}$ and $x > 1$
	With G.C.	1. Sketch the 2 different inequalities using G.C. 2. Calculate the x-intercepts (or points of intersection). 3. Determine the solution, ensuring it does not include values which will make the denominators of the original inequality zero.	

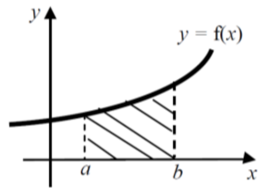
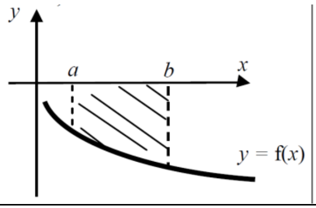
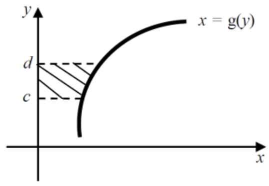
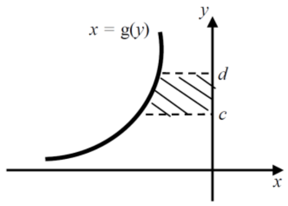
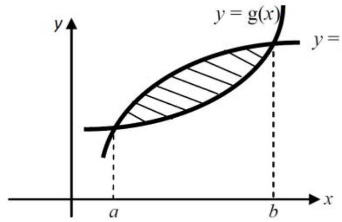
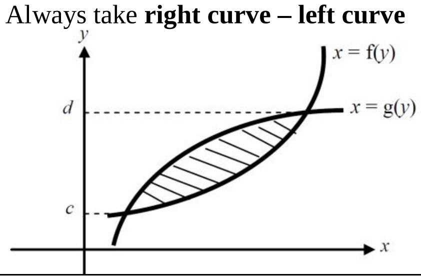
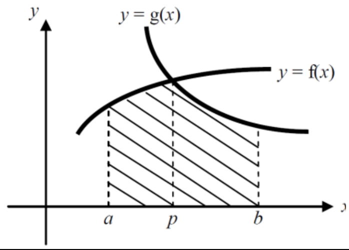
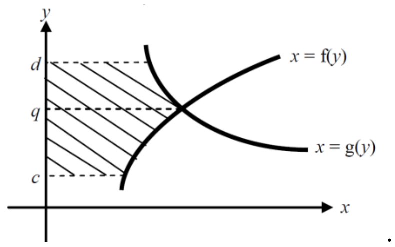
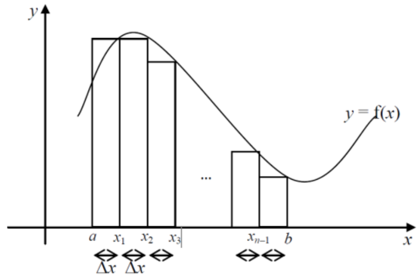
Properties of |x|	$\sqrt{x^2} = \|x\|$	$\|ab\| = \|a\|\|b\|$ $\left\|\frac{a}{b}\right\| = \frac{\|a\|}{\|b\|}, b \neq 0$	$\|x\| = \|k\| \Rightarrow x = \pm k$	$\|x\| > k \Leftrightarrow$ $x > k$ or $x < -k$	$\|x\| < k \Leftrightarrow$ $-k < x < k$
Solving sim. linear eqn using GC	Use the PlySmlt2 SIMULT EQN SOLVER in TI-84 GC.				

Chapter 7: Differentiation and its Applications

Name	Formula		
Rules of differentiation	$\frac{\mathrm{d}}{\mathrm{d}x}(u \pm v)=\frac{\mathrm{d}u}{\mathrm{d}x}\pm\frac{\mathrm{d}v}{\mathrm{d}x}$	$\frac{\mathrm{d}}{\mathrm{d}x}(uv)=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)-u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)}{v^2}$
Chain Rule	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$	$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$
Higher order derivatives	$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}'(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{f}''(x)$ (used for finding **nature of points**) **If f''(x) > 0, curve is u shaped.** **If f''(x) < 0, curve is n shaped.**	
Trigonometric Functions	$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x$	$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)=\sec^2 x$
	$\frac{\mathrm{d}}{\mathrm{d}x}(\operatorname{cosec} x)=-\operatorname{cosec} x\cot x$	$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)=\sec x\tan x$	$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x)=-\operatorname{cosec}^2 x$
Exponential Functions	$\frac{\mathrm{d}}{\mathrm{d}x}(e^x)=e^x$	$\frac{\mathrm{d}}{\mathrm{d}x}(a^x)=\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\ln a\ x})=a^x\ln a$	
Logarithmic Functions	$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x}$	$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x)=\frac{1}{x}\log_a e$	
Implicit Functions	$\frac{\mathrm{d}}{\mathrm{d}x}\ \mathrm{f}(y)=\frac{\mathrm{d}}{\mathrm{d}y}\ \mathrm{f}(y)\times\frac{\mathrm{d}y}{\mathrm{d}x}$	Note: whenever there is a *y* function, there will be a **dy/dx** function in the answer.	
Inverse trigonometric functions	$\frac{\mathrm{d}}{\mathrm{d}x}\ \sin^{-1}\mathrm{f}(x)=\frac{\mathrm{f}'(x)}{\sqrt{1-x^2}}$	$\frac{\mathrm{d}}{\mathrm{d}x}\ \cos^{-1}\mathrm{f}(x)=-\frac{\mathrm{f}'(x)}{\sqrt{1-x^2}}$	$\frac{\mathrm{d}}{\mathrm{d}x}\ \tan^{-1}\mathrm{f}(x)=\frac{\mathrm{f}'(x)}{1+\ \mathrm{f}(x)\ ^2}$
Parametric Differentiation	$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}=\frac{\mathrm{d}y}{\mathrm{d}x}\times\frac{\mathrm{d}x}{\mathrm{d}t}$	When given parametric eqn, just find dy/dx and the points from there	
Tangent and normal	$y-y_1=m\ \ x-x_1$	$y-y_1=-\frac{1}{m}\ \ x-x_1$	
First derivative test			
Second derivative test	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ *(use table form to determine nature)*	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}>0$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0$
Increasing functions		For increasing functions, f'(x) > 0. If **concave downwards**, f''(x) < 0. If **concave upwards**, f''(x) > 0. Shape: **n** shape. *Strictly increasing*: No flat points allowed.	
Decreasing functions		For decreasing functions, f'(x) < 0. If **concave downwards**, f''(x) < 0. If **concave upwards**, f''(x) > 0. Shape: **u** shape. *Strictly decreasing*: No flat points allowed.	
Graphs of y = f'(x)	Horizontal/oblique asy. becomes *y* = 0, vertical asy. remains. S Stationary points at **(h, k)** all become *x* intercepts at **(h, 0)**		

Chapter 8: Integration and its Applications

Integration	$\int \mathrm{f}(x)\ \mathrm{d}x = \mathrm{F}(x) + c$		
Definite integral	$\int_a^b \mathrm{f}(x)\ \mathrm{d}x = \mathrm{F}(b) - \mathrm{F}(a)$		
Properties of definite integral	$\int_a^a \mathrm{f}(x)\ \mathrm{d}x = 0$	$\int_a^b \mathrm{f}(x)\ \mathrm{d}x = -\int_b^a \mathrm{f}(x)\ \mathrm{d}x$	$\int_a^b k\,\mathrm{f}(x)\ \mathrm{d}x = k\int_a^b \mathrm{f}(x)\ \mathrm{d}x$
	$\int_a^b \mathrm{f}(x) \pm \mathrm{g}(x)\ \mathrm{d}x = \int_a^b \mathrm{f}(x)\ \mathrm{d}x \pm \int_a^b \mathrm{g}(x)\ \mathrm{d}x$	$\int_a^b \mathrm{f}(x)\ \mathrm{d}x = \int_a^c \mathrm{f}(x)\ \mathrm{d}x + \int_c^b \mathrm{f}(x)\ \mathrm{d}x$	
Integrals involving <u>linear</u> algebraic functions	$\int (ax+b)^n\ \mathrm{d}x = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ where $n \neq 1$	$\int \frac{1}{ax+b}\ \mathrm{d}x = \frac{1}{a}\ln\lvert ax+b \rvert + c$ (include modulus sign)	
	$\int_a^c \lvert \mathrm{f}(x) \rvert\ \mathrm{d}x = \int_a^b -\mathrm{f}(x)\ \mathrm{d}x + \int_b^c \mathrm{f}(x)\ \mathrm{d}x$ whereby b is where $\mathrm{f}(x) = 0$.		
Reverse chain rule	$\int [\mathrm{f}(x)]^n\, \mathrm{f}'(x)\ \mathrm{d}x = \frac{[\mathrm{f}(x)]^{n+1}}{n+1} + c$ where $n \neq 1$	$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\ \mathrm{d}x = \ln\lvert \mathrm{f}(x) \rvert + c$	
	Note: the derivative may need to be manipulated (by adding **constants**, **NOT variables** outside and inside the integral sign (not the final value of the integral cannot be changed)		
Exponential Functions	$\int \mathrm{f}'(x) e^{\mathrm{f}(x)}\ \mathrm{d}x = e^{\mathrm{f}(x)} + c$	$\int \mathrm{f}'(x) a^{\mathrm{f}(x)}\ \mathrm{d}x = \frac{e^{\mathrm{f}(x)}}{\ln a} + c$	
Trigonometric functions	$\int \sin x\ \mathrm{d}x = -\cos x + c$	$\int \cos x\ \mathrm{d}x = \sin x + c$	
	$\int \sec^2 x\ \mathrm{d}x = \tan x + c$	$\int \sec x\ \tan x\ \mathrm{d}x = \sec x + c$	
	$\int \mathrm{cosec}\, x \cot x\ \mathrm{d}x = -\mathrm{cosec}\, x + c$	$\int \mathrm{cosec}^2\, x\ \mathrm{d}x = -\cot x + c$	
	$\int \tan x\ \mathrm{d}x = \ln\lvert \sec x \rvert + c$	$\int \cot x\ \mathrm{d}x = \ln\lvert \sin x \rvert + c$	
	$\int \sec x\ \mathrm{d}x = \ln\lvert \sec x + \tan x \rvert + c$	$\int \mathrm{cosec}\, x\ \mathrm{d}x = -\ln\lvert \mathrm{cosec}\, x + \cot x \rvert + c$	
Trigonometric identities	$\int \sin^2 x\ \mathrm{d}x = \int \frac{1-\cos 2x}{2}\ \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + c$	$\int \cos^2 x\ \mathrm{d}x = \int \frac{1+\cos 2x}{2}\ \mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + c$	
	$\int \tan^2 x\ \mathrm{d}x = \int \sec^2 x - 1\ \mathrm{d}x = \tan x - x + c$	$\int \cot^2 x\ \mathrm{d}x = \int \mathrm{cosec}^2 x - 1\ \mathrm{d}x = -\cot^2 x - x + c$	
Factor Formulae	$\sin P + \sin Q = 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$	$\sin P - \sin Q = 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$	
	$\cos P + \cos Q = 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$	$\cos P - \cos Q = -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$	
Involving fractions	$\int \frac{1}{(px+q)^2 + a^2}\ \mathrm{d}x = \frac{1}{ap}\tan^{-1}\left(\frac{px+q}{a}\right) + c$	$\int \frac{1}{\sqrt{a^2 - (px+q)^2}}\ \mathrm{d}x = \frac{1}{p}\sin^{-1}\left(\frac{px+q}{a}\right) + c$	
	$\int \frac{1}{x^2 - a^2}\ \mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{x-a}{x+a} \right\rvert$	$\int \frac{1}{a^2 - x^2}\ \mathrm{d}x = \frac{1}{2a}\ln\left\lvert \frac{a+x}{a-x} \right\rvert$	
Partial Fractions	$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}$	$\frac{px+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{(cx+d)^2}$	
	$\frac{px^2+qx+r}{(ax+b)(x^2+c)} = \frac{A}{ax+b} + \frac{Bx+C}{x^2+c}$		
Integration by substitution	1. Change all x to u. 2. Change du to dx 3. Integrate (should be easy) 4. Change all u back to x.	Note: Only use integration by substitution if the **question says so**.	
Integration by parts	$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\ \mathrm{d}x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}$ • Choice of function to be differentiated u follows the order: **LIATE** (logarithmic, inverse trigo, algebraic, trigo, exponential) • It can be applied twice on certain integrals.	Can be used for: • Integrating **simple functions** (logarithmic and inverse trigo functions) such as ln x, $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ • Product of **different types of functions**.	

	$\mathrm{f}(x) \geq 0$ (**above *x-axis***)	$\mathrm{f}(x) \leq 0$ (**below *x-axis***)	$\mathrm{f}(x) \geq 0$ on $[a,c]$, $\mathrm{f}(x) \leq 0$ on $[c,b]$
Area bounded by curve w.r.t. x-axis	$A=\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ 	$A=-\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ 	$A=\int_a^b \mathrm{f}(x)\ \mathrm{d}x-\int_b^c \mathrm{f}(x)\ \mathrm{d}x$ (the negative area must be **negated**)
	$\mathrm{f}(x) \geq 0$ (**right of *y-axis***)	$\mathrm{f}(x) \leq 0$ (**left of *y-axis***)	$\mathrm{f}(x) \geq 0$ on $[a,c]$, $\mathrm{f}(x) \leq 0$ on $[c,b]$
Area bounded by curve w.r.t. y-axis	$A=\int_c^d \mathrm{f}(y)\ \mathrm{d}y$ 	$A=-\int_c^d \mathrm{f}(y)\ \mathrm{d}y$ 	$A=\int_c^d \mathrm{f}(y)\ \mathrm{d}y-\int_b^c \mathrm{f}(y)\ \mathrm{d}y$ (the negative area must be **negated**)
	Note: Remember to (1) change **f(x) to f(y) if required**, (2) change the coordinates (3) write d*y*.		
Area between 2 curves	For f(*x*) > g(*x*), $A=\int_a^b [\mathrm{f}(x)-\mathrm{g}(x)]\ \mathrm{d}x$ Always take **top curve – bottom curve** 	For f(*x*) > g(*x*), $A=\int_c^d [\mathrm{f}(y)-\mathrm{g}(y)]\ \mathrm{d}y$ Always take **right curve – left curve** 	
Area with one point of intersection	$A=\int_a^b \mathrm{f}(x)\ \mathrm{d}x+\int_b^c \mathrm{g}(x)\ \mathrm{d}x$ where intersection at *p*. 	$A=\int_c^q \mathrm{f}(y)\ \mathrm{d}y+\int_q^d \mathrm{g}(y)\ \mathrm{d}y$ where intersection at *q* 	
Area under parametric curve	$A=\int_a^b y\ \mathrm{d}y=\int_{t_a}^{t_b} \mathrm{g}(t)\mathrm{f}'(t)\ \mathrm{d}t$ with *x* = f(t) and *y* = g(t)	$A=\int_c^d x\ \mathrm{d}y=\int_{t_c}^{t_d} \mathrm{f}(t)\mathrm{g}'(t)\ \mathrm{d}t$ with *x* = f(t) and *y* = g(t)	
Area as a limit of sum of areas of rectangles		**(1)** Express total area of rectangles in terms of f(*x*). $A=(\Delta x)\mathrm{f}(x_1)+(\Delta x)\mathrm{f}(x_2)+...+(\Delta x)\mathrm{f}(x_{n-1})+(\Delta x)\mathrm{f}(x_b)$ **(2)** Express total area in **sigma** notation. $A=\sum_{r=1}^{n}\mathrm{f}(x_r)\Delta x$ **(3)** Find limit of the sum of rectangles as $n\to\infty$. $A=\lim_{n\to\infty}\sum_{r=1}^{n}\mathrm{f}(x_r)\Delta x=\int_a^b \mathrm{f}(x)\ \mathrm{d}x$	
Volume of revolution	$V=\pi\int_a^b [\mathrm{f}(x)]^2\ \mathrm{d}x$	$V=\pi\int_c^d [\mathrm{f}(y)]^2\ \mathrm{d}y$	
Volume of rev^n bounded by 2 graphs	$V=\pi\int_a^b [\mathrm{f}(x)]^2-[\mathrm{g}(x)]^2\ \mathrm{d}x$ Always take **top curve – bottom curve**	$V=\pi\int_c^d [\mathrm{f}(y)]^2-[\mathrm{g}(y)]^2\ \mathrm{d}y$ Always take **right curve – left curve**	

Chapter 9: Differential Equations

Direct Integration	$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}(x)\Rightarrow y=\int \mathrm{f}(x)\ \mathrm{d}x$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{f}(x)\Rightarrow y=\mathrm{f}''(x)$
Separation of variables	$\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{f}(y)\Rightarrow \int \frac{1}{\mathrm{f}(y)}\ \mathrm{d}y=\int 1\ \mathrm{d}x$	
By substitution	$y=vx\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=v+x\frac{\mathrm{d}v}{\mathrm{d}x}$	$z=x+y\Rightarrow \frac{\mathrm{d}z}{\mathrm{d}x}=1+\frac{\mathrm{d}y}{\mathrm{d}x}$
	To obtain back the general solution, remember to replace the new variable back to y.	
Applications of D.E.	$\frac{\mathrm{d}P}{\mathrm{d}t}\propto P\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t}=kP\Rightarrow \int \frac{1}{P}\ \mathrm{d}P=\int k\ \mathrm{d}t$	$\frac{\mathrm{d}P}{\mathrm{d}t}\propto \frac{1}{P}\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t}=\frac{k}{P}\Rightarrow \int P\ \mathrm{d}P=\int k\ \mathrm{d}t$

Chapter 10: Macluarin's Series

Power series representation	$\mathrm{f}(x)=\mathrm{f}(0)+x\mathrm{f}'(0)+\frac{x^2}{2!}\mathrm{f}''(0)+...+\frac{x^n}{n!}\mathrm{f}^{(n)}0+...$	• Differentiate to find $\mathrm{f}'(x),\mathrm{f}''(x)$ and $\mathrm{f}'''(x)$ • Substitute x = 0 into the differentiated terms and then apply the formula.
Standard Maclaurin series	$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+...+\frac{n(n-1)...(n-r+1)}{r!}x^r+...\ (\lvert x\rvert<1)$	Only use the standard series from **MF15** if derivation is not required.
	$\mathrm{e}^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...+\frac{x^r}{r!}+...$ (all x)	$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5}-\frac{(-1)^r x^{2r+1}}{(2r+1)!}+...$ (all x)
	$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-...+\frac{(-1)^r x^{2r}}{(2r)!}+...$ (all x)	$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-...+\frac{(-1)^{r+1}x^r}{r}+...\ (-1<x\le 1)$

Small angle approximations	When x is **small** and measured in **radians**, and x^3 onwards is **neligible**, the following approximations are valid:		
	$\sin x\approx x$	$\cos x\approx 1-\frac{x^2}{2}$	$\tan x\approx x$

Compound angle formulae	$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$	You **cannot** assume $x\pm\alpha\approx x$ for some constant α. However you can assume $\sin(\alpha x)\approx\alpha x$ Note: Always select the value of x **closer to 0** if there are 2 answers.
	$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$	
	$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$	

Chapter 11: Permutations and Combinations

Counting principles	*Addition principle*: If there are two or more **non–overlapping** categories of ways to perform an operation, and there are *m* ways in the first category, *n* ways in the second category and *k* ways in the last category, there are $(\boldsymbol{m}+\boldsymbol{n}+\ldots+\boldsymbol{k})$ to perform it.		*Multiplication principle*: If an operation *A* can be completed in *m* ways, a second operation *B* performed in *n* ways and the last operation in *k* ways, the successive operations can be completed in $\boldsymbol{m}\times\boldsymbol{n}\times\ldots\times\boldsymbol{k}$ ways.	
Permutations	A *permutation* is an **ordered arrangement of objects** mainly concerned to find the **total number of ways** to **arrange objects**.	No. of ways to arrange *n* distinct objects in a straight line…		$\boldsymbol{n!}$
		…where *p* of them are **identical**.	Note: If there are 2 objects that **must be together**, treat them as a **single unit** and multiply 2!. If objects must be separated, place them between the **gaps** of the row.	$\frac{n!}{p!}$
		…and *q* of them are **identical**.		$\frac{n!}{p!q!}$
		No. of ways to arrange *r* objects out of *n* **distinct** objects in a straight line		$^n\mathbf{P}_r$
Combinations	A *combination* is an **unordered selection** of a **number of objects** from a **given set**.	No. of ways to choose *r* objects from *n* distinct objects		$^n\mathbf{C}_r$
		If *r* different balls are distributed to *n* different urns (such that any urn can contain any number of balls), then number of outcomes…		$\mathbf{2}^n$
		<u>Note</u>: When **restrictions** are given, they must be satisfied first before the number of combinations is selected. This can be done by breaking down into different cases, calculating separately and adding/subtracting them. Some useful techniques: **1.** Complementary technique **2.** Grouping technique (order must be taken into account) **3.** Insertion technique (order must be taken to account – use for 3 or more objects) <u>Note</u>: When subdividing into groups of **equal number**, remember to divide by the **number of groups**.		
Permutations in a circle	No. of ways of arranging *n* distinct objects in a circle if:			
	The positions are **indistinguishable**: $\frac{n!}{n}=(n-1)!$		The positions are **distinguishable**: $n!=n\times(n-1)!$	
	Note that seats become distinguishable if seats are **numbered**, **different colour** and **different shapes/sizes**.			

Chapter 12: Probability

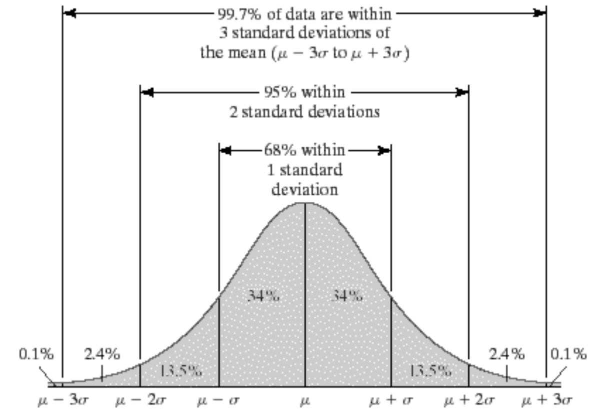
Probability of an event	$\mathrm{P}(A)=\frac{\text{No. of outcomes event occurs}}{\text{No. of possible outcomes}}=\frac{\mathrm{n}(A)}{\mathrm{n}(S)}$				
Complimentary Events	$\mathrm{P}(A)+\mathrm{P}(A')=1$				
Addition Law	$\mathrm{P}(A\text{ or }B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\text{ and }B)$		$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)-\mathrm{P}(A\cap B)$		
Mutually exclusive events	2 events are mutually exclusive if **they cannot occur together** (within a situation)				
	$\mathrm{P}(A\cap B)=0$	$\mathrm{P}(A\cup B)=\mathrm{P}(A)+\mathrm{P}(B)$		$\mathrm{P}(A\mid B)=0$ or $\mathrm{P}(B\mid A)=0$	
Conditional Probability	$\mathrm{P}(B\mid A)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A)}$	$\mathrm{P}(A\mid B)=\frac{\mathrm{P}(B\cap A)}{\mathrm{P}(B)}$	If *A* and *B* are mutually exclusive, P(*A* | *B*)=0 (Since *B* occurs, *A* wont occur, vice versa)		
Independent Events	2 events are independent if the occurrence of one does not affect the other (between diff. situations)				
	$\mathrm{P}(A\mid B)=\mathrm{P}(A)$	$\mathrm{P}(B\mid A)=\mathrm{P}(B)$		$\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)$ *	
Other useful approaches	Few stages with few outcomes: **<u>Tree diagram</u>**	Diff. probabilities for diff. events: **<u>Venn diagram</u>**	Small sample space: **<u>Table of outcomes</u>**	P&C method: **<u>Only WITHOUT</u>** replacement	**<u>Sequences and Series</u>**: For turn-by-turn situations
Tree diagram	How to use a tree diagram: **1.** Multiply the probabilities along the branches to get the end results **2.** On any set of branches that meet up at a point, the probabilities must **add up to 1**. **3.** Check that all end results add up to 1. **4.** To answer any question, find the relevant end results. If more than one satisfy the requirements, **add** these end results together.				
Other formulas	$\mathrm{P}(A\cup B)=1-\mathrm{P}(A'\cap B')$	$\mathrm{P}(A\cap B)=1-\mathrm{P}(A'\cup B')$	$\mathrm{P}(A)=\mathrm{P}(A\cap B)+\mathrm{P}(A\cap B')$	$\mathrm{P}(A'\mid B)+1-\mathrm{P}(A\mid B)$	
Notes	"and" means $\cap$ (i.e. multiply) whereas "or" means $\cup$ (i.e. add)				

Chapter 13: Binomial and Possion Distributions

Use **pdf** for equal, **cdf** for less than or equal.

Binomial Distribution	$X \sim \text{B}(n, p)$	Where X is the **random variable**, n is the no. of trials and p is the probability of **success**.	$\text{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$
	To find **mode** (value of x with highest probability), use the **TABLE** function in G.C. to get highest value.		
	Conditions for Binomial Distribution: **1.** There are n **independent** trials,* **2.** Each trial has **exactly two possible outcomes:** a **"success"** or a **"failure"** **3.** The **probability** of a "success", denoted by p is the **same for each trial**.*		
Poisson Distribution	$X \sim \text{Po}(\lambda)$	Where λ is the **mean no. of occurences**.	$\text{P}(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}, x = 0, 1, 2, 3$
	Conditions for poisson distribution: **1.** The events occur at **random** and are **independent of each other** in a **given interval of time or space***, **2.** The **average number of events per interval** is **constant** throughout the interval* **3.** The average number of events per interval is **proportional** to the size of the interval.		
Expectation	The **mean value (average)** of a distribution.		
	Binomial Distribution: $\text{E}(X) = np$		Poisson Distribution: $\text{E}(X) = \lambda$
Variance	**Spread of values** of X from its **mean**.		
	Binomial Distribution: $\text{Var}(X) = \sigma^2 = np(1-p)$		Poisson Distribution: $\text{Var}(X) = \sigma^2 = \lambda$
Standard Deviation	**Spread of values** of X.		
	Binomial Distribution: $\sigma = \sqrt{np(1-p)}$		Poisson Distribution: $\sigma = \sqrt{\lambda}$
Additive properties	If $X \sim \text{Po}(\lambda)$ and $Y \sim \text{Po}(\mu)$, and X and Y are **independent**, then $X + Y \sim \text{Po}(\lambda + \mu)$		

Chapter 14: Normal Distribution

<table>
<tr><td>***Normal Distribution***</td><td>$X \sim \mathrm{N}(\mu, \sigma^2)$</td><td colspan="2">Where μ and σ are the **mean** (average) and the **variance** (spread of values) respectively.</td><td rowspan="4">
</td></tr>
<tr><td>***Standard normal distribution***</td><td>$Z \sim \mathrm{N}(0,1)$</td><td colspan="2">Where $\mu = 0$ and $\sigma = 1$.</td></tr>
<tr><td rowspan="2">***Area under normal curve*** (by symmetry)</td><td colspan="3">$\mathrm{P}(a < X < b) = \mathrm{P}(a < X \le b)$
$= \mathrm{P}(a \le X < b) = \mathrm{P}(a \le X \le b)$</td></tr>
<tr><td colspan="3">$\mathrm{P}(X > a) = 1 - \mathrm{P}(X < a)$
$\mathrm{P}(X < \mu - a) = \mathrm{P}(X > \mu + a)$
$\mathrm{P}(X < \mu + a) = \mathrm{P}(X > \mu - a)$</td></tr>
<tr><td>***Transforming to standard normal***</td><td colspan="4">$Z = \dfrac{X-\mu}{\sigma}$ (used for solving unknown μ or σ) – use **invnorm** function.</td></tr>
<tr><td rowspan="6">***Properties of expectation and variance***</td><td colspan="2">$\mathrm{E}(a) = a$</td><td colspan="2">$\mathrm{Var}(a) = 0$</td></tr>
<tr><td colspan="2">$\mathrm{E}(aX) = a\,\mathrm{E}(X)$</td><td colspan="2">$\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$</td></tr>
<tr><td colspan="2">$\mathrm{E}(aX \pm b) = a\,\mathrm{E}(X) \pm b$</td><td colspan="2">$\mathrm{Var}(aX \pm b) = a^2\,\mathrm{Var}(X)$</td></tr>
<tr><td colspan="2">$\mathrm{E}(X_1 + ... + X_n) = \mathrm{E}(X_1) + ... + \mathrm{E}(X_n) = n\,\mathrm{E}(X)$</td><td colspan="2">$\mathrm{Var}(X_1 + ... + X_n) = \mathrm{Var}(X_1) + ... + \mathrm{Var}(X_n) = n\,\mathrm{Var}(X)$</td></tr>
<tr><td colspan="2">$\mathrm{E}(aX \pm bY) = a\,\mathrm{E}(X) \pm b\,\mathrm{E}(Y)$
When $a = b = 1$,
$\mathrm{E}(X \pm Y) = \mathrm{E}(X) \pm \mathrm{E}(Y)$</td><td colspan="2">$\mathrm{Var}(aX \pm bY) = a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y)$
When $a = b = 1$,
$\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$</td></tr>
<tr><td colspan="4">If X and Y are **two independent normal variables**, then $aX + bY$ will also be a **normal variable**.</td></tr>
<tr><td>***Sample mean***</td><td>$\mathrm{E}(\overline{X}) = \mu$</td><td>$\mathrm{Var}(\overline{X}) = \dfrac{\sigma^2}{n}$</td><td colspan="2">Distribution: $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ for any size n</td></tr>
<tr><td rowspan="2">***Central Limit Theorem (CLT)***</td><td colspan="4">If $X_1, X_2 ... X_n$ is a random sample of size n taken from a **non-normal** or unknown distribution with mean μ and variance σ^2, then for sufficiently large n, the sample mean is $\overline{X} \sim \mathrm{N}\left(\mu, \dfrac{\sigma^2}{n}\right)$ approximately.</td></tr>
<tr><td colspan="2">Large samples are usually of size **50**.</td><td colspan="2">$X_1 + X_2 + ... + X_n \sim \mathrm{N}(n\mu, n\sigma^2)$ approximately by CLT.</td></tr>
<tr><td rowspan="4">***Approximation of binomial and poisson distributions***</td><td>***Using poisson distributions***</td><td colspan="3">If $X \sim \mathrm{B}(n, p)$, then **$X \sim \mathrm{Po}(np)$ approximately** if n is large (>50) and **$np < 5$**.</td></tr>
<tr><td>***Using normal distributions***</td><td colspan="3">If $X \sim \mathrm{B}(n, p)$, then **$X \sim \mathrm{N}(np, npq)$ approximately** if n is large, **$np > 5$** and **$nq > 5$**
***Continuity correction** must be applied. E.g.
$\mathrm{P}(5 < X \le 8) \xrightarrow{\text{c.c.}} \mathrm{P}(5.5 < X \le 8.5)$
(note: convert to more/less than and equal to before c.c.)</td></tr>
<tr><td>***When n is large (>50), np > 5 but nq < 5***</td><td colspan="3">**Redefine** X as $X' \sim \mathrm{B}(n, q) \rightarrow X' \sim \mathrm{Po}(nq)$ approximately.</td></tr>
<tr><td>***Using normal distribution***</td><td colspan="3">If $X \sim \mathrm{Po}(\lambda)$, then $X \sim \mathrm{N}(\lambda, \lambda)$ **approximately** if $\lambda > 10$
***Continuity correction** must be applied.</td></tr>
</table>

Chapter 15: Sampling

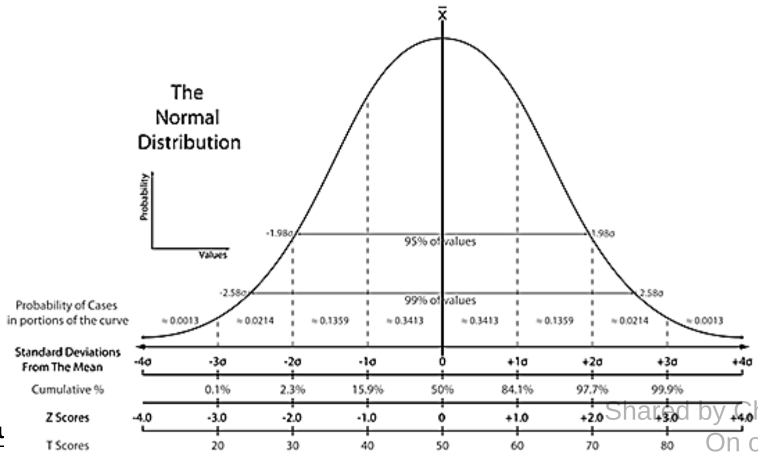
Random sampling	: Every member of population has equal chance of being chosen			
Types of sampling	**Type**	**Steps**	**Advantages**	**Disadvantages**
	Simple random sampling	1. Create a list of the population (sampling frame) 2. Create a random sample (using random selection)	Free from bias	• Difficult/impossible to identify every member of the population • Not able to get access to some members chosen from the sample
	Systematic sampling	1. List the populaton in some order 2. From the first k elements select a member randomly. 3. Select every kth member of the next k elements	Could lead to more precise inferences concerning population because the sample chosen is **spread evenly** throughout the entire population.	• There could be **bias** caused by the effect of **periodic/cyclic pattern** of the population. • Not always possible to **list the members** of the population in some order.
	Stratified sampling	1. Divide the population into a number of **non-overlapping subpopulations** (age / gender). 2. Take a **random sample** from each stratum with sample size proportional to the size of the stratum.	• Likely to give a **sample representative** of the population. • Each strata can be **treated separately**, hence sampling may be more **convenient** and more **accurate**.	Strata may not be **clearly defined. (overlapping of strata)**
	Non-random sampling: Every member does **not** have an equal chance of being chosen			
	Quota sampling	1. Divide the population into a number of **non-overlapping subpopulations** (age / gender). 2. Samples taken from each stratum are **non-random** (by choice)	Information can be **collected quickly.**	• Not representative of the population as compared to other types of sampling • **Biased** (non-random) as not everyone gets an **equal chance** of being selected. • **No sampling frame**

Point estimation		***Population***	***Sample***	***Unbiased estimate***
	Mean	μ	$\bar{x}=\frac{1}{n}\sum x=c+\frac{1}{n}\left[\sum(x-c)\right]$	$\mu=\bar{x}=\frac{1}{n}\sum x=c+\frac{1}{n}\left[\sum(x-c)\right]$
	Variance	σ	$\sigma_x^2=\frac{1}{n}\sum(x-\bar{x})^2$ (biased!) $=\frac{1}{n}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]$ $=\frac{1}{n}\left\{\sum(x-c)^2-\frac{\left[\sum(x-c)\right]^2}{n}\right\}$	$s^2=\frac{n}{n-1}\sigma_x^2=\frac{1}{n-1}\sum(x-\bar{x})^2$ $=\frac{1}{n-1}\left[\sum x^2-\frac{\left(\sum x\right)^2}{n}\right]$ $=\frac{1}{n-1}\left\{\sum(x-c)^2-\frac{\left[\sum(x-c)\right]^2}{n}\right\}$

Chapter 16: Hypothesis Testing

Important definitions	H_0 (null hypothesis)	Particular **claim** for a value for the **population mean.** *(In hypothesis testing, we try to **reject H_0** as far as possible as it is a more reliable claim compared to **do not reject H_0.**)*
	H_1 (alternative hypothesis)	Range of values that **excludes** the value specified by null hypothesis.
	Test statistic	Random variable whose value is calculated from **sample data.**
	Critical region	Range of values of the test static that leads to the **rejection of H_0** The value of c which determines the critical region is known as the *critical value*.
	Level of significance, α	Probability of rejecting H_0 given that H_0 is true. (usually around 0.05) • If a result is **significant** at α%, then it is **also significant** at any level **greater** than α%. • If a result is **not significant** at α%, then it is also **not significant** at any level < α%. • The *p*–value is the **smallest level of significance** at which H_0 can be rejected.
	p-value	Probability of observing a value of the test statistic as **extreme** or **more extreme** than the one obtained, given that H_0 is true. *p*-value $\le \alpha$: Test statistic lies in the critical region, hence we **reject H_0.** *p*-value $> \alpha$: Test statistic does not lie in the critical region, hence we **do not reject H_0.**
Hypothesis tests	Types of tests	***Left-tail test***: A change in the **decrease direction**. H_1: $\mu < \mu_0$ ***Right-tail test***: A change in the **increase** direction. H_1: $\mu > \mu_0$ ***2-tail test***: A difference in **either direction.** H_1: $\mu \neq \mu_0$. Note that α (and *p*-value) is **divided equally** between the 2 tails of the critical region.
	z-test	• True when population distribution is **Normal** and the population variance is known this holds for **any sample size**, large or small. • Also true when population size is **large** when population distribution is **unknown**. When *n* is large >50, by **central limit theorem**, … Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ (approximately), test statistic $Z = \frac{\overline{X} - \mu_0}{\sqrt{\sigma^2 / n}}$ (approximately). However, when population *variance* is **unknown**, we use s^2 instead of σ^2. Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{s^2}{n}\right)$ approximately, test statistic $Z = \frac{\overline{X} - \mu_0}{\sqrt{s^2 / n}}$ approximately where $s^2 = \frac{1}{n-1}\left[\sum x - \frac{\left(\sum x\right)^2}{n}\right] = \frac{1}{n-1}\sum (x - \bar{x})^2$
	T-test	Use when *n* is small and population variance is unknown. Given (or assume) that the population distribution is **normal**, Under H_0, $\overline{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$ approximately, test statistic $T = \frac{\overline{X} - \mu_0}{\sqrt{S^2 / n}} \sim t(n-1)$.

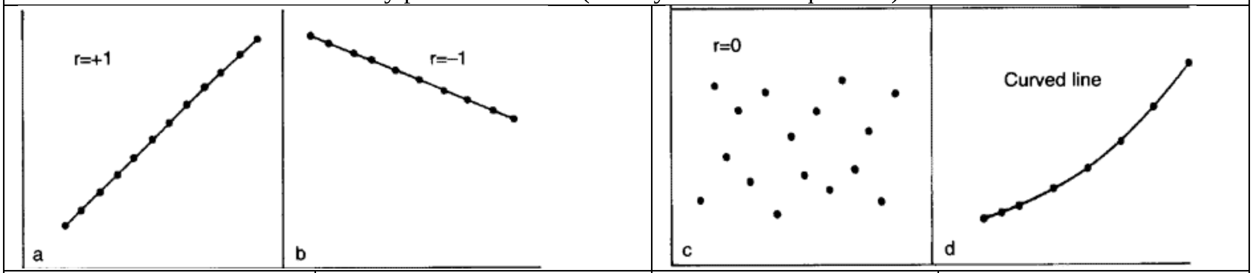
General hypothesis testing procedure	***Critical region method*** (if $\bar{x}, \mu_0$ or n is unknown)	***p-value method***
	1. Write down H_0 and H_1. 2. Determine an approximate *test statistic* and its distribution. 3. Identify the *level of significance*, *α* (from qn) 4. Determine the *critical region*. **Multiply** by 2 if there is a 2-tail test. 5. Compute the *test statistic* (using G.C.) 6. Determine if the test static lies *within the critical region*, and state conclusion **in context of the problem**. (since *z* = ____ (< or >) _____, we (reject / do not reject H_0 at ___% of significance and conclude that there is sufficient evidence that …)	1. Write down H_0 and H_1. 2. Determine an approximate *test statistic* and its distribution. 3. Identify the *level of significance*, *α* (from qn) 4. Compute the *test statistic* (using G.C.) 5. Find the *p*-value (from G.C.). **Multiply** by 2 if there is a 2-tail test. 6. Determine if the test static lies *within the critical region*, and state conclusion **in context of the problem**. (since *p* = ___ (< or >) α = ___, we (reject / do not reject H_0 at ___% of significance and conclude that there is sufficient evidence that …)



Chapter 17: Correlation and Regression

Scatter diagrams

- A sketch where each axis represents a **variable** and each point represents an **observation.**
- **Controlled variable** is usually plotted at x-axis (usually stated from question).



Positive linear correlation	**Negative** linear correlation	**No** linear correleation	**Non**-linear correleation

Product moment correlation coefficient

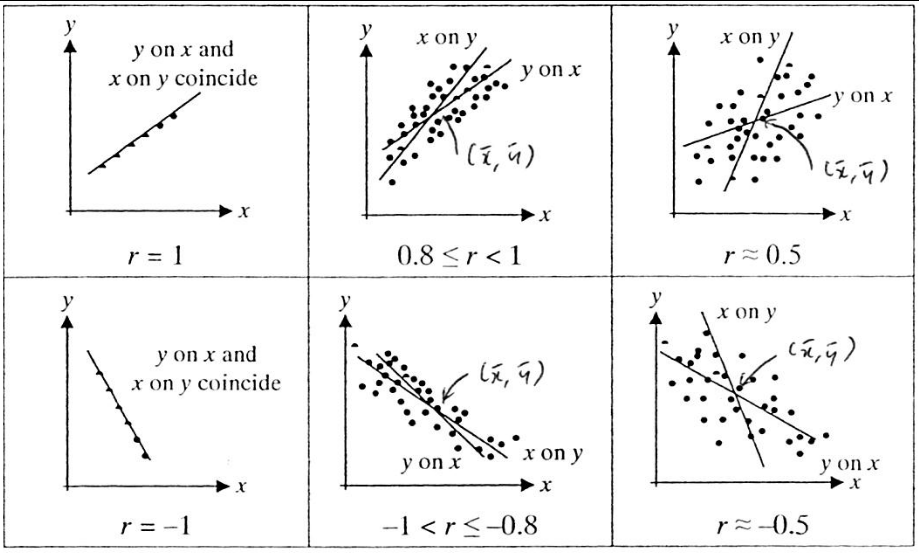
- The *product moment correleation coefficient*, denoted by r, measures the **strength and direction** of a **linear correleation** between the 2 variables.
- $-1 \le r \le 1$
- **Anomalies** are removed from a set of bivariate data to calculate an accurate value of r.
- There may not be direct **cause-effect relationship** between the 2 variables.

$$r = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sqrt{\left\{\sum\left(x-\bar{x}\right)^2\right\}\left\{\sum\left(y-\bar{y}\right)^2\right\}}} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sqrt{\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)\left(\sum y^2 - \frac{\left(\sum y\right)^2}{n}\right)}}$$

Regression lines

The *linear regression* fits a straight line in the scatter diagram.

Least squares regression line of y on x		Least squares regression line of x on y
$y-\bar{y} = b(x-\bar{x})$ where $b = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (x-\bar{x})^2} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum x^2 - \frac{\left(\sum x\right)^2}{n}}$	Both regression lines intersect at $(\bar{x}, \bar{y})$	$x-\bar{x} = d(y-\bar{y})$ where $d = \frac{\sum (x-\bar{x})(y-\bar{y})}{\sum (y-\bar{y})^2} = \frac{\sum xy - \frac{\sum x \sum y}{n}}{\sum y^2 - \frac{\left(\sum y\right)^2}{n}}$



Choice of regression line

Case	*Estimate y given x*	*Estimate x given y*
x is controlled, y is random	Use y on x	
y is controlled, x is random	Use x on y	
Both x and y is random	Use y on x	Use x on y

Interpolation extrapolation

Interpolating: Estimation using a value **within the given range** of data.	*Extrapolation*: Estimation using a value **outside the given range** of data (unreliable and should be avoided)

Linear law

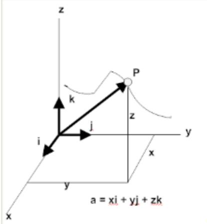
Non-linear model	*Variable Y*	*Variable X*
$y = a + bx^2$	y	x^2
$y = a + \frac{b}{x}$	y	$\frac{1}{x}$
$y = ax^b$	$\ln y$	$\ln x$

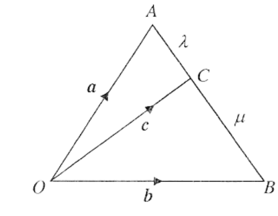
Chapter 18: Vectors

1. Definitions, Geometric Representations and Properties of Vectors

Modulus of vector	Modulus of vector **a** is its magnitude/length, denoted as |**a**|. If **a** = $\overrightarrow{PQ}$, its length = $\|\overrightarrow{PQ}\|$.
Unit vector	Vector with **magnitude 1**. Unit vector of **a** is $\frac{1}{\|\mathbf{a}\|}\mathbf{a}$, denoted as $\hat{\mathbf{a}}$.
Negative vectors	Negative vector of **a**, denoted by **–a**, is the vector of magnitude |**a**| with **opposite direction**.
Null vector	Vector that has **no magnitude or direction**. Denoted by **0**.
Addition of vectors	If $\overrightarrow{AB}=\mathbf{u}$ and $\overrightarrow{BC}=\mathbf{v}$, $\overrightarrow{AC}=\mathbf{w}=\mathbf{u}+\mathbf{v}$. **w** is known as the **vector sum** of **u** and **v**.
Subtraction of vectors	If $\overrightarrow{AB}=\mathbf{u}$ and $\overrightarrow{AD}=\mathbf{v}$, $\overrightarrow{DB}=\mathbf{u}-\mathbf{v}$
Proving a parallelogram	To show that *A, B, C, D* is a parallelogram, just show $\overrightarrow{AB}=\overrightarrow{DC}$ or $\overrightarrow{AD}=\overrightarrow{BC}$
Parallel vectors	If **a** and **b** are **non-zero** vectors, $\mathbf{a} \,//\, \mathbf{b} \Leftrightarrow \mathbf{a}=\lambda\mathbf{b}$ for some $\lambda\in\mathbb{R}\setminus\{0\}$.
Collinear points	If 3 points *A*, *B* and *C* are collinear (**lie in a straight line**), $\Leftrightarrow \overrightarrow{AB}=\lambda\overrightarrow{AC}$ or $\overrightarrow{AB}=\lambda\overrightarrow{BC}$ or $\overrightarrow{BC}=\lambda\overrightarrow{AC}$ for some $\lambda\in\mathbb{R}\setminus\{0\}$.
Multiplication of vector by scalar	For all vectors **a** and **b**, and $\lambda,\ \mu\in\mathbb{R}$, $\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}$ $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$ $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$

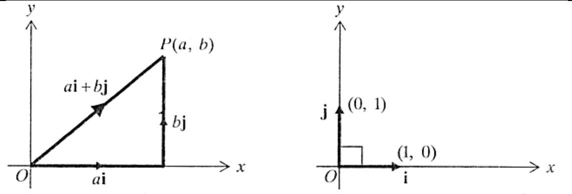
Position vector	***Ratio Theorem***
Position vector of a point *P* **with reference to origin O** is the vector $\overrightarrow{OP}$. Given 2 points *P* and *Q*, we have $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$.	If point *C* divides the line *AB* in the ratio $\lambda:\mu$, then the position vector of *C* is given by: $\overrightarrow{OC}=\frac{\mu\overrightarrow{OA}+\lambda\overrightarrow{OB}}{\lambda+\mu}$; $\mathbf{c}=\frac{\mu\mathbf{a}+\lambda\mathbf{b}}{\mu+\lambda}$





Vectors in 2-D space

Given a point *P*(*a*, *b*) in the *x*-*y* plane, the position vector of *P* with respect to *O* can be written as $\overrightarrow{OP}=\begin{pmatrix}a\\b\end{pmatrix}$ or $a\mathbf{i}+b\mathbf{j}$ where **i, j** are **unit vectors** of the positive *x*-axis and positive *y*-axis respectively.

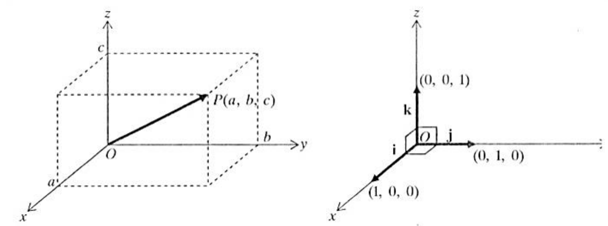


Length from origin to point $P=\left|\overrightarrow{OP}\right|=\left|\begin{pmatrix}a\\b\end{pmatrix}\right|=\sqrt{a^2+b^2}$

Unit vector in direction of $\overrightarrow{OP}=\frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|}=\frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}a\\b\end{pmatrix}$

Vectors in 3-D space

Given a point *P*(*a*, *b*, *c*) in the *x*-*y*-*z* plane, the position vector of *P* with respect to *O* can be written as $\overrightarrow{OP}=\begin{pmatrix}a\\b\\c\end{pmatrix}$ or $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ where **i, j** and **k** are **unit vectors** of the positive *x*-axis, positive *y*-axis and positive *z*-axis respectively.



Length from origin to point $P=\left|\overrightarrow{OP}\right|=\left|\begin{pmatrix}a\\b\\c\end{pmatrix}\right|=\sqrt{a^2+b^2+c^2}$

Unit vector in direction of $\overrightarrow{OP}=\frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|}=\frac{1}{\sqrt{a^2+b^2+c^2}}\begin{pmatrix}a\\b\\c\end{pmatrix}$

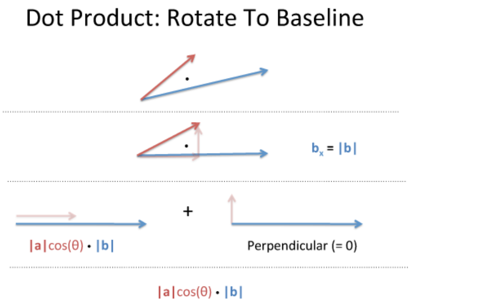
2. Scalar Product

Definition

If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$, then the scalar product of **a** and **b**, denoted as **a•b**, is defined as

$$\mathbf{a}\bullet\mathbf{b}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\bullet\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}=a_1b_1+a_2b_2+a_3b_3.$$

Note that the product is a <u>**scalar**</u>.



Also, it can be proven that $\mathbf{a}\bullet\mathbf{b}=\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos\theta$ where θ is the (**diverging or converging**) angle between **a** and **b**.

Parallel vectors	For 2 parallel vectors, $\mathbf{a}\bullet\mathbf{b}=\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos0^\circ=\left|\mathbf{a}\right|\left|\mathbf{b}\right|$	For 2 opposite vectors, $\mathbf{a}\bullet\mathbf{b}=\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos180^\circ=-\left|\mathbf{a}\right|\left|\mathbf{b}\right|$	$\mathbf{a}\bullet\mathbf{a}=\left|\mathbf{a}\right|^2$	$\mathbf{i}\bullet\mathbf{i}=\mathbf{j}\bullet\mathbf{j}=\mathbf{k}\bullet\mathbf{k}=1$.
Perpendicular vectors	If **a** and **b** are perpendicular, then $\mathbf{a}\bullet\mathbf{b}=\left|\mathbf{a}\right|\left|\mathbf{b}\right|\cos90^\circ=0\ \therefore\ \mathbf{a}\perp\mathbf{b}\Leftrightarrow\mathbf{a}\bullet\mathbf{b}=0$	$\mathbf{i}\bullet\mathbf{j}=\mathbf{j}\bullet\mathbf{k}=\mathbf{k}\bullet\mathbf{i}=0$	Other properties: **a•b = b•a**, **a•(b+c) = a•b + a•c**, **a•(λb) = λ(a•b)**	

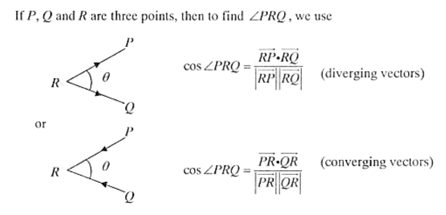
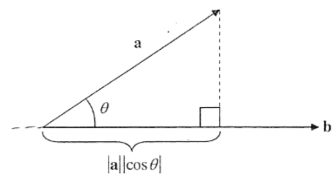
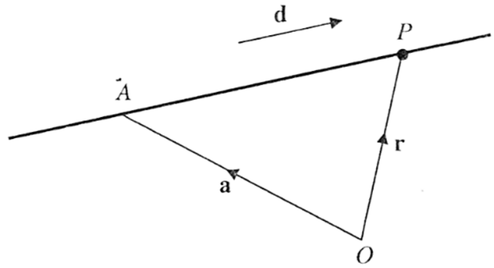
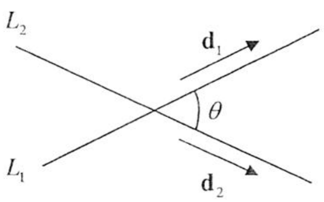
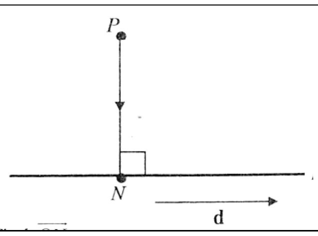
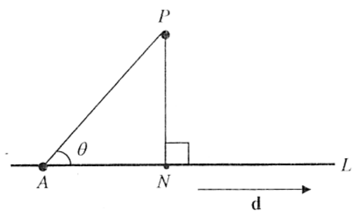
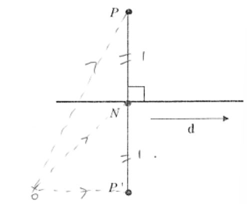
Scalar product (cont')	***Angle between 2 vectors*** $\cos\theta = \dfrac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ or $\theta = \cos^{-1}\dfrac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ $= \dfrac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$		***Length of projection*** Length of projection of **a** on **u** $= \lvert\mathbf{a}\rvert\lvert\cos\theta\rvert$ $= \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\lvert\cos\theta\rvert$ $= \lvert\mathbf{a}\bullet\mathbf{b}\rvert = \dfrac{\mathbf{a}\bullet\mathbf{b}}{\lvert\mathbf{b}\rvert}$ 
3. Vector equation of straight line	***Vector equation***	Consider a line which is parallel to vector **b** and which passes through a fixed point A with position vector **a**. If P is a point on the line, then its position vector **r** is given by **a** + λ**d** where $\lambda \in \mathbb{R}$	
	Line through 2 fixed pts	Let A and B be 2 points with position vectors **a** and **b** respectively. • A **direction vector** of the line through A and B is $\overrightarrow{AB}$. • A **vector equation** of the line is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ where $\mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\lambda \in \mathbb{R}$.	
	Parametric Equations	If a line passes through $A(x_1, y_1, z_1)$ and is parallel to $\mathbf{d} = \alpha\mathbf{i} + \beta\mathbf{j} + \gamma\mathbf{k}$, then the position vector of any point $P(x, y, z)$ on the line is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, i.e. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + \lambda\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ where $\lambda \in \mathbb{R}$. Equating the **i**, **j** and **k** components, we get $x = x_1 + \lambda\alpha, y = y_1 + \lambda\beta, z = z_1 + \lambda\gamma$.	
	Cartesian Equation	Equating the value of λ, we get $\dfrac{x - x_1}{\alpha} = \dfrac{y - y_1}{\beta} = \dfrac{z - z_1}{\gamma} = \lambda$.	
	Relationship between 2 lines	***Parallel***: If L_1 // L_2 $\Leftrightarrow \mathbf{d}_1 // \mathbf{d}_2$ ***Intersecting***: If L_1 and L_2 intersect, there are unique values of λ and μ such that $\mathbf{a}_1 + \lambda\mathbf{d}_1 = \mathbf{a}_2 + \lambda\mathbf{d}_2$ ***Skew***: $\mathbf{d}_1$ and $\mathbf{d}_2$ are **not parallel** and there are **no unique values** of λ and μ for $\mathbf{a}_1 + \lambda\mathbf{d}_1 = \mathbf{a}_2 + \lambda\mathbf{d}_2$. <u>**Note**</u>: Use the G.C. to determine if the system of linear equations is consistent and to find the solution if it exists.	
	Acute angle between 2 lines	Consider 2 lines L_1 and L_2 where $L_1 : \mathbf{a}_1 + \lambda\mathbf{b}$ and $L_2 : \mathbf{c} + \lambda\mathbf{d}$ The acute angle between the 2 lines is the same as the acute angles between **b** and **d.** $\theta = \cos^{-1}\dfrac{\lvert\mathbf{d}_1\bullet\mathbf{d}_2\rvert}{\lvert\mathbf{d}_1\rvert\lvert\mathbf{d}_2\rvert}$	
	Foot of ⊥ from point P to a line	Let vector equation of line L be $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$. Since N is on the line, $\overrightarrow{ON} = \mathbf{a} + \lambda\mathbf{b}$ Since $\overrightarrow{PN} \perp L \Rightarrow \overrightarrow{PN} \perp \mathbf{d}$. Hence $\overrightarrow{PN}\bullet\mathbf{d} = 0$ [by ***perpendicular vectors***] Solve the equation for λ and substitute into the vector equation to find $\overrightarrow{ON}$.	
	Distance from a point to a line	Let vector equation of line L be $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ and **a** is the position vector of the point A on the line L. The distance PN can be found using: (1) Find $\overrightarrow{ON}$ then find $\lvert\overrightarrow{PN}\rvert$ (2) Find AN (***length of projection*** of $\overrightarrow{AP}$ on **d**, then use Pythagoras theorem: $\overrightarrow{PN} = \sqrt{AP^2 - AN^2}$ (3) Find θ (angle between $\overrightarrow{AP}$ and **d**). Then find $\overrightarrow{PN} = \lvert\overrightarrow{AP}\rvert\sin\theta$ (use this if θ is found or known) (4) Use $\overrightarrow{PN} = \dfrac{\lvert\overrightarrow{AP}\times\mathbf{d}\rvert}{\mathbf{d}}$	

Image of a point P by reflection in the line L

First find $\overrightarrow{ON}$ (using one of the 4 methods stated above)
Since N is the **mid-point** of PP', using ***ratio theorem***, we have

$$\overrightarrow{ON} = \frac{\overrightarrow{OP} + \overrightarrow{OP'}}{2}$$

Then find $\overrightarrow{OP}$ by making it the subject.



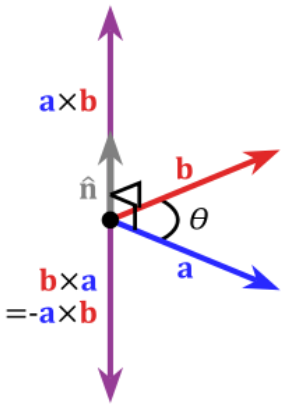
4. Vector Product

Definition

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, vector product of **a** and **b** denoted by

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ -(a_1b_3 - a_3b_1) \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

Also, it can be proven that $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\ \mathbf{n}$ where θ is the angle between **a** and **b**. **n** is the **unit vector** perpendicular to both **a** and **b**, the direction which follows the **right hand grip rule**.
Note that **a** and **b** must be **perpendicular** on the **same plane**.
Also, $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$



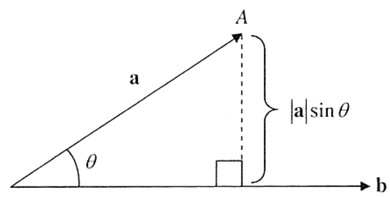
⊥ distance

Perpendicular distance from A to **b**

$$= |\mathbf{a}|\sin\theta = \frac{|\mathbf{a}||\mathbf{b}|\sin\theta}{|\mathbf{b}|} = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|} \text{ (or } |\mathbf{a} \times \mathbf{b}| \text{)}$$

This can be used to find:

- Area of parallelogram = $|\mathbf{a} \times \mathbf{b}|$ where **b** is the length,
- Area of triangle = ½$|\mathbf{a} \times \mathbf{b}|$

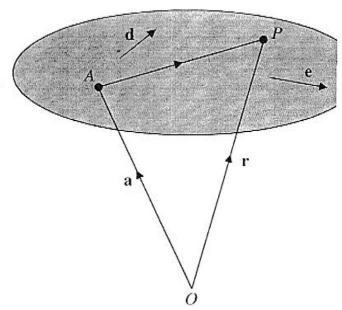


5. Planes

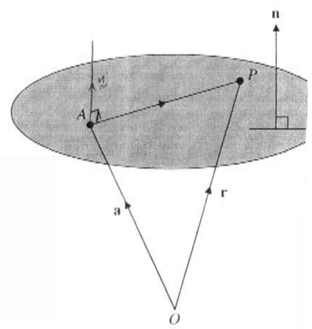
Equation of Planes

The ***vector equation*** of a plane is given by
$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d} + \mu\mathbf{e},\ \lambda, \mu \in \mathbb{R}$ where

- Vectors **d** and **e** are **parallel** to the plane and **non-parallel** to each other
- **a** gives the **point** of the vector while λ**d** and μ**e** gives the direction of the vector.



In a ***scalar product*** form, we let $\overrightarrow{AP} \perp \mathbf{n}$.
$\therefore \overrightarrow{AP} \perp \mathbf{n} = 0, (\mathbf{r} - \mathbf{a})\bullet\mathbf{n} = 0, \mathbf{r}\bullet\mathbf{n} = \mathbf{a}\bullet\mathbf{n} = d$
Note that the position vector **r** of **every point** on the plane will satisfy the equation **r•n = d**.
d can be obtained from **a•n**

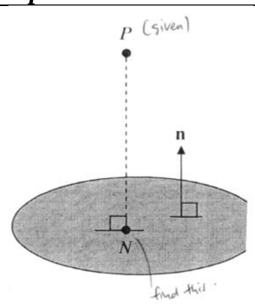


Cartesian Equation of a Plane

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$. Then $\mathbf{r}\bullet\mathbf{n} = d \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d \therefore n_1x + n_2y + n_3z = d$

Foot of ⊥ from a point to plane

From the figure, the vector equation of PN: $\mathbf{r} = \overrightarrow{OP} + \lambda\mathbf{n}$.
Since N is the **point of intersection** between PN and the plane, we can equate the above with **r•n = d.**
$\therefore \overrightarrow{ON}\bullet\mathbf{n} = d$ and $\overrightarrow{ON} = \overrightarrow{OP} + \lambda\mathbf{n}$ for some $\lambda \in \mathbb{R}$.
$\Rightarrow (\overrightarrow{OP} + \lambda\mathbf{n})\bullet\mathbf{n} = d$. Solve for λ and find $\overrightarrow{ON}$.

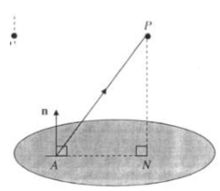


Distance from point P to a plane

Let Π be a plane with vector equation **r•n** = d. The **perpendicular distance** PN can be found by:

(1) Choose any point A on the plane. Then PN = length of projection of $\overrightarrow{AP}$ on **n**.

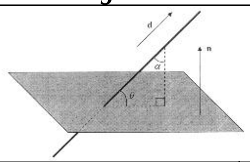
(2) $\overrightarrow{PN} = |\overrightarrow{PN}|$



Point of intersection of line & plane

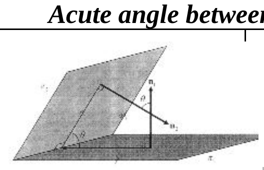
Line has vector equation **r = a + λd**
Plane has vector eqn **r•n = d.**
Equating them, $(\mathbf{a} + \lambda\mathbf{d})\bullet\mathbf{n} = d$

Acute angle between line & plane



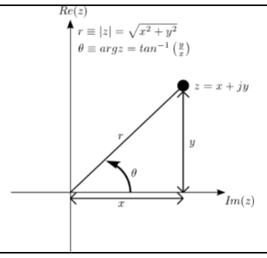
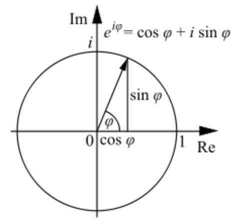
$$\sin\theta = \frac{|\mathbf{d}\bullet\mathbf{n}|}{|\mathbf{d}||\mathbf{n}|}$$

Acute angle between 2 planes



$$\cos\theta = \frac{|\mathbf{n}_1\bullet\mathbf{n}_2|}{|\mathbf{n}_1||\mathbf{n}_2|}$$

Chapter 19: Complex Numbers

Definition	***Imaginary numbers***	$\mathbf{i}=\sqrt{-1}$; $\mathbf{i}^2=-1$; Denoted by $\mathbb{C}=\{a+b\mathbf{i}:a,b\in\mathbb{R}\}$
	Forms	A complex number is a number of the form ***a + bi***, where $a,b\in\mathbb{R}$
	Cartesian form	*a* is called the **real part of *z***, denoted as **Re(*z*)**. ; *b* is called the **imaginary part of *z***, denoted as **Im(*z*)**.
	Simple operations	$(a+b\mathbf{i})\pm(c+d\mathbf{i})=(a\pm c)+(b\pm d)\mathbf{i}$; $\mathbf{i}\times\mathbf{i}=\mathbf{i}^2=-1$; For **division**, rationalise the denominator.
	Complex conjugate	The *complex conjugate* of the complex number $z=(a+b\mathbf{i})$ is denoted as $z^*=(a-b\mathbf{i})$ **.**
		$z+z^*=2a$ $z-z^*=2b\mathbf{i}$; $zz^*=a^2+b^2$ $(z_1\pm z_2)^*=z_1{}^*+z_2{}^*$; $(z_1z_2)^*=z_1{}^*z_2{}^*$; $\left(\frac{z_1}{z_2}\right)^*=\frac{z_1{}^*}{z_2{}^*}$
	Equality	2 complex no. are equal only if they have the **same real & imaginary parts**, $a+b\mathbf{i}=c+d\mathbf{i}\Leftrightarrow a=c,b=d$
Complex roots		$b^2-4ac\geq 0\Rightarrow 2$ real roots ; The complex roots of any polynomial equation with **real** coefficients will always **occur in conjugate pairs.** $b^2-4ac<0\Rightarrow 2$ complex conjugate roots
Multiple powers		$\mathrm{i}^{4n-3}=\mathrm{i}$, e.g. $\mathrm{i},\mathrm{i}^5,\mathrm{i}^9,\mathrm{i}^{13}=\mathrm{i}$; $\mathrm{i}^{4n-2}=-1$, e.g. $\mathrm{i}^2,\mathrm{i}^6,\mathrm{i}^{10}=-1$; $\mathrm{i}^{4n-3}=-1$, e.g. $\mathrm{i}^3,\mathrm{i}^7,\mathrm{i}^{11}=-\mathrm{i}$; $\mathrm{i}^{4n}=1$ e.g. $\mathrm{i}^4,\mathrm{i}^8,\mathrm{i}^{12}=1$
Fundamental Theorem of algebra		If $P(z)=0$ has *n* solutions $\alpha_1,\alpha_2,\alpha_3,...,\alpha_n$, it can be expressed in the form $P(z)=(z-\alpha_1)(z-\alpha_2)...(z-\alpha_n)$.
Argand diagram	***Diagrams***	 
	Conjugate	Represented by $P'(x,-y)$ (reflection of *P* in the **real axis**)
	Modulus	$\|z\|=\sqrt{x^2+y^2}=r$
	Argument	$\arg(z)=\theta$. For a positive no. *a*,
		$\arg(a)=0$; $\arg(a\mathbf{i})=\frac{\pi}{2}$; $\arg(-a)=\pi$; $\arg(-a\mathbf{i})=-\frac{\pi}{2}$
		Note: must follow principal argument: $-\pi<\theta\leq\pi$
		Modulus-argument form: Since (by *parametric form*), $x=r\cos\theta, y=r\sin\theta$ $\therefore z=x+iy=r(\cos\theta+i\sin\theta)$ • $\|z^*\|=\|z\|$ • $\arg(z^*)=-\arg(z)$ • $zz^*=x^2+y^2=\|z\|^2$ Multiplication by **i** rotates 90º.
Modulus-argument form	***Operations***	$z_1z_2=r_1r_2\left[\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\right]$; $\frac{z_1}{z_2}=\frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$
	Powers	$\|z_1z_2\|=\|z_1\|\|z_2\|$; $\left\|\frac{z_1}{z_2}\right\|=\frac{\|z_1\|}{\|z_2\|}$; $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$ $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$; $\|z^n\|=\|z\|^n$; $\arg(z^n)=n\arg(z)$
Euler's formula		$re^{i\theta}=r(\cos\theta+i\sin\theta)$, where θ is in radians.
Locus	***Definition***	A locus is a **set of points** that **satisfy given conditions**. Note: always take *z* = *x* + i*y*.
	Circle	The equation $\|z-z_1\|=r$ represents a circle with centre at (x_1, y_1) and radius *r*. The Cartesian equation of the locus is $(x-x_1)^2+(y-y_1)^2=r^2$; Notes: • Check if the circle passes through *O* by comparing the radius with the distance between the centre of the circle and *O*. • Check whether the circle crosses the axes by comparing the radius and the distance.
	Perpendicular bisector	The equation $\|z-z_1\|=\|z-z_2\|$ represents the perpendicular bisector of the line joining the points (x_1, y_1) and (x_2, y_2). The Cartesian equation will be a linear equation.
	Half-line	The equation $\arg(z-z_1)=\theta$ represents a half-line from *A*(x_1, y_1) (excluding *A*) making an angle θ with the positive real axis.
	Others	Assume *z* = *x* + i*y* and then deduce its nature.
De Moivre's Theorem		If $z=r(\cos\theta+i\sin\theta)$, then $z^n=r^n(\cos n\theta+i\sin n\theta)$ for $n\in\mathbb{Q}$ (rational no.)
Nth roots		$z=\sqrt[n]{R}e^{i\left(\frac{\alpha+2k\pi}{n}\right)}, k=0,1,2,3,...,(n-1)$