RAFFLES INSTITUTION
2012 Year 6 Preliminary Examination

# MATHEMATICS

Paper 1

**9740/01**

**12 September 2012**

**3 hours**

Additional materials: Answer Paper
List of Formulae (MF15)

## READ THESE INSTRUCTIONS FIRST

Write your name and CT group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the test, fasten all your work securely together.

This document consists of 5 printed pages.

**1** A study conducted by the Department of Health classified a population of size 21000 into three categories: underweight, normal and overweight. At the start of the study, it was found that $x$, $y$ and $z$ (in thousands) were the number of individuals in the population who were underweight, normal and overweight respectively. At that time, the number of individuals who were overweight was three times the number of individuals who were underweight.

Two years later, the Department found that 10% of the individuals who were underweight and 20% of the individuals who were overweight became normal but 5% of the individuals who were normal became overweight. The weight categories of the rest of the individuals remain unchanged. However, the number of individuals who were overweight was still three times the number of individuals who were underweight.

Assuming that the population remains unchanged during the period of study, find the values of $x$, $y$ and $z$. [3]

**2** Expand $\dfrac{1}{(1+2x^2)^2}$ as a series in ascending powers of $x$, up to and including the term in $x^6$, giving the coefficients in their simplest form. [2]

Find the coefficient of $x^{2r}$ and the range of values of $x$ for the expansion to be valid. [3]

**3** A graphic calculator is **not** to be used in answering this question.

By writing $w = a + \mathrm{i}b$ where $a, b \in \mathbf{R}$, solve the equation $w^2 = 5 + 12\mathrm{i}$.
Hence find the roots of the equation $z^2 - z - (1+3\mathrm{i}) = 0$. [6]

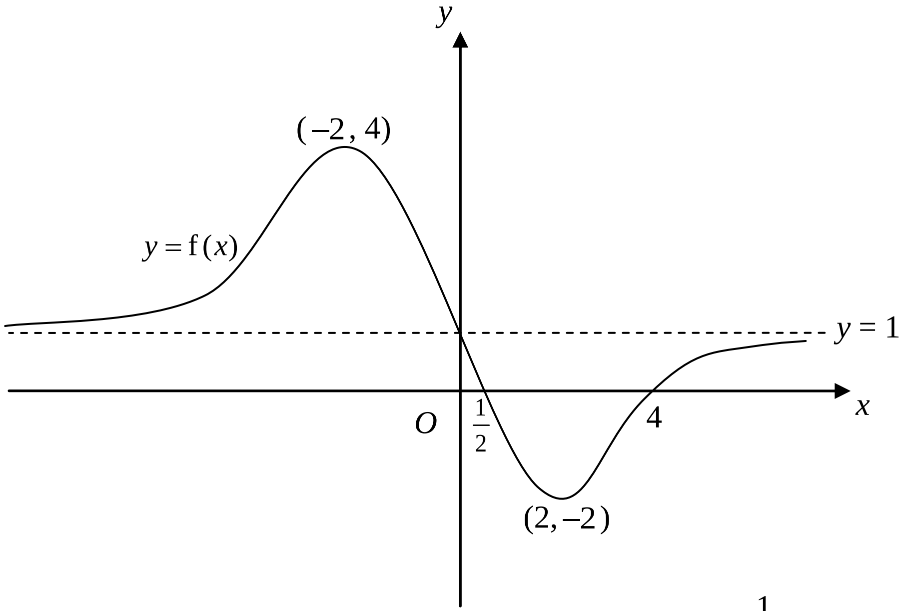
**4** Find how many positive integers between 811 and 2013 are
**(i)** even numbers;
**(ii)** even numbers which are divisible by 7.

Find the sum of the even numbers between 811 and 2013 which are not divisible by 7. [6]

**5(a)** Find $\displaystyle\int \ln\left(2\mathrm{e}^{\frac{1}{\sqrt{1-4x^2}}}\right)\,\mathrm{d}x$ . [2]

**(b)** Using the substitution $u = \sqrt{x}$, find the exact value of $\displaystyle\int_1^4 \frac{1}{1+2\sqrt{x}+x}\,\mathrm{d}x$. [5]

**6**




In the diagram above, the curve $y=\mathrm{f}(x)$ cuts the $x-$axis at $x=\dfrac{1}{2}$ and $x=4$, has turning points at $(-2,4)$ and $(2,-2)$, and has a horizontal asymptote $y=1$.

Sketch, on separate diagrams, the graphs of

**(i)** $y=\mathrm{f}(1-x)$, [3]

**(ii)** $y=\dfrac{1}{\mathrm{f}(x)}$, [3]

stating the equations of any asymptotes and the coordinates of any turning points and points of intersection with the axes.

**7(a)** The position vectors of the points $A$ and $B$ relative to an origin $O$ are **a** and **b** respectively where **a** and **b** are non-zero, non-parallel vectors.
The points $P$ and $Q$ have position vectors $\mathbf{a}+\mathbf{b}$ and $3\mathbf{a}-3\mathbf{b}$ respectively.

Find, in terms of **a** and **b**, the position vector of the point $R$ on $PQ$ such that $3PR=RQ$. What can you say about the points $O$, $A$ and $R$? [3]

**(b)** The planes $p_1$ and $p_2$ , which meet in the line $l$, have equations $x-2y+2z=0$ and $2x-2y+z=0$ respectively.

**(i)** Find an equation of $l$ in Cartesian form. [2]

The plane $p_3$ has equation $\left(x-2y+2z\right)+c\left(2x-2y+z\right)=d$.

**(ii)** Given that $d=0$, show that all 3 planes meet in the line $l$ for any constant $c$. [2]

**(iii)** Given instead that the 3 planes have no point in common, what can be said about the value of $d$? [1]

**8** The curve $C$ has equation $y = \tan^{-1}(x)$ and the line $l$ has equation $y = -2x - 2 - \frac{\pi}{4}$. The region $R$ is bounded by the curve $C$, the line $l$ and the $x$-axis.

**(i)** Verify that the curve $C$ and the line $l$ meet at the point where $x = -1$ and find the exact area of the region $R$. [5]

**(ii)** Write down the equations of the curve $C$ and the line $l$ when each of them is translated by $\frac{\pi}{8}$ units in the negative $x$-direction. [2]

Hence find the volume of solid formed when $R$ is rotated completely about the line $x = \frac{\pi}{8}$, giving your answer correct to 2 decimal places. [2]

**9** By using the substitution $z = ye^{2x}$, find the general solution of the differential equation $\frac{dy}{dx} + 2y = (x+1)e^{-2x}$, expressing your answer in the form $y = f(x)$. [4]

It is given that $y = 1$ when $x = 0$.

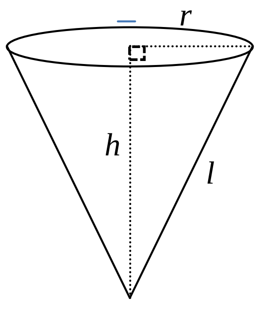
**(i)** Find the particular solution. [1]

**(ii)** By repeated differentiation of the given differential equation, find the Maclaurin expansion for $y$ up to and including the term in $x^3$. [4]

**(iii)** Without carrying out the calculation, describe briefly how you would use the answer in **(i)**, to check the correctness of your answer in **(ii)**. [2]

**10(a)** A paper drinking cup in the shape of a cone (as shown in the diagram below) is to hold 120 $\text{cm}^3$ of water. Use differentiation to find the height $h$ cm and radius $r$ cm of the cup that will require the least amount of paper. [7]

[You do not need to verify that the amount of paper required is the least.]




[Volume of cone $V = \frac{1}{3}\pi r^2 h$; Curved surface area of cone $S = \pi r l$]

**(b)** The curve $C$ has parametric equations $x = 5\cos t,\ y = 3\sin t, 0 \le t \le 2\pi$.

The normal to $C$ at the point $P(5\cos t,\ 3\sin t)$ is denoted by $l$.

**(i)** Find an equation of $l$. [3]

The normal $l$ meets the $x$ and $y$ axes at the points $A$ and $B$ respectively, and $M$ is the mid-point of $AB$.

**(ii)** Find the Cartesian equation of the locus of $M$ as $P$ varies. Sketch this locus. [4]

**11(a)** It is given that

$$\mathrm{h}(x)=\begin{cases} x+1 & \text{for } -1\le x<1, \\ 2 & \text{for } \ 1\le x<3, \\ -x+5 & \text{for } \ 3\le x\le 5. \end{cases}$$

and that $\mathrm{h}(x)=\mathrm{h}(x+6)$ for all real values of $x$.

**(i)** Sketch the graph of $y=\mathrm{h}(x)$ for $-1\le x\le 11$. [2]

**(ii)** It is given that $\int_0^a \mathrm{h}(x)\,\mathrm{d}x=\frac{63}{2}$. Determine the value of $a$. [2]

**(b)** Functions f and g are defined by

$\mathrm{f}: x\to a(x-2)^2+3,\ x\in\mathbb{R}$ where $a$ is a positive constant;

$\mathrm{g}: x\to \dfrac{x}{x-5},\ x\in\mathbb{R},\ x\ne 5.$

**(i)** Sketch the graph of $y=\mathrm{f}(x)$, indicating the coordinates of the stationary point and intersections with the axes if any. [2]

**(ii)** The function k, a restriction of the function f is defined by

$\mathrm{k}: x\to a(x-2)^2+3,\ x\in\mathbf{R}, x\ge 7.$

Given that the composite function gk exists, find the range of values for $a$. [3]

**(iii)** Given that $a=\frac{1}{2}$, solve the inequality $\mathrm{f}(x)>\mathrm{g}(x)$. Hence write down the solution of $\frac{1}{2}(x+2)^2+3>\dfrac{x}{x+5}$. [3]

**12(a)** Given that $z=\left(\sqrt{3}+\mathrm{i}\right)^{\frac{2}{7}}$, find the exact values of $zz^*$ and the smallest positive value of $\arg(z)$. [4]

**(b)** The complex numbers $z$ and $w$ are such that

$$\left|z+\sqrt{3}-\mathrm{i}\right|=2 \quad \text{and} \quad \arg\left(\mathrm{i}w-\frac{\mathrm{i}}{\sqrt{3}}+1\right)=-\frac{\pi}{6}.$$

Sketch on a single Argand diagram the loci of points representing $z$ and $w$. [5]

Hence, find in exact form

**(i)** the least value of $\left|z-\dfrac{1}{\sqrt{3}}-\mathrm{i}\right|$, [2]

**(ii)** the range of values of $\arg\left(z-\dfrac{1}{\sqrt{3}}-\mathrm{i}\right)$. [2]

**END OF PAPER**