RAFFLES INSTITUTION
H3 Mathematics (9820)

Chapter 1: Methods of Proof

SYLLABUS INCLUDES

Knowledge of terms such as ‘Definition’ and ‘Theorem’

Conditional Statements (such as ‘if P then Q’ and ‘P if and only if Q’)

Necessary and sufficient conditions

Existential and universal quantifiers (such as ‘there exists’, ‘for each’ and ‘for all’)

Logical connectives (such as ‘and’, ‘or’, ‘not’, ‘implies’)

Converse, inverse, contrapositive and negation of statements

Set notation and language

Use of direct proof, proof by mathematical induction, disproof by counterexample, proof by contradiction, proof of existence, proof by construction, pigeonhole principle, symmetry principle

CONTENT

1 Introduction

So far, most of the Mathematics formally taught in secondary school has been "computational". The objective of this Chapter is to help us now think about Mathematics a little more carefully or rigorously. We will emphasize the role of "definitions" and the demonstration of "proof". Many of the ideas discussed here will be re-visited in later Chapters.

Anyone who has studied both elementary Euclidean geometry and an experimental science like Physics, should be aware of the very different ways in which the propositions of these two disciplines are established. In the Physical sciences, the propositions or "laws" are accepted because they are confirmed by observation. Mathematics is often used though to "describe" these laws but it is the acceptance of these Mathematical descriptions by empirical observations. In Euclidean geometry, propositions or "theorems" are accepted because they are deduced by means of a logical proof from previously established truths. It was the ancient Greeks who first used this axiomatic method to give geometry a formal structure. This method consists of accepting *without proof* certain propositions, known as axioms. Then all the other statements (called theorems) of the system are derived from the axioms by principles of logic.

The treatment here is brief and is mainly intended to give a "flavor" of things to come. At the end of this Chapter we will focus on "induction" which is an important and useful "method of proof" to establish the truth of statements involving subsets of the positive integers.

1.1 Elements of Logic

1.1.1 Statements

A ***statement***, or a proposition, is a sentence which is either true or false, but not both.

Example 1 The following are examples of statements.

1. Everyone in this room is taller than 160cm.
2. $e^2 \neq 0$
3. There are integers x and y such that $2x+3y=0$.
4. If x and y are any odd integers, then xy is an odd integer.
5. For all real values x, $x^2>0$ and $\sin x \le 1$.

Given a statement p, $\sim p$ (read as "not p") is the negation of the statement p.
For example, consider the following statement, p:"I have a headache."
The statement, "I do not have a headache" is the negation of p.

The negation of statement 2. $q: e^2 \neq 0$ is $\sim q: e^2 = 0$
What is the negation of 3, 4 and 5 ?

Negation of 3 : There are **no** integers x and y such that $2x+3y=0$.
Negation of 4 : There **exist some *x* and *y*** which are odd integers, and xy is an **even** integer.
Negation of 5 : There **exists** a real values x, such that **either** $x^2 \le 0$ **or** $\sin x > 1$.

1.1.2 Quantifiers

Let us introduce the logical quantifiers:

$\forall$ which symbolizes "for all", and $\exists$ which symbolizes "there exists"

Example 2

Forms of quantified statements as typically seen in Mathematical statements.

1. $\forall x \in \mathbb{R}, x^2 \geq 0.$
- For any real number x, $x^2 \geq 0$.

2. $\exists x \in \mathbb{R}, x^2 = 2.$
- There exists a real number such that $x^2 = 2$.

Negation of Statements

Negating statements with quantifiers can be tricky. Before we try negating the 2 statements above in Example 2, let us look at a more concrete example.

Example 3

Now, consider the following statement,
p: The height of everyone in the room is at least 150cm.

For this statement to be true, the height of every single person in the room must be at least 150. For it to be false, it suffices for there are to be at least one person in the room whose height is less than 150cm.

p: $\forall i \in \{\text{Alan, Beth,... ,Zach}\},\ h_i \geq 150$; $\sim p$: $\exists i \in \{\text{Alan, Beth,... ,Zach}\},\ h_i < 150$

In other words p: $\sim(\forall x, p(x)) \equiv (\exists x, \sim p(x))$

Now let us look back at **Example 2** and try to negate the two statements.

1. p : $\forall x \in \mathbb{R}, x^2 \geq 0.$

   $\sim p$:

2. q : $\exists x \in \mathbb{R}, x^2 = 2.$

   $\sim q$:

Write the negation of the following statements as a simple exercise.

1. $\exists x \in \mathbb{R}, -2 < x < 3.$
2. $\forall x \in \mathbb{Z}$, if x is odd, then x^2 is odd.

1.1.3 Conditional Statements

An important connective is the conditional statement $p \Rightarrow q$
The conditional statement p implies q ($p \Rightarrow q$) means that if p is a true statement then q is also a true statement. $p \Rightarrow q$ fails to hold only when p is true and q is false !
If p is false then the statement $p \Rightarrow q$ is said to be ***vacuously*** true.

Example 4

p: $x > 2$
q: $x > 1$

Any number which is greater than 2 is also greater than 1, so $p \Rightarrow q$.

Note:
(i) $p \Rightarrow q$ does not require that p is true or p is false, but only that if p is true then q is true.

Comments

(ii) $p \Rightarrow q$ does not mean that $q \Rightarrow p$ (converse) or $\sim p \Rightarrow \sim q$.
(iii) It does, however follow that $\sim q \Rightarrow \sim p$ (contrapositive).

A statement and its contrapositive are either both true or both false, i.e. $p \Rightarrow q$ and $\sim q \Rightarrow \sim p$ are equivalent statements.

There are several ways in which $p \Rightarrow q$ can be expressed in Mathematics.
We say p is the hypothesis of the conditional statement and q is the conclusion.

(i) "if p then q"
(ii) "p only if q"
(iii) "p is a sufficient condition for q"
(iv) "q is a necessary condition for p"

p is a necessary and sufficient condition for q means p if and only if q (p iff q) or symbolically, $p \Leftrightarrow q$.

Example 5

(a) p: $y = x^2$; q: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2x$

Which of the following statements are correct?

A p is a necessary condition for q.
B p is a necessary and sufficient condition for q.
C ($\sim p$) is a sufficient condition for ($\sim q$).
D ($\sim q$) is a necessary condition for ($\sim p$).
E q is a necessary condition for p.

(b) $p:(x-1)(x-2)>0;\ q:x>2$

Which of the following statements are correct?

A $q \Rightarrow p$

B $\sim p \Rightarrow \sim q$

C q is a sufficient condition for p.

D q is a necessary condition for p.

E p is a necessary condition for q.

(c) Insert the correct conditional symbol between the given statements.

That is, $p \Rightarrow q, p \Leftarrow q$ or $p \Leftrightarrow q$

(i) $p:x^2=4;\ q:x=2$

(ii) $p:\dfrac{x^2}{x-1}\le 0;\ q:x<1$

(iii) $p:\sin\theta=0;\ q:\tan\theta=0$

(iv) $p:\sin\theta=0;\ q:\cos\theta=1$

2 Methods of proof

2.1 Direct proof

This is probably the form of proof you are most familiar with. Many statements can be established directly by assuming the given data and these only.

Definitions play an important role in Mathematics. A direct proof of a proposition is often a demonstration that the proposition follows logically from certain definitions.

Example 6: Prove that if x and y are odd integers, then xy is an odd integer.

Solution: Use the definition of "odd" in the hypothesis "if x and y are odd integers"

$x=2m+1$ and $y=2n+1$ for some $m,n\in\mathbb{Z}$

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn+2m+2n+1 \\ &= 2(2mn+m+n)+1 \end{aligned}$$

Our conclusion holds, by definition.

Exercise 1: Show that for all rational numbers a and b

(i) $a+b$ is rational (ii) ab is rational

Thought Process: What is a rational number? How do we characterize (express) rational numbers? The rest follows easily.

Example 7: Show that for all real values a and b, $\max(a,b)=\dfrac{a+b+|a-b|}{2}$.

Solution: Use the definition of "| |" and "max". We simply check,

for $a \le b$, $LHS = \max(a,b) = b$

$$RHS = \frac{a+b+|a-b|}{2} = \frac{a+b-(a-b)}{2} = b$$

for $a > b$ (left to reader)

Remark: $\min(a,b) = -\max(-a,-b)$

Exercise 2: Show that for all non-negative real values a and b, $\dfrac{a+b}{2} \ge \sqrt{ab}$. When does equality hold ? Deduce also that if $x>0$, then $x+\dfrac{1}{x} \ge 2$.

Thought Process: Can you manipulate the required result into a form that will make the conclusion obvious?

Exercise 3: Prove that, if $a > 0$, the equation $x^3 + ax + b = 0$, has only one real root.

Thought Process:

- We have a cubic equation. What do we know about the roots of a cubic equation? In fact all cubic equations have at least one real root (convince yourself using a graph).
- What are the different 'kinds' of cubic graphs? What is so special about those with only one real root?
- How do we make use of the condition $a > 0$? Is it frivolous?

2.2 Proof by Contradiction *(Reductio Ad Absurdum)*

A contradiction is a statement that is always false. So any statement of the form '*q* and ~*q*' is false. To verify a statement *p* by contradiction, we start with ~*p* and deduce a statement of the form '*q* and ~*q*'. As this is false, we can argue that ~*p* is false, hence *p* is true.

Example 8: Prove that $\log_2 3$ is irrational.

Solution: Assume that $\log_2 3$ is rational, that is, $\log_2 3 = \frac{m}{n}$ where *m* and *n* are integers.

Then $2^{m/n} = 3$, or $2^m = 3^n$. But 2^m is even while 3^n is odd, so we have a contradiction. Thus our assumption that $\log_2 3$ is rational must be wrong, and $\log_2 3$ is irrational.

Exercise 4:

(a) If $p+q$ is irrational then at least one of p, q is irrational.

(b) If pq is irrational then at least one of p, q is irrational.

Exercise 5: Prove that $\sqrt{2}$ is irrational.

Hint: Assume that $\sqrt{2}$ is rational, that is, $\sqrt{2} = \frac{m}{n}$ where *m* and *n* have no common factors.

2.3 Use of Contrapositives

Sometimes, when it is difficult to prove a statement in the form $p \Rightarrow q$, a direct proof of the contrapositive statement, $\sim q \Rightarrow \sim p$ can be used.

Exercise 6:

1. Prove that if n is a natural number such that n^2 is even then n is even.

2. Prove that if n is a perfect number, then n is not a prime number.
   (A perfect number is equal to the sum of its positive divisors excluding itself, e.g. 6 = 3 + 2 + 1.)

2.4 Use of Counter examples

If a statement is suspected of being false, then one single counter example is sufficient to prove this fact.

Exercise 7: Determine if the following statements are ***true or false***, justifying your answer.

1. If $a > b, c > d$ then $ac > bd$.

2. If $f''(x) = 0$ when $x = a$ then the curve $y = f(x)$ has a point of inflexion when $x = a$.

3. If a function f defined on [0,1] attains its maximum at $c \in [0,1]$, then $f'(c) = 0$.

4. If $\lim_{n\to\infty} u_n = 0$, then the series $u_1 + u_2 + ...$ converges.

5. If $u_1, u_2, ...$ is bounded then the sequence $u_1, u_2, ...$ converges.

2.5 Mathematical Induction

Mathematical induction (MI) is a particular method of proof in whose validity is easy to believe and which becomes, with a little practice, a handy tool for the problem solver. That is not to say that all proofs by MI are easy to create, but the structure of MI is so simple that one can concentrate on the mathematical ideas without concern for the logic behind the proof: the logic will be validated by the method itself.

Not all problems succumb to MI; yet, when they do, the results are very satisfying: not only does our problem hold for the first ten or the first ten billion cases, it holds for all of the infinitely many possibilities.

2.5.1 Conjectures

Let us explore several introductory examples to sharpen your sense for patterns. Each example starts with the first few cases of a mathematical "event". Your task is to find some common feature and formulate a conjecture which you believe holds true in all cases.

Conjecture 1 Observe the sum of consecutive odd numbers, starting with 1. Can you guess a formula for this sum which will work in all cases?

Conjecture 2 Which is larger:

(a) the natural numbers n or the powers of 2, 2^n?

(b) the cubes of natural numbers n^3, or the powers of 2, 2^n?

Conjecture 3 If n is any natural number, find a common divisor

(a) for all differences $n^3 - n$

(b) for all sums $2^{n+2} + 7^n$

Looking at the previous conjectures, we can see that each assertion can be put in the form: P_n true for all integers $n \geq n_0$, where P_n is a statement involving the integer n, and n_0 is the "starting point". There are 2 forms of induction, "standard (or weak)" and "total (or strong)".

Conjecture 1:

Conjecture 2:

Conjecture 3:

2.5.2 Weak Induction

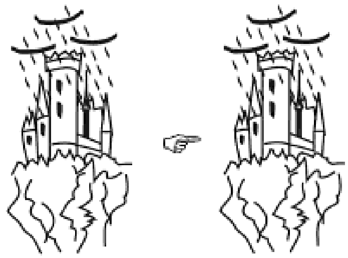
1. Establish the truth of P_{n_0}. This is called the "base case", and it is usually (but not always) an easy exercise.
2. Assume truth of P_k for some arbitrary integer *k*. This is called the inductive hypothesis. Then show that inductive hypothesis implies that P_{k+1} is also true. In other words, show that the conditional statement $P_k \Rightarrow P_{k+1}$, is true.

This is sufficient to prove P_n true for all integers $n \geq n_0$, since if P_{n_0} is true, by **2**. P_{n_0+1} is true which implies again that P_{n_0+2} is true, etc.
In other words we can establish P_n is true after $n-n_0$ implications.

Another way to look at this standard form of induction is the following:

Imagine the following unbelievable situation. Upon being defeated and forced to abandon his castle forever, an evil magician casts a spell: if it rains one day over the castle, then it will rain the next day too. It happens to rain there today. Will it rain forever over the castle?

The answer, of course, is Yes: if it rains today, then the spell will make it rain tomorrow, and again by the spell it will rain the day after tomorrow, and so on and so forth without stopping! To formalize mathematically, we break the desired statement "It will rain forever over the castle" into a sequence of statements P_n for n = 0, 1, 2, ... :



P_0 : It rains today over the castle

P_1 : It rains tomorrow over the castle

P_2 : It rains the day after tomorrow over the castle

$\vdots$

P_n: It rains on the nth day into the future over the castle

The magic spell can be encoded by the implication $P_n \Rightarrow P_{n+1}$ for all $n \geq 0$.

Now, even with the magic spell, until it happens to rain on some particular day, we won't be able to start off our argument. What triggers the infinite sequence of events $\{P_n\}$ is the extra information that "It rains today", i.e. that P_0 is true. Mathematicians refer to P_0 as the basis case or **base case**; the magic spell is the inductive rule or **inductive hypothesis**; and the method justifying that all P_n's are true is called **Mathematical Induction**.

Let us see how we will use it formally to prove **Conjecture 1.**

As we saw, Conjecture 1 breaks up into the statements

$$P_n : 1+3+...+(2n-1) = n^2 \text{ for all } n = 1, 2, 3, ...$$

Basis Step. P_1 : $1 = 1^2$ is obviously true.

Inductive Step. Assume that P_n is true for some $n \geq 1$, i.e. $1+3+...+(2n-1) = n^2$.

We need to prove that P_{n+1} is also true, i.e., $1+3+...+(2n+1) = (n+1)^2$.

How do we do that? Obviously, we need to somehow use P_n.

Finding a relationship between P_{n+1} and P_n is the most important step of the inductive step. If sums are involved, usually P_{n+1} has one or more terms than P_n, and we write $L_{n+1} = L_n +$ these additional terms, where L_n denotes the left hand side of the statement P_n.

In the context of Conjecture 1, we have

$$L_{n+1} = L_n + (2n+1) = n^2 + (2n+1) = (n+1)^2 = R_{n+1}.$$

We just established $L_{n+1} = R_{n+1}$ which means P_{n+1} is true and completes the inductive step. The method of Mathematical Induction now allows us to conclude that all statements P_n are true for all n =1, 2, 3, ...

Now let us prove Conjecture 2 and 3:

Conjecture 2:

Conjecture 3:

Example 9: A sequence is defined by $u_1 = 1$ and $u_{n+1} = \dfrac{n^2}{n+1}u_n$ for $n = 1, 2, 3, \ldots$

Prove that $u_n = \dfrac{(n-1)!}{n}$ by mathematical induction.

Solution: Let P_n be the statement $u_n = \dfrac{(n-1)!}{n}$, where $n \in \mathbb{Z}^+$.

When $n = 1$, LHS = $u_1 = 1$

RHS = $\dfrac{(1-1)!}{1} = 0! = 1$

$\therefore$ LHS = RHS Hence, P_1 is true.

Assume that P_k is true for **some** $k \in \mathbb{Z}^+$, i.e. assume $u_k = \dfrac{(k-1)!}{k}$

To prove that P_{k+1} is true, i.e. to prove $u_{k+1} = \dfrac{(k+1-1)!}{k+1} = \dfrac{k!}{k+1}$.

Now, L.H.S. = u_{k+1} $= \dfrac{k^2}{k+1}u_k$

$= \left(\dfrac{k^2}{k+1}\right)\left(\dfrac{(k-1)!}{k}\right)$

$= \dfrac{k!}{k+1}$ = R.H.S.

Hence, $P_k \Rightarrow P_{k+1}$. Since P_1 is true, by Mathematical Induction, P_n is true for **all** $n \in \mathbb{Z}^+$.

Exercise 8: The terms of the sequence $x_1, x_2, \ldots$ satisfy $x_{n+1} = rx_n + B$ for $n \in \mathbb{Z}^+$ and $|r| \neq 1$, where $x_1 \neq 0$ and $B \neq 0$. Show that $x_n = \dfrac{B(1-r^{n-1})}{1-r} + x_1 r^{n-1}$ for $n \in \mathbb{Z}^+$ and $|r| \neq 1$,

Exercise 9: Prove that $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ for all positive integers n. Deduce the corresponding formula for $(x+y)^n$.

Remark: This result is known as the Binomial Theorem. Here $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ and you will need the identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, which you should prove too.

Exercise 10: Show that an equilateral triangle can be dissected into n equilateral triangles for $n \geq 6$.

2.5.3 Strong Induction

Strong induction gets its name because we use a "stronger" inductive hypothesis.
After establishing the base case, we assume that for some k,

$$P_{n_0},\ P_{n_0+1},\ ...,\ P_{k-1},\ P_k \text{ are all true.}$$

and we use this assumption to prove that P_{k+1} is true. Sometimes, strong induction will work (in a more straightforward manner) when standard induction doesn't.

Interestingly, it can be shown that strong and weak induction are actually equivalent.

Let's look at some examples of strong induction first.

Example 10: Show that any positive integer $n \geq 2$ can be written as a product of primes.

Solution: Base case is $n = 2,$ which is a prime.

Assume the result is true for all $n \leq k$ for some $k \in \mathbb{Z}^+, k \geq 2.$

For $n = k+1,$ we have 2 possibilities:

if $k+1$ is prime, there is nothing to prove;
if $k+1$ is not prime, then $k+1 = mn$ where $m \leq k$ and $n \leq k$
Since $m \leq k$ and $n \leq k,$ both m and n can be written as a product of primes.
Thus $k+1 = mn$ can be written as a product of primes.

By strong induction, any positive integer $n \geq 2$ can be written as a product of primes.

Exercise 11 Show that every integer $n \geq 12$ can be expressed as a sum of fours and fives.

2.6 Method of Infinite Descent

The technique of infinite descent (descent infini) was developed by the great amateur mathematician Pierre de Fermat (1601-1665). Besides using the technique to prove negative results such as the equation $x^4 + y^4 = z^2$ has no nontrivial integer solution, which we will discuss in a later tutorial, he also used the technique to prove positive results.

For instance, he knew that an odd prime p can be expressed as the sum of two integer squares if and only if p is of the form $4k + 1$. To show that a prime of the form $4k+3$ is not a sum of two squares is not hard. In fact, every square gives a remainder of 0 or 1 when divided by 4, thus no matter what possibilities, the sum of two squares cannot be of the form $4k + 3$, which gives a remainder of 3 when divided by 4.

To prove a prime of the form $4k + 1$ is the sum of two squares, he assumed that if there is a prime of the form $4k + 1$ which is not the sum of two squares, then there will be another (smaller) prime of the same nature, and hence a third one, and so on. Eventually he would come to the number 5, which should not be the sum of two squares. But we know $5 = 1^2 + 2^2$ a sum of two squares, a contradiction!

The idea of infinite descent may be described as follows. Mainly it is because a finite subset of natural numbers must have a smallest member. So if A is a subset of the natural numbers **N**, and if we need to prove, for every $a \in A$, the statement P(a) is valid. Suppose by contradiction, the statement is not valid for all $a \in A$, i.e. there exists a non-empty subset of A, denoted by B, and such that P(x) is not true for any $x \in B$. Now because B is non-empty, there exists a smallest element of B, denoted by b and such that P(b) is not valid. Using the given conditions, if we can find a still smaller $c \in A$ ($c < b$), and such that P(c) is not valid, then this will contradict the minimal nature of b. The conclusion is thus that P(a) must be valid for all $a \in A$.

There are variations of this scenario. For instance, suppose there is a positive integer a_1 such that P(a_1) is valid, and from this, if we can find a smaller positive integer a_2 such that P(a_2) is valid, then we can find a still smaller positive integer a_3 such that P(a_3) is valid, and so on. Hence we can find an infinite and decreasing chain of positive integers (infinite descent)

$$a_1 > a_2 > a_3 > \ldots.$$

This is clearly impossible. So the initial hypothesis P(a_1) cannot be valid.

This may sound confusing, but let us look at how we can use the method of Infinite Descent to provide another proof of the irrationality of $\sqrt{2}$ **(Exercise 5)**.

Assume that $\sqrt{2}$ is rational, that is, $\sqrt{2} = \frac{m}{n}$. We do not assume here that m and n have no common factors.

Then, we must have $2n^2 = m^2$. Similar to the proof in Exercise 5, we must have m to be an even integer, and so we let $m = 2m_1$ for some integer m_1. Then substituting into the original equation we have $n^2 = 2m_1^2$. This will lead us to similarly conclude that n is even, and so we let $n = 2n_1$ for some integer n_1. Substituting back into the previous equation we obtain $2n_1^2 = m_1^2$.

At this stage we see that if positive integers *m*, *n* satisfies the equation $2n^2 = m^2$, then we must be able to find positive integers m_1, n_1 satisfying the same equation. In addition, we also have $m_1 = \frac{m}{2} < m$, $n_1 = \frac{n}{2} < n$. Continuing this process we must be able to find positive integers $m > m_1 > m_2 > \ldots$ and $n > n_1 > n_2 > \ldots$

However, this cannot continue forever as the set of positive integers contains a smallest element (which is 1) and thus we arrive at a contradiction.

So the method of descent is essentially another form of induction. Recall that in mathematical induction, we start from a smallest element a of a subset of natural numbers (base case), and prove the so-called inductive step. So we can go from P(a) to P(a + 1), then P(a + 2) and so on.

2.7 Pigeonhole Principle

Think of any four integers. Would you be surprised if I say that among these four integers, there are two whose difference is divisible by 3?

In this section, we shall introduce a fundamental principle in combinatorics, known as the Pigeonhole Principle, from which we will be able to deduce the existence of a certain kind of quantity, pattern or arrangement such as those in the opening assertion.

If 4 pigeons are to be put into 3 compartments (pigeonholes), you will certainly agree that one of the compartments will have at least 2 pigeons in it. A much more general statement of this simple observation, known as the Pigeonhole Principle, is given below:

The Pigeonhole Principle (PP)

Let k and n be any two positive integers. If at least $kn + 1$ objects are distributed among n boxes, then one of the boxes must contain at least $k + 1$ objects.

Proof

If no boxes contain $k + 1$ or more objects, then every box contains at most k objects. This implies that the total number of objects put into the n boxes is at most kn, a contradiction. □

Note: In particular, when $k = 1$, we see that if at least $n + 1$ objects are distributed among n boxes, then one of the boxes must contain at least 2 objects.

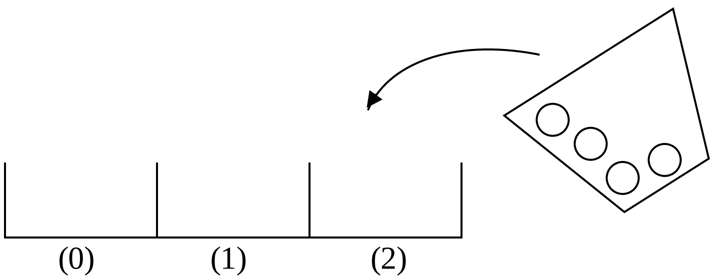
Another form of PP:

If N objects are placed into n boxes, then there is a box containing at least $\left\lceil \frac{N}{n} \right\rceil$ objects.

(PP) has important applications in number theory and is also known as the *Dirichlet Drawer Principle*, after the German Mathematician Peter G. L. Dirichlet (1805 – 1895), who had used it to prove some results in number theory. (PP) may seem almost trivial; however, it is a surprisingly useful and powerful device that proves many "existence" statements in Mathematics.

Example 11
Show that among any 4 integers, there are two whose difference is divisible by 3.

Solution
We treat the 4 integers as 4 objects and we create 3 boxes, namely:
Box (0) for integers which are divisible by 3,
Box (1) for integers which leave a remainder of 1 when divided by 3,
Box (2) for integers which leave a remainder of 2 when divided by 3,



By (PP), there is at least one box with at least two integers. If the two integers are in box (i), where $i = 0$, 1 or 2, then we may express the two integers as $3x + i$ and $3y + i$, where x and y are integers. We may assume $x \geq y$.

Thus the difference between these two integers is $(3x + i) - (3y + i) = 3(x - y)$, which is divisible by 3. □

(PP) looks almost trivial like the other counting principles before it, but just like them, it makes itself most useful by suggesting an approach to tackle a problem. The Pigeonhole Principle suggests that to prove some 'existence' statements in Mathematics, we try to transform the problem partly into one of distributing a number of objects into a number of boxes. The questions to focus on then becomes "*What are the objects*?" and "*What are the boxes*?".

Example 12
There must be at least two women in Singapore with the same number of hairs on their heads. (A typical head of hair has about 110,000 to 150,000 hairs.)

Solution

Clearly there are at least 2 million women in Singapore, so assuming that you don't have more than 1 million hairs on your head (that is stretching it really), by Pigeonhole Principle, there must be at least 2 women in Singapore with the same number of hairs on their heads.

Example 13
A box contains 4 red socks and 6 blue socks. Suppose you can take any number of socks without looking, what is the minimum number of socks you'd have to pull out to guarantee a pair of the same color?

Exercise 12

If there are n (where $n > 1$) people who can shake hands with one another, then there are at least two of them who shake hands with the same number of people.

Exercise 13

Let B = { b_1, b_2, b_3, b_4, b_5 } be a set of 5 distinct positive integers. Show that for **any** permutation $b_{i_1}b_{i_2}b_{i_3}b_{i_4}b_{i_5}$ of B, the product $(b_{i_1} - b_1)(b_{i_2} - b_2)$ ... $(b_{i_5} - b_5)$ is always even.

For example, if B = {26, 7, 1, 22, 11} and a permutation of B is {1, 22, 26, 11, 7}. Then the product (1 – 26)(22 – 7)(26 – 1)(11 – 22)(7 – 11) is even.

Exercise 14

Let n be a positive integer. Show that if you have n integers, then either one of them is a multiple of n or a sum of several of them is a multiple of n.

Exercise 15

Prove that from a set of 10 distinct two-digit numbers, it is possible to select two disjoint subsets whose members have the same sum.

Tutorial

1. Prove that $\sqrt{6}$ is irrational. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.

2. Show that a square can be dissected into n smaller squares for $n \geq 6$.

3. If $p_1 p_2 = 2(q_1 + q_2)$, show that at least one of the equations $x^2 + p_1 x + q_1 = 0$ and $x^2 + p_2 x + q_2 = 0$ has real roots.

4. Which of the following statements are true and which are false? Justify your answers.

(i) $a^{\ln b} = b^{\ln a}$ for all $a, b > 0$.

(ii) $\cos(\sin\theta) = \sin(\cos\theta)$ for all real θ.

(iii) There exists a polynomial $p(\theta)$ such that $\left|p(\theta) - \cos\theta\right| \leq 10^{-6}$ for all real θ.

(iv) $x^4 + 3 + x^{-4} \geq 5$ for all $x > 0$.

5. Prove that

$$\left(1-a_1\right)\left(1-a_2\right)\cdots\left(1-a_n\right) \geq 1 - a_1 - a_2 - \cdots - a_n,$$

where $0 < a_i < 1,\ \ i = 1, 2, \ldots, n$

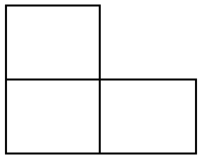
6. Use Mathematical Induction to prove that $9^n - 1$ is a multiple of 8 for all $n \in \mathbb{Z}^+$.

7. The terms of the sequence $x_1, x_2, \ldots$ satisfy $x_{n+1} = 5 - \dfrac{6}{x_n}$ for $n \geq 0$

Prove that if $2 < x_1 < 3$ then the sequence is increasing and bounded above by 3.

8. Show that if f and g are n times differentiable, then $(\mathrm{fg})^{(n)} = \sum_{k=0}^{n} \binom{n}{k} \mathrm{f}^{(k)} \mathrm{g}^{(n-k)}$ for all positive integers n, where $\mathrm{f}^{(n)}$ denotes the nth derivative of f.

9. A square grid consists of $2^n \times 2^n$ smaller squares, where $n \geq 1$. One of the smaller squares is randomly covered. Use Mathematical Induction to show that it is always possible to cover the remaining squares with L-shaped trominoes. (An L-shaped tromino consists of 3 squares, in an L-shape, as shown below)



10. Let n be a positive integer. Choose any $(n + 1)$-element subset of $\{1, 2, ..., 2n\}$. Show that this subset must contain two integers, one of which divides the other. Is the statement still true if instead of $n + 1$ elements, only n elements are chosen?

11. People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the room (Lazy Susan), and this platform can rotate. Each person ordered a different entrée, and it turns out that no one has the correct entrée in front of him. Show that it is possible to rotate the platform so that at least two people will have the correct entrée in front of them.

12. Inside a 1 by 1 square, 101 points are placed. Show that some three of them form a triangle with area no more than 0.01.

13. Show that the decimal expansion of a rational number must eventually become periodic.

14. **[2013/9824/2(modified)]**

Given that $y = e^x \sin x$, prove by induction that

$$\frac{d^n y}{dx^n} = 2^{\frac{1}{2}n} e^x \sin\left(x + \frac{1}{4}n\pi\right)$$

for every positive integer n. Hence find the Maclaurin's series for $e^x \sin x$ up to and including the term in x^4. On a single diagram, sketch the graphs of $y = 4e^x \sin x$ and $y = \frac{d^4}{dx^4}\left(e^x \sin x\right)$.

15. **[013/9824/1]**

Define φ to be the number $\dfrac{1+\sqrt{5}}{2}$ and let $S = \{a\varphi + b \mid a \text{ and } b \text{ are positive integers}\}$.

(i) Express φ^2 as an element of S.

(ii) For any positive integers c and d, express $\varphi(c\varphi + d)$ as an element of S.

(iii) Hence express φ^5 as an element of S.

The function F is defined on the positive integers by F(1) = F(2) = 1 and F(n + 2) = F(n + 1) + F(n) for all positive integers n.

(iv) Write down the values of F(3), F(4) and F(5).

(v) For any positive integer n, $n \geq 2$, use the function F to express φ^n as an element of S.

Assignment 1: Methods of Proof

1. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Using an argument by contradiction, show that abc must be even.

2. Prove that $1 < \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{3n+1} < 2$ for all positive integers n.

3. A triangulation of a polygon is a partition of the polygon into triangles, all whose vertices are vertices of the original polygon. Given such a triangulation, call two vertices adjacent if they are joined by the edge of a triangle. Suppose we decide to colour the vertices of a triangulated polygon. Show that three colours is sufficient to ensure that no two adjacent vertices have the same colour.

4. The integers from 1 to 10 are randomly distributed around a circle. Prove that there must be three consecutive numbers whose sum is at least 17.

(i) Replace the number 17 with 18. Is the result still true?

(ii) Replace the number 17 with 19. Is the result still true?