FINAL EXAMINATION 2023
YEAR THREE EXPRESS ADDITIONAL MATHEMATICS PAPER 2 SOLUTIONS

1 It is given that a and b are the roots of the quadratic equation $x^2-2x-1=0$ and that $a>b$.
Show that $\frac{a}{b}=-3-2\sqrt{2}$. [5]

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-1)}}{2(1)}=\frac{2\pm\sqrt{8}}{2}$$
$$=\frac{2\pm2\sqrt{2}}{2}=1\pm\sqrt{2}$$
$$\therefore a=1+\sqrt{2} \text{ and } b=1-\sqrt{2}$$

$$\frac{a}{b}=\frac{1+\sqrt{2}}{1-\sqrt{2}}\times\frac{1+\sqrt{2}}{1+\sqrt{2}}$$
$$=\frac{1+2\sqrt{2}+2}{1-2}=\frac{3+2\sqrt{2}}{-1}$$
$$=-3-2\sqrt{2}$$

MARKER'S COMMENTS:

Students should infer from the question that exact answers are required. Therefore, they should not be relying on non-exact roots from the calculator.

2 A circle with centre O passes through the points $P(-1,7)$ and $Q(0,8)$.

(i) State the relationship between the perpendicular bisector of PQ and the point O. [1]

(ii) Find the coordinates of O, given that the line $y=2x-2$ passes through the centre of the circle. [5]

(iii) Hence find the equation of the circle. [2]

(i) The perpendicular bisector of the chord PQ passes through O, the centre of the circle.

COMMON MISTAKES:

1) The perpendicular bisector is the radius.

No, the perpendicular bisector is a continuous line.

2) A line drawn through point O is perpendicular to the perpendicular bisector.

No, there are so many ways to draw this line.

(ii) $m_{PQ}=\frac{8-7}{0-(-1)}=1$

$\Rightarrow m_{\perp}=-1$

Midpoint of $PQ=\left(\frac{-1+0}{2},\frac{7+8}{2}\right)$
$=\left(-\frac{1}{2},\frac{15}{2}\right)$

Equation of perpendicular bisector of PQ: $y-\frac{15}{2}=-\left(x-\left(-\frac{1}{2}\right)\right)$

COMMON MISTAKE:

Finding the gradient of the line perpendicular to $y=2x-2$.

$\therefore y = -x + 7$ ------------(1)

$y = 2x - 2$ -------------(2)

Sub (1) into (2): $-x + 7 = 2x - 2$

$\therefore x = 3\,;\ y = 2(3) - 2 = 4$

$\therefore O(3,4)$

MARKER'S COMMENTS:

- Students should remember to write all their coordinates in brackets.
- Some students used an alternative method by equating the distance from P and Q to the centre *O*. This is not recommended as it is highly tedious and likely to lead to careless mistakes.

(iii) Radius of circle $= \sqrt{(3-0)^2 + (4-8)^2} = 5$ units

Equation of circle: $(x-3)^2 + (y-4)^2 = 25$

COMMON MISTAKE:

Using the midpoint of PQ to find the radius. This point is not even on the circle.

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3. The polynomial $f(x)$ is given by $f(x) = 9x^3 - 30x^2 - 23x - 4$

(a) Factorise $f(x)$ completely. [4]

(b) Hence, prove that the equation $9x\sqrt{x} - 23\sqrt{x} = 4 + 30x$ has only one real root. Find the solution. [3]

(a) <u>By Trial and Error</u>

Let $x = 4$: $f(x) = 9(4)^3 - 30(4)^2 - 23(4) - 4 = 0$

$\Rightarrow (x-4)$ is a factor.

	9	−30	−23	−4
4		36	24	4
	9	6	1	0

$f(x) = (x-4)(9x^2 + 6x + 1)$

$= (x-4)(3x+1)^2$

MARKER'S COMMENTS:

Factor theorem working was not clearly shown by many students, who likely just used their calculator to get the relevant roots.

(b) For $f(x) = 0$, $x = 4, -\frac{1}{3}$

$9x\sqrt{x} - 23\sqrt{x} = 4 + 30x \quad \Rightarrow 9\sqrt{x^3} - 30\sqrt{x^2} - 23\sqrt{x} - 4 = 0$

Replace "x" with "$\sqrt{x}$" $\Rightarrow \sqrt{x} = 4$ or $-\frac{1}{3}$ (NA) $\quad \therefore x = 16 \Rightarrow$ one real root

COMMON MISTAKES:

- Instead of squaring $\sqrt{x}$ to get x, many students applied square root again.
- Many students rejected $\sqrt{x} = -\frac{1}{3}$ by claiming that you cannot square root a negative number.
  In this context, the rejection reason should be that a surd cannot be negative (in this case, the surd is a square root; the square root of a non-negative real number is a non-negative real number, while the square root of a negative real number is a complex number). There was also no need to waste time substituting all the roots back into the original equation to check.

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4. (a) Solve the equation $2^{x+1} + 2^{-x} = 3$. [4]

(b) The equation of a curve is $y = 2x^2 - 4ax + b$ where a and b are constants. Explain why $y > 0$ if $b > 2a^2$. [3]

(a) $2^{x+1} + 2^{-x} = 3 \Rightarrow 2^x \times 2 + \frac{1}{2^x} = 3$

Let $a = 2^x$: $2a + \frac{1}{a} = 3$

$2a^2 - 3a + 1 = 0$

$(2a-1)(a-1) = 0$

$\therefore a = \frac{1}{2}$ or 1

$\Rightarrow 2^x = 2^{-1}$ or $2^x = 2^0$

$\therefore x = -1$ or 0

MARKER'S COMMENTS:

- Students need to improve on their indices laws.
- There was no need to use logarithm in this question.
- The factorisation step for solving quadratic equations must be clearly shown.

(b) $b^2 - 4ac = (-4a)^2 - 4(2)(b)$

$= 16a^2 - 8b$

$= 8(2a^2 - b)$

If $8(2a^2 - b) < 0$

$2a^2 - b < 0$

$\Rightarrow b > 2a^2$

$\therefore$ When $b > 2a^2$, $b^2 - 4ac < 0$.

And since $a = 2 > 0$, $\therefore y > 0$ when $b > 2a^2$

COMMON MISTAKES:

- Misconception that $b^2 - 4ac > 0$.
- Forgot to include the condition a > 0.
- Substituting $b = 2a$ before finding discriminant.
- $16a^2 - 8b = 2a^2 - b$.

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5. (a) Find an expression for *x*, in terms of *e*, for which $\ln(3-x)=\ln x + 3$. [3]

(b) Solve the equation $\log_5 x+\log_{25} x=4$. [3]

(a)
$$\ln(3-x)=\ln x + 3$$
$$\ln(3-x)=\ln x + \ln e^3$$
$$\ln(3-x)=\ln\left(xe^3\right)$$
$$3-x=xe^3$$
$$3=xe^3+x$$
$$3=x\left(e^3+1\right)$$
$$\therefore x=\frac{3}{\left(e^3+1\right)}$$

COMMON MISTAKES:

- $\ln(3-x)=\ln 3-\ln x$
- $\ln(3-x)=\dfrac{\ln 3}{\ln x}$
- $\ln x+3=\ln(x+3)$
- $\ln x+\ln e^3=\ln\left(x+e^3\right)$
- Expressing *e* in terms of *x*.

(b)
$$\log_5 x+\log_{25} x=4$$
$$\log_{25} x^2+\log_{25} x=4$$
$$\log_{25} x^3=4$$
$$x^3=25^4$$
$$x=\left(25^4\right)^{\frac{1}{3}}$$
$$\therefore x=73.1$$

MARKER'S COMMENTS:

- When using shortcuts to the change-of-base law, many students get confused and make careless mistakes
- Students are required to simplify their answers to 3s.f. as instructed on the cover page and not leave final answer as $5^{\frac{8}{3}}$.

6. (ai) Prove the trigonometric identity: $\dfrac{\operatorname{cosec}^2 A + 2\cot A}{(\cos A + \sin A)^2} = \operatorname{cosec}^2 A$ [4]

(aii) Hence solve the equation: $\operatorname{cosec}^2 A + 2\cot A = 4(\cos A + \sin A)^2$ for $0 \le A \le 360°$. [4]

(ai) LHS: $\dfrac{\operatorname{cosec}^2 A + 2\cot A}{(\cos A + \sin A)^2}$

$= \dfrac{1 + \cot^2 A + 2\cot A}{(\cos A + \sin A)^2}$

$= \dfrac{(1 + \cot A)^2}{(\cos A + \sin A)^2}$

$= \dfrac{\left(1 + \dfrac{\cos A}{\sin A}\right)^2}{(\cos A + \sin A)^2}$

$= \dfrac{\left(\dfrac{\sin A + \cos A}{\sin A}\right)^2}{(\cos A + \sin A)^2}$

$= \dfrac{(\sin A + \cos A)^2}{(\sin A)^2} \times \dfrac{1}{(\cos A + \sin A)^2}$

$= \dfrac{1}{(\sin A)^2}$

$= \operatorname{cosec}^2 A$ = RHS (proven)

Alternatively,

LHS: $\dfrac{\operatorname{cosec}^2 A + 2\cot A}{(\cos A + \sin A)^2}$

$= \dfrac{\dfrac{1}{\sin^2 A} + \dfrac{2\cos A}{\sin A}}{\cos^2 A + \sin^2 A + 2\cos A \sin A}$

$= \dfrac{\dfrac{1 + 2\cos A \sin A}{\sin^2 A}}{1 + 2\cos A \sin A}$

$= \dfrac{1}{\sin^2 A}$

$= \operatorname{cosec}^2 A$ = RHS (proven)

MARKER'S COMMENTS:

- All working must be clearly shown in a proving question. Quite a few students skipped from: $(\cos A + \sin A)^2 = 1 + 2\sin A \cos A$

COMMON MISTAKE:

- Did not apply identities correctly when expanding: $(\cos A + \sin A)^2 = \cos^2 A + \sin^2 A = 1$ ✗

(aii) $\dfrac{\operatorname{cosec}^2 A + 2\cot A}{(\cos A + \sin A)^2} = 4$

$\operatorname{cosec}^2 A = 4$

$\dfrac{1}{\sin^2 A} = 4$

$\sin^2 A = \dfrac{1}{4}$

$\sin A = \pm\dfrac{1}{2}$

$\alpha = \sin^{-1}\dfrac{1}{2} = 30°$

$\therefore A = 30°, 150°, 210°, 330°$

MARKER'S COMMENTS:

- Students must clearly write the degree symbol as otherwise the angle will be treated as radians.

COMMON MISTAKE:

- Forgot to take $\pm\sqrt{\ }$.

6b. Given that $\dfrac{1+\sin x+2\cos x}{1+2\sin x+\cos x}=1$ and x is acute, find the exact value of $\cos x$. [4]

6b. $\dfrac{1+\sin x+2\cos x}{1+2\sin x+\cos x}=1$

$1+\sin x+2\cos x=1+2\sin x+\cos x$

$\cos x=\sin x$

$\dfrac{\sin x}{\cos x}=1 \quad \Rightarrow \quad \tan x=1$

$\alpha=\tan^{-1}1=45°$

$\therefore x=45°$

$\therefore \cos x=\dfrac{1}{\sqrt{2}}$

COMMON MISTAKE:

- Students did not leave answers in exact form.

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7. (ai) Write down the first four terms in the expansion of $(1-4x)^6$. [2]

(aii) Hence find he coefficient of x^3 in the expansion of $(3+x^2)(1-4x)^6$. [2]

COMMON MISTAKES:

- Missing out all the negative signs.
- Expanding up to the 5th term (forgetting that 1 is the 1st term).

(ai) $(1-4x)^6=1+{}^6C_1(-4x)^1+{}^6C_2(-4x)^2+{}^6C_3(-4x)^3+\dots$

$=1+6(-4x)+15(16x^2)+20(-64x^3)+\dots$

$=1-24x+240x^2-1280x^3+\dots$

(aii) $(3+x^2)(1-4x)^6=(3+x^2)(1-24x+240x^2-1280x^3+\dots)$

$=\dots+(3)(-1280)x^3+(1)(-24)x^3+\dots$

$=\dots-3840x^3-24x^3+\dots$

$=\dots-3864x^3+\dots$

$\therefore$ Coefficient of x^3 is -3864.

MARKER'S COMMENTS:

- Students should avoid full expansion in this question as it leads to several careless mistakes, especially with the negative signs.

7b. Explain why there is no constant term in the expansion of $\left(2x+\dfrac{1}{x^3}\right)^{18}$. [3]

<u>General Term</u>

$$T_{r+1}={}^{18}C_r(2x)^{18-r}\left(\frac{1}{x^3}\right)^r$$

$$={}^{18}C_r(2)^{18-r}(x)^{18-r}(x^{-3})^r$$

$$={}^{18}C_r(2)^{18-r}x^{18-4r}$$

Constant Term: $18-4r=0 \quad \Rightarrow r=\dfrac{9}{2}$.

Since r is a non-integer when the combined powers of x is zero, there is no constant term.

8. A solid was heated and left to cool in a container in a room. The difference between its temperature and the surrounding room temperature at time *t* hours was $T°C$. The table shows some recorded values of *t* and *T*.

***t* (hours)**	5	10	15	22	25
$T\,(°C)$	15.1	8.4	5.2	2.7	1.6

The variables *t* and *T* are related by the equation $T = ae^{bt}$, where *a* and *b* are constants.

(a) On the grid opposite, draw a straight line graph of $\ln T$ against *t*. [3]

(b) Use your graph to estimate the value of *a* and of *b*. [4]

(c) With reference to (a), explain why the value of *T* cannot be zero. [1]

(a) Given $T = ae^{bt}$

t **(hours)**	5	10	15	22	25
$\ln T\left(°C\right)$	2.71	2.13	1.65	0.99	0.47

COMMON MISTAKES:

- Plotted *T* against *t*, mostly resulting in a curve.
- Inverted the axes, plotting ln*t* against *T*.
- Did not start either axis from 0.
- Scale was poorly planned, resulting in too much or too little space.
- Due to the previous point, many lines of best fit could be extended to find the vertical intercept, which will affect part (b).

(b)

Linearizing:

$\ln T = \ln a + \ln e^{bt}$

$\ln T = \ln a + bt \ln e$

$\ln T = bt + \ln a$

$\Rightarrow Y = mX + c$

where gradient $= b$

and $\ln T$-intercept $= \ln a$

From the graph, the $\ln T$-intercept is 3.25 $\Rightarrow \ln a \simeq 3.25$

$\therefore a \simeq e^{3.25} \simeq 25.8$

From the graph, gradient = *b*.

$$\Rightarrow b \simeq \frac{3-0.75}{2.5-23}$$

$$\simeq -0.109$$

MARKER'S COMMENTS:

- Note that any working that uses algebraic calculation instead of **using the graph** to find gradient and y-intercept will receive no credit.
- Students who obtained a positive gradient should keep in mind that a decreasing line has negative gradient.

(c) $T = 0 \Rightarrow \ln T$ is undefined. Hence the value of T cannot be zero.

MARKER'S COMMENTS:

- Students overthought this question, with many answers trying to explain in terms of the context. Do keep in mind that the question asked you to refer to part (a), so there should be a direct explanation with regards to ln*T*.

(8a)