RAFFLES INSTITUTION
H2 Further Mathematics (9649)
2019 Year 6

Chapter 13B: Linear Spaces

SYLLABUS INCLUDES

- Linear spaces and subspaces, and the axioms (restricted to spaces of finite dimension over the field of real numbers only)
- Linear independence
- Basis and dimension (in simple cases)

PRE-REQUISITES

- Vectors, Matrices, Functions

CONTENT

1 Real Vector Spaces (Real Linear Spaces)

Definition 1.1

A **non-empty** set V of vectors equipped with two operations:

vector addition, denoted by $\oplus$ and **scalar multiplication** denoted by $\otimes$,

is called a **vector space over** $\mathbb{R}$ (or **linear space over** $\mathbb{R}$ or **real vector space** or **real linear space)** if the following 10 axioms are satisfied:

[A1] $\mathbf{u} \oplus \mathbf{v} \in V$ for all $\mathbf{u}, \mathbf{v} \in V$. (We say V is **closed under** $\oplus$.)

[A2] $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}, \mathbf{v} \in V$. (We say vector addition, $\oplus$, is **commutative**.)

[A3] There exists a zero element , denoted by **0**, **in** V such that

$$\mathbf{0} \oplus \mathbf{u} = \mathbf{u} \quad \text{and} \quad \mathbf{u} \oplus \mathbf{0} = \mathbf{u} \quad \text{for all } \mathbf{u} \in V.$$

[A4] For each $\mathbf{u} \in V$, there exists an **additive inverse**, denoted by $-\mathbf{u}$, **in** V such that

$$\mathbf{u} \oplus (-\mathbf{u}) = (-\mathbf{u}) \oplus \mathbf{u} = \mathbf{0}.$$

[A5] $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w} \in V$.
(We say vector addition, $\oplus$, is **associative**.)

[M1] $\alpha \otimes \mathbf{u} \in V$ for all $\alpha \in \mathbb{R}$ and $\mathbf{u} \in V$. (We say V is **closed under** $\otimes$.)

[M2] $1 \otimes \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

[M3] $\alpha \otimes (\beta \otimes \mathbf{u}) = (\alpha\beta) \otimes \mathbf{u}$ for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in V$.

[D1] $\alpha \otimes (\mathbf{u} \oplus \mathbf{v}) = (\alpha \otimes \mathbf{u}) \oplus (\alpha \otimes \mathbf{v})$ for all $\alpha \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$. **(distributive)**

[D2] $(\alpha+\beta) \otimes \mathbf{u} = (\alpha \otimes \mathbf{u}) \oplus (\beta \otimes \mathbf{u})$ for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in V$. **(distributive)**

A ***vector space*** V, equipped with ***vector addition*** $\oplus$ and ***scalar multiplication*** $\otimes$, is denoted by $(V, \oplus, \otimes)$.

Remarks :

(a) When we say vector space "**over** $\mathbb{R}$", it means the scalar used in the scalar ~~product~~ multiplication is taken from the set of real numbers. For our syllabus, we will only consider real scalar values. So, we will omit the phrase "over $\mathbb{R}$" from now onwards.

(b) Usually, the symbol $\otimes$ is omitted , i.e. $\alpha \otimes \mathbf{u}$ is simply denoted by $\alpha\mathbf{u}$, if there is no possibility of confusion.

(c) If there is a possibility of confusion, you could denote zero element and additive inverse by other notations such as **e** and **f**. Whatever notation you used for zero element and additive inverse, you should first define them in your solution.

Example 1

$(\mathbb{R}^2,+,\bullet)$ is a vector space over $\mathbb{R}$ where the set $\mathbb{R}^2 = \left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R}\right\}$ and + and • are the **standard operations** on $\mathbb{R}$ defined as follows :

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}, \quad \alpha \bullet \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

Proof

To prove that $(\mathbb{R}^2,+,\bullet)$ is a vector space over $\mathbb{R}$, all we need to do is to check that all the 10 axioms are satisfied.

First of all, note that $\mathbb{R}^2$ is a non-empty set since $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

[A1] $\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \in \mathbb{R}^2$ for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^2$ since $u_i + v_i \in \mathbb{R}$ for $i = 1, 2$

[A2] $\underset{\sim}{u} + \underset{\sim}{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} = \begin{pmatrix} v_1 + u_1 \\ v_2 + u_2 \end{pmatrix} = \underset{\sim}{v} + \underset{\sim}{u}$ for all $\underset{\sim}{u}, \underset{\sim}{v} \in \mathbb{R}^2$

[A3] There exist a zero element $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ such that
$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 + u_1 \\ 0 + u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ for all $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$

[A4] For each $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathbb{R}^2$, there exists an additive inverse $\begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} \in \mathbb{R}^2$ such that
$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} -u_1 \\ -u_2 \end{pmatrix} = \begin{pmatrix} u_1 + (-u_1) \\ u_2 + (-u_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$
where $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the zero element wrt the standard operation on $\mathbb{R}^2$.

[A5] For all $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w} \in \mathbb{R}^2$,
$\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 + w_1 \\ u_2 + v_2 + w_2 \end{pmatrix}$
$(\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w} = \left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} u_1 + w_1 + v_1 \\ u_2 + w_2 + v_2 \end{pmatrix}$

Hence $\underset{\sim}{u} + (\underset{\sim}{v} + \underset{\sim}{w}) = (\underset{\sim}{u} + \underset{\sim}{v}) + \underset{\sim}{w}$ for all $\underset{\sim}{u}, \underset{\sim}{v}$ and $\underset{\sim}{w} \in \mathbb{R}^2$.

Remarks :

[M1] $a \cdot \underset{\sim}{u} = a\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} a u_1 \\ a u_2 \end{pmatrix} \in \mathbb{R}^2$ for all $a \in \mathbb{R}$ and $\underset{\sim}{u} \in \mathbb{R}^2$ since $a u_1 \in \mathbb{R}$ for

[M2]

[M3]

[D1]

[D2]

Since $(\mathbb{R}^2,+,\cdot)$ satisfies all the 10 axioms, $(\mathbb{R}^2,+,\cdot)$ is a vector space over $\mathbb{R}$.

Remarks :

Let n be a positive integer. $(\mathbb{R}^n,+,\cdot)$ is a vector space where the set $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix} : x_1,\ x_2,\ ...,\ x_n \in \mathbb{R} \right\}$

and $+$ and $\cdot$ are the **standard operations** on $\mathbb{R}^n$ are defined as follows :

$$\begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ : \\ . \\ y_n \end{pmatrix} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \\ : \\ . \\ x_n+y_n \end{pmatrix}, \quad \alpha \bullet \begin{pmatrix} x_1 \\ x_2 \\ : \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ : \\ . \\ \alpha x_n \end{pmatrix}, \quad \alpha \in \mathbb{R}.$$

In particular, the set of real numbers, $\mathbb{R}$, is a vector space with respect to $+$ and $\cdot$ where $+$ and $\cdot$ are the usual real number addition and multiplication, and

$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x,\ y, z \in \mathbb{R} \right\}$ is a vector space with respect to the standard operations on $\mathbb{R}^3$.

Example 2

Let V be the set of all positive real numbers. Define $\oplus$ by $\mathbf{u} \oplus \mathbf{v} = \mathbf{uv}$ and $\otimes$ by $\alpha \otimes \mathbf{v} = \mathbf{v}^{\alpha}$,

where $\mathbf{u}, \mathbf{v} \in V$ and $\alpha \in \mathbb{R}$. Prove that V is a vector space over $\mathbb{R}$.

Proof :	

Example 3

(a) Let $V=\{\mathbf{0}\}$. Define $\mathbf{0} \oplus \mathbf{0}=\mathbf{0}$ and $\alpha \otimes \mathbf{0}=\mathbf{0}$ for any $\alpha \in \mathbb{R}$.
Then $(V, \oplus, \otimes)$ forms a vector space called the **zero vector space** (or **zero linear space**).

(b) Let P_n denote the set of all polynomials with real coefficients of degree at most n, i.e. $a_0+a_1x+a_2x^2+\ldots+a_nx^n$ where $a_i \in \mathbb{R}$, with addition $\oplus$ and scalar multiplication $\otimes$ defined as the usual addition and scalar multiplication of polynomials:

$$\left(a_0+a_1x+a_2x^2+\ldots+a_nx^n\right)\oplus\left(b_0+b_1x+b_2x^2+\ldots+b_nx^n\right)$$
$$=\left(a_0+b_0\right)+\left(a_1+b_1\right)x+\left(a_2+b_2\right)x^2+\ldots+\left(a_n+b_n\right)x^n$$
$$\alpha\otimes\left(a_0+a_1x+a_2x^2+\ldots+a_nx^n\right)=\left(\alpha a_0\right)+\left(\alpha a_1\right)x+\left(\alpha a_2\right)x^2+\ldots+\left(\alpha a_n\right)x^n$$

Then $(P_n, \oplus, \otimes)$ forms a vector space.

Proof :

(a) V is non-empty since $\mathbf{0}\in V$.

[A1] $\mathbf{0}\in V, \mathbf{0}\in V \Rightarrow \mathbf{0} \oplus \mathbf{0}=\mathbf{0}\in V$. Hence V is closed under $\oplus$.

[A2] $\mathbf{0} \oplus \mathbf{0}=\mathbf{0}=\mathbf{0} \oplus \mathbf{0}$ for $\mathbf{0} \in V$. The vector addition is commutative.

[A3] $\mathbf{0}$ is the zero element w.r.t. $\oplus$ since

[A4] The additive inverse of $\mathbf{0} \in V$ is $\mathbf{0}$ since

[A5] For $\mathbf{0} \in V$, $\mathbf{0} \oplus (\mathbf{0} \oplus \mathbf{0})=\mathbf{0} \oplus \mathbf{0}=\mathbf{0}$;
$(\mathbf{0} \oplus \mathbf{0}) \oplus \mathbf{0}=\mathbf{0} \oplus \mathbf{0}=\mathbf{0}$.
Hence $\mathbf{0} \oplus (\mathbf{0} \oplus \mathbf{0})=(\mathbf{0} \oplus \mathbf{0}) \oplus \mathbf{0}$ for $\mathbf{0} \in V$.
The vector addition is associative.

[M1] $\alpha\otimes\mathbf{0}=\mathbf{0} \in V$ for all $\alpha\in \mathbb{R}$, $\mathbf{0}\in V$. Hence V is closed under $\otimes$.

[M2] $1\otimes\mathbf{0}=\mathbf{0}$ for $\mathbf{0} \in V$.

[M3] For all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{0}\in V$, $\alpha\otimes(\beta\otimes\mathbf{0})=\alpha\otimes\mathbf{0}=\mathbf{0}$; $(\alpha\beta)\otimes\mathbf{0}=\mathbf{0}$
Hence $\alpha\otimes(\beta\otimes\mathbf{0})=(\alpha\beta)\otimes\mathbf{0}$ for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{0}\in V$.

[D1] For all $\alpha \in \mathbb{R}$ and $\mathbf{0} \in V$,
$a \otimes (0 \oplus 0) = \alpha \otimes 0 = 0$

Hence

[D2] For all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{0} \in V$,

Hence

$(\{\mathbf{0}\}, \oplus, \otimes)$ forms a vector space with respect to the operations $\oplus$ and $\otimes$.

(b) P_n is non-empty since $0+0x+0x^2+\ldots+0x^n=0\in P_n$.

Let $\sum_{i=0}^{n}a_ix^i, \sum_{i=0}^{n}b_ix^i, \sum_{i=0}^{n}c_ix^i \in P_n$

[A1] $\left(\sum_{i=0}^{n}a_ix^i\right)\oplus\left(\sum_{i=0}^{n}b_ix^i\right)=\sum_{i=0}^{n}(a_i+b_i)x^i\in P_n$, since $a_i+b_i\in\mathbb{R}$ for all $i\in\{0,1,2,\ldots,n\}$

Hence P_n is closed under $\oplus$.

[A2] $\left(\sum_{i=0}^{n}a_ix^i\right)\oplus\left(\sum_{i=0}^{n}b_ix^i\right)$

$= \sum_{i=0}^{n}(a_i+b_i)x^i$

$= \sum(b_i+a_i)x^i = \left(\sum ax\right)\oplus\left(\sum bx\right)$

[A3] 0 is the zero element w.r.t. $\oplus$ since

$0\oplus\left(\sum_{i=0}^{n}a_ix^i\right)=\left(\sum 0x^i\right)\oplus\left(\sum a_ix^i\right)=\sum(0+a_i)x^i$

$=\sum_{i=0}^{n}a_ix^i$

[A4] The additive inverse of $\sum_{i=0}^{n}a_ix^i$ is $\sum_{i=0}^{n}(-a_i)x^i\in P_n$ since

$\left(\sum_{i=0}^{n}a_ix^i\right)\oplus\left(\sum_{i=0}^{n}(-a_i)x^i\right)=\sum_{i=0}^{n}(a_i-a_i)x^i=\sum_{i=0}^{n}0x^i$

$=0$

$=\sum_{i=0}^{n}(-a_i+a_i)x^i=\left(\sum_{i=0}^{n}(-a_i)x^i\right)\oplus\left(\sum_{i=0}^{n}a_ix^i\right)$

[A5] $\sum_{i=0}^{n}a_ix^i\oplus\left(\sum_{i=0}^{n}b_ix^i\oplus\sum_{i=0}^{n}c_ix^i\right)$

$= \sum(a_i+b_i+c_i)x^i$

$= \sum(a_i+b_i)x^i \oplus \sum c_ix^i$

$= \left(\sum a_ix^i\oplus\sum b_ix^i\right)\oplus\sum c_ix^i$

[M1] $\alpha \otimes \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (\alpha a_i) x^i \in P_n$ for all $\alpha \in \mathbb{R}$ since $\alpha a_i \in \mathbb{R}$ for all $i = 0,1,2,\ldots,n$

Hence P_n is closed under $\otimes$.

[M2] $1 \otimes \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} (1 \cdot a_i) x^i = \sum_{i=0}^{n} a_i x^i$

[M3] $\alpha \otimes \left(\beta \otimes \sum_{i=0}^{n} a_i x^i \right) = \alpha \otimes \sum_{i=0}^{n} (\beta a_i) x^i = \sum_{i=0}^{n} (\alpha \beta a_i) x^i = (\alpha\beta) \otimes \sum_{i=0}^{n} a_i x^i$

[D1]

$$\begin{aligned}
\alpha \otimes \left(\sum_{i=0}^{n} a_i x^i \oplus \sum_{i=0}^{n} b_i x^i \right) &= \alpha \otimes \sum_{i=0}^{n} (a_i + b_i) x^i \\
&= \sum_{i=0}^{n} \alpha (a_i + b_i) x^i = \sum_{i=0}^{n} (\alpha a_i + \alpha b_i) x^i \\
&= \sum_{i=0}^{n} (\alpha a_i) x^i \oplus \sum_{i=0}^{n} (\alpha b)_i x^i = \left(\alpha \otimes \sum_{i=0}^{n} a_i x^i \right) \oplus \left(\alpha \otimes \sum_{i=0}^{n} b_i x^i \right)
\end{aligned}$$

[D2]

$$\begin{aligned}
(\alpha + \beta) \otimes \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} ((\alpha + \beta) a_i) x^i &= \sum_{i=0}^{n} (\alpha a_i + \beta a_i) x^i \\
&= \left(\sum_{i=0}^{n} \alpha a_i x^i \right) \oplus \left(\sum_{i=0}^{n} \beta a_i x^i \right) \\
&= \left(\alpha \otimes \sum_{i=0}^{n} a_i x^i \right) \oplus \left(\beta \otimes \sum_{i=0}^{n} a_i x^i \right)
\end{aligned}$$

$(P_n\ ,\ \oplus\ ,\ \otimes)$ forms a vector space.

Not all sets with given operations form a vector space, as some of the 10 axioms may not satisfy.

Example 4

Determine whether each of the following sets with the given operations forms a vector space over $\mathbb{R}$.

(a) The set of integers $\mathbb{Z}$ with the standard integer addition and real number multiplication.

(b) The set of all real numbers with the operations : $u \oplus v = u - v + 1$, $\alpha \otimes u = \alpha u$

(c) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x + y + z = 1 \right\}$ with the standard operations on $\mathbb{R}^3$.

(d) The set $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 : a > 0 \right\}$ under the standard operations on $\mathbb{R}^2$.

Solution :

Notes :

1) To show V is **not closed** under $\oplus$, find an example of a $\mathbf{u} \in V$ and a $\mathbf{v} \in V$ such that $\mathbf{u} \oplus \mathbf{v} \notin V$.

2) To show $\oplus$ is **not commutative**, find an example of a $\mathbf{u} \in V$ and a $\mathbf{v} \in V$ such that $\mathbf{u} \oplus \mathbf{v} \neq \mathbf{v} \oplus \mathbf{u}$.

3) To show **[A3]** fails, either show that there is **no** zero element $\mathbf{0}$ such that $\mathbf{0} \oplus \mathbf{u} = \mathbf{u}$ and $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$;

or show that there is a zero element, $\mathbf{0}$, with respect to $\oplus$, but $\mathbf{0} \notin V$.

4) To show **[A4]** fails, either find an example of a $\mathbf{u} \in V$ such that its additive inverse **does not exist** or find an example of a $\mathbf{u} \in V$ such that its additive inverse exists but $\notin V$.

5) To show $\oplus$ is **not associative**, find an example of a $\mathbf{u} \in V$, a $\mathbf{v} \in V$ and a $\mathbf{w} \in V$ such that $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) \neq (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$.

6) To show V is **not closed** under $\otimes$, find an example of a $\mathbf{u} \in V$ and a $\alpha \in \mathbb{R}$ such that $\alpha \otimes \mathbf{u} \notin V$.

7) To show **[M2]** fails, find an example of a $\mathbf{u} \in V$ such that $1.\mathbf{u} \neq \mathbf{u}$.

8) To show **[M3]** fails, find an example of a $\mathbf{u} \in V$, a $\alpha \in \mathbb{R}$ and a $\beta \in \mathbb{R}$ such that

$$\alpha \otimes (\beta \otimes \mathbf{u}) \neq (\alpha\beta) \otimes \mathbf{u}.$$

9) To show **[D1]** fails, find an example of a $\mathbf{u} \in V$, a $\mathbf{v} \in V$ and a $\alpha \in \mathbb{R}$ such that $\alpha \otimes (\mathbf{u} \oplus \mathbf{v}) \neq (\alpha \otimes \mathbf{u}) \oplus (\alpha \otimes \mathbf{v})$.

10) To show **[D2]** fails, find an example of a $\mathbf{u} \in V$, a $\alpha \in \mathbb{R}$ and a $\beta \in \mathbb{R}$ such that

$$(\alpha+\beta) \otimes \mathbf{u} \neq (\alpha \otimes \mathbf{u}) \oplus (\beta \otimes \mathbf{u}).$$

Useful Result

Let V be a vector space and let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be vectors in V. If $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \oplus \mathbf{w}$ or $\mathbf{v} \oplus \mathbf{u} = \mathbf{w} \oplus \mathbf{u}$, then $\mathbf{v} = \mathbf{w}$.

Proof:

2 Vector Subspaces

Definition 2.1

Let $(V, \oplus, \otimes)$ be a real vector space and U be a **non-empty** subset of V.

U is called a **vector subspace** (or simply called a **subspace**) of V if U itself is also a vector space over $\mathbb{R}$ under the same operations $\oplus$ and $\otimes$.

For example, let us consider $V = \mathbb{R}^2$ and $U = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$. From Examples 1 and 3(a), we have shown that $(V,+,\cdot)$ and $(U,+,\cdot)$ are vector spaces under the same standard operations. Clearly, $U \subset V$.

So, we can say that $\left(\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},+,\cdot \right)$ is a subspace of $(\mathbb{R}^2,+,\cdot)$.

To show that U is a subspace, we must show that U satisfies all the 10 axioms of a vector space. However, since U is part of a larger vector space V, only the closure for vector addition and multiplication in U need to be checked since the rest of the properties are inherited from V.

Theorem 2.2 (Subspace Criteria)

Given a **non-empty subset** U of a real vector space $(V, \oplus, \otimes)$, There's a zero

if $\mathbf{u} \oplus \mathbf{v} \in U$ for all $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \otimes \mathbf{u} \in U$ for all $\alpha \in \mathbb{R}$ and $\mathbf{u} \in U$,

then U is a **subspace** of V with respect to the same operations $\oplus$ and $\otimes$.

Proof

If $\mathbf{u} \oplus \mathbf{v} \in U$ for all $\mathbf{u}, \mathbf{v} \in U$ and $\alpha \otimes \mathbf{u} \in U$ for all $\alpha \in \mathbb{R}$ and $\mathbf{u} \in U$,

then the axioms **[A1]** and **[M1]** hold. Hence we need only show that U satisfies the remaining 8 axioms.

Axioms **[A2]**, **[A5]**, **[M2]**, **[M3]**, **[D1]**, **[D2]** are automatically satisfied by the vectors in U since they are satisfied by all vectors in V. Therefore, to complete the proof, we need only verify that axioms **[A3]** and **[A4]** are satisfied by vectors in U. See tutorial 13b Qn 4

Notes :

- If a non-empty subset U of a real vector space $(V, \oplus, \otimes)$ does not satisfy any one of the 10 axioms as stated in **Definition 1.1**, then U is not a vector space with respect to the same operations $\oplus$ and $\otimes$, and so U is **NOT** a subspace of V.

Remarks

(a) Every vector space V has at least 2 subspaces, namely V itself and the zero vector space $\{\mathbf{0}\}$.

(b) Every subspace of V must contain the zero element $\mathbf{0}$.

Example 5

For each of the following subsets of $\mathbb{R}^3$, determine whether or not it is a subspace with respect to the standard operations on $\mathbb{R}^3$. Give reasons for your answers.

(i) $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = 1 \right\}$; **(ii)** $T = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x = y = z \right\}$;

(iii) $U = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 7x = y \right\}$; **(iv)** $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : x^2 + y^2 = 1 \right\}$;

Describe geometrically the sets T and U.

NOTE :

For $\mathbb{R}^n$, if nothing is mentioned about the operations + and $\cdot$, we shall take the operations to be the **standard** operations defined on $\mathbb{R}^n$.

Solution :

(i) The zero element in $\mathbb{R}^3$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, but $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \notin S$ as the x-component is 0, and not 1. Since S does not contain the zero element, S with the standard operations is not a subspace.

(ii) T is non-empty as $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \in T$ since $0 = 0 = 0$.

Let $\begin{pmatrix} x \\ x \\ x \end{pmatrix}, \begin{pmatrix} y \\ y \\ y \end{pmatrix} \in T$. Then,

$$\begin{pmatrix} x \\ x \\ x \end{pmatrix} + \begin{pmatrix} y \\ y \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \\ x+y \end{pmatrix} \in T \quad \textbf{(Closed under vector addition)}$$

$$\alpha \begin{pmatrix} x \\ x \\ x \end{pmatrix} = \begin{pmatrix} \alpha x \\ \alpha x \\ \alpha x \end{pmatrix} \in T \text{ for any } \alpha \in \mathbb{R} \quad \textbf{(Closed under scalar multiplication)}$$

Hence T is a subspace of $\mathbb{R}^3$.

T is a line which passes through the origin with direction vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(iii) U is non empty as $\begin{pmatrix}0\\0\\0\end{pmatrix} \in U$ since $7(0)=0$

Let $\begin{pmatrix}x\\7x\\z\end{pmatrix}$, $\begin{pmatrix}a\\7a\\b\end{pmatrix} \in U$

$$\begin{pmatrix}x\\7x\\z\end{pmatrix} + \begin{pmatrix}a\\7a\\b\end{pmatrix} = \begin{pmatrix}x+a\\7(x+a)\\z+b\end{pmatrix} \in U$$

$$d\begin{pmatrix}x\\7x\\z\end{pmatrix} = \begin{pmatrix}dx\\7dx\\dz\end{pmatrix} \in U \text{ for any } d \in \mathbb{R}$$

Hence U is a subspace of $\mathbb{R}^3$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\7x\\z\end{pmatrix} = x\begin{pmatrix}1\\7\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$$

U is a plane that passes thru origin with direction vectors $\begin{pmatrix}1\\7\\0\end{pmatrix}$ and $\begin{pmatrix}0\\0\\1\end{pmatrix}$

(iv) ~~V is non empty as $\begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{pmatrix} \in V$ since $\frac{1}{2}+\frac{1}{2}=1$~~

The zero element in $\mathbb{R}^3$ is $\begin{pmatrix}0\\0\\0\end{pmatrix}$ but $\begin{pmatrix}0\\0\\0\end{pmatrix} \notin V$ as $0+0 \neq 1$

3 Linear Span

3.1 Linear Combination

Let $(V, \oplus, \otimes)$ be a vector space and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n elements in V.

Then for any **real** scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, the vector

$$(\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \ \ldots \ \oplus (\alpha_n \otimes \mathbf{v}_n)$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

For example, $\begin{pmatrix}3\\3\\4\end{pmatrix}$ can be written as a linear combination of $\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\0\\5\end{pmatrix}$ as

$$\begin{pmatrix}3\\3\\4\end{pmatrix} = 2\begin{pmatrix}2\\1\\3\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} + 0\begin{pmatrix}3\\0\\5\end{pmatrix}.$$

Example 6

Let $\mathbf{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\mathbf{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ be vectors in $\mathbb{R}^3$. Determine whether each of the following vectors can be written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$.

(a) $\mathbf{u} = \begin{pmatrix}2\\3\\0\end{pmatrix}$, **(b)** $\mathbf{v} = \begin{pmatrix}1\\2\\1\end{pmatrix}$.

(a) By observation, $\underset{\sim}{u} = \begin{pmatrix}2\\3\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3\begin{pmatrix}0\\1\\0\end{pmatrix} = 2e_1 + 3e_2$

~~Thus, u can be written as a lin~~

Thus, **u** can be written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$.

Proof by contradictn

(b) Suppose **v** can be written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$, that is, there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2$.

Then, $\begin{pmatrix}1\\2\\1\end{pmatrix} = \alpha_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\\0\end{pmatrix}$

As the z-component is not consistent, $\underset{\sim}{v}$ cannot be written as a linear combinatn of e_1 and e_2.

Example 7

Determine whether each of the following vectors can be written as a linear combination of $\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\0\\5\end{pmatrix}$.

(a) $\begin{pmatrix}0\\0\\0\end{pmatrix}$, **(b)** $\begin{pmatrix}1\\2\\4\end{pmatrix}$.

Solution :

(a) Consider $\begin{pmatrix}0\\0\\0\end{pmatrix}=a\begin{pmatrix}2\\1\\3\end{pmatrix}+b\begin{pmatrix}1\\-1\\2\end{pmatrix}+c\begin{pmatrix}3\\0\\5\end{pmatrix}$, where $a,b,c\in\mathbb{R}$.

From GC, we get $a=-t, b=-t, z=t$ where $t\in\mathbb{R}$

In particular, $\begin{pmatrix}0\\0\\0\end{pmatrix}=-\begin{pmatrix}2\\1\\3\end{pmatrix}-\begin{pmatrix}1\\-1\\2\end{pmatrix}+\begin{pmatrix}3\\0\\5\end{pmatrix}$

So, $\begin{pmatrix}0\\0\\0\end{pmatrix}$ can be written as a linear combinatn of $\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\0\\5\end{pmatrix}$.

(b) Consider $\begin{pmatrix}1\\2\\4\end{pmatrix}=a\begin{pmatrix}2\\1\\3\end{pmatrix}+b\begin{pmatrix}1\\-1\\2\end{pmatrix}+c\begin{pmatrix}3\\0\\5\end{pmatrix}$, where $a,b,c\in\mathbb{R}$.

From GC, there is no soln to this eqn.

So $\begin{pmatrix}1\\2\\4\end{pmatrix}$ cannot be written as a linear combinatn of $\begin{pmatrix}2\\1\\3\end{pmatrix}$, $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}3\\0\\5\end{pmatrix}$

3.2 Linear Span of a Finite Set of Vectors

Let $(V, \oplus, \otimes)$ be a vector space over $\mathbb{R}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n elements in V.

The **linear span of** $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, denoted by $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, is the set of **all** elements of the form $(\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \cdots \oplus (\alpha_n \otimes \mathbf{v}_n)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$.

(basically, it means all the possible linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$)

That is, $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{(\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \cdots \oplus (\alpha_n \otimes \mathbf{v}_n) : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}\}$

Note : Since $(V, \oplus, \otimes)$ is a vector space over $\mathbb{R}$, all elements of the form

$$(\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \ldots \oplus (\alpha_n \otimes \mathbf{v}_n) \in V \text{ if } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \in V.$$

Example 8

Find the linear span of **(a)** $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$, **(b)** $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\0\\-2\end{pmatrix}\right\}$ in $\mathbb{R}^3$, **(c)** $\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\}$ in $\mathbb{R}^3$.

For each case, describe the locus.

Solution :

(a) $\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\right\} = \left\{\alpha\begin{pmatrix}1\\0\\1\end{pmatrix} : \alpha \in \mathbb{R}\right\} = \left\{\begin{pmatrix}\alpha\\0\\\alpha\end{pmatrix} : \alpha \in \mathbb{R}\right\}$

The locus is a line which passes through the origin with direction vector $\begin{pmatrix}1\\0\\1\end{pmatrix}$.

(b) $\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}-2\\0\\-2\end{pmatrix}\right\} = \left\{\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}-2\\0\\-2\end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\} = \left\{\begin{pmatrix}\alpha_1-2\alpha_2\\0\\\alpha_1-2\alpha_2\end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\}$

Notice that $\begin{pmatrix}-2\\0\\-2\end{pmatrix} = -2\begin{pmatrix}1\\0\\1\end{pmatrix}$, and so we can simplify the linear combination as

$\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}-2\\0\\-2\end{pmatrix} = \alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix} - 2\alpha_2\begin{pmatrix}1\\0\\1\end{pmatrix} = (\alpha_1 - 2\alpha_2)\begin{pmatrix}1\\0\\1\end{pmatrix} = \beta\begin{pmatrix}1\\0\\1\end{pmatrix}$, where $\beta = \alpha_1 - 2\alpha_2 \in \mathbb{R}$

(c) $\text{span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}\right\} = \left\{\alpha_1\begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\\2\end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\} = \left\{\begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_1+2\alpha_2\end{pmatrix} : \alpha_1, \alpha_2 \in \mathbb{R}\right\}$

The locus is a plane which passes thru the origin with directions vectors $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\2\end{pmatrix}$

Theorem 3.1

Let $(V, \oplus, \otimes)$ be a vector space over $\mathbb{R}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a set of n elements in V. Then the linear span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subspace of V over $\mathbb{R}$ with respect to the same operations $\oplus$ and $\otimes$.

Definition 3.2

Let $(V, \oplus, \otimes)$ be a vector space over $\mathbb{R}$.

Suppose there exists a finite set of elements $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ such that linear span of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = V$.

(In other words, every element in V can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.)

Then we say that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **spans** V

OR V **is spanned by** $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

OR $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **finite spanning set** for V.

Example 9

(a) $\mathbb{R}^2 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R}\right\}$, the set of all real 2-dimensional vectors, with respect to the standard operations, is spanned by $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$ since any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbb{R}^2$ can be written as $x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ where $x, y \in \mathbb{R}$

OR $\text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\} = \left\{\alpha\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\} = \left\{\begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}\right\} = \mathbb{R}^2$

(b) The vector space $\mathbb{R}^n$, with respect to the standard operations, is equal to the linear span of

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}\right\}$$

This is because any vector in $\mathbb{R}^n$, say $\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$, can be written as

$$u_1\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + u_2\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + u_3\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ldots + u_{n-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix} + u_n\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

where $u_i \in \mathbb{R}$, $i = 1, 2, \ldots n$.

In particular, $\mathbb{R}$ = linear span of $\{1\}$;

$$\mathbb{R}^3 = \text{linear span of } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\};$$

$$\mathbb{R}^4 = \text{linear span of } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

3.3 Method to Check Whether a Real Vector Space V is Spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$

Let $(V, \oplus, \otimes)$ be a real vector space and $\mathbf{u}$ **be any** element in V.

Suppose $\mathbf{u} = (\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \ldots \oplus (\alpha_n \otimes \mathbf{v}_n)$.

If there is a real solution for every α_i, then V is spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

If there is no real solution for some of the α_i's, or there is a restriction on $\mathbf{u}$, then V is **not** spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.

Example 10

Determine whether $\mathbb{R}^3$ is spanned by

(a) $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$; (b) $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Solution :

(a) Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for some $\alpha, \beta, \gamma \in \mathbb{R}$.

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0 \end{pmatrix} = 2 \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2 - 2 = 0$$

$\therefore$ Not unique

Augmented Matrix is $\left(\begin{array}{ccc|c} 2 & 1 & 1 & x \\ 1 & 0 & 1 & y \\ 2 & 2 & 0 & z \end{array}\right)$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & x \\ 1 & 0 & 1 & y \\ 2 & 2 & 0 & z \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & y \\ 2 & 1 & 1 & x \\ 2 & 2 & 0 & z \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & y \\ 0 & 1 & -1 & x-2y \\ 0 & 2 & -2 & z-2y \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & y \\ 0 & 1 & -1 & x-2y \\ 0 & 0 & 0 & z-2x+2y \end{array}\right)$$

The system is consistent if $z - 2x + 2y = 0$

ie. there will be solutions for α, β & γ provided $z - 2x + 2y = 0$ (only of the form $\begin{pmatrix} x \\ y \\ 2x-2y \end{pmatrix}$)

So any other form will not have soln for α, β, γ.

Thus $\mathbb{R}^3$ is not spanned by $\underset{\sim}{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$, $\underset{\sim}{w} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and $\underset{\sim}{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

(b) Let $\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3$ such that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}0\\1\\1\end{pmatrix}$ for some $\alpha,\beta,\gamma\in\mathbb{R}$.

$$\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

using GC, $\det\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}=-2$

Thus the system has a unique soln. $\Rightarrow \left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ spans V

$$\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}=\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}^{-1}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&-\frac{1}{2}&\frac{1}{2}\\-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\frac{x+y-z}{2}\\\frac{x+y-z}{2}\\\frac{x+y-z}{2}\end{pmatrix}$$

Hence $\begin{pmatrix}x\\y\\z\end{pmatrix}=\frac{x+y-z}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}+\frac{x+y-z}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}+\frac{x+y-z}{2}\begin{pmatrix}0\\1\\1\end{pmatrix}$

Hence $\mathbb{R}^3$ is spanned by $\underset{\sim}{v},\underset{\sim}{w},\underset{\sim}{u}$

Note :

Some vector spaces have a finite spanning set. Likewise, some vector spaces have an **infinite spanning set. For the current syllabus, we will only look at finite spanning sets.**

4 Linear Dependence and Independence

4.1 Linear Dependence and Independence

Let $(V, \oplus, \otimes)$ be a real vector space and $\mathbf{0}$ be the zero element in V.

The set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a **linearly dependent** set of elements if at least one of the vectors in S can be written as a linear combination of the other vectors in S.
If no vector in S can be written as a linear combination of the other vectors in S, then S is called a **linearly independent set**

If $S = \{\mathbf{v}_1\}$ (that is S is a singleton), then S is linearly independent if $\mathbf{v}_1 \neq \mathbf{0}$ and linearly dependent if $\mathbf{v}_1 = \mathbf{0}$.

For example, this set $\left\{ \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \\ -1 \end{pmatrix} \right\}$ is linearly dependent since

The set $S = \left\{ \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent, since it is clear that $\begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Remark

(a) If a set S has exactly two nonzero vectors, then S is linearly independent if and only if any one of the vectors is not a scalar multiple of the other vector.

(b) Consider the set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right\}$ in $\mathbb{R}^3$ and we want to check if the set is linearly independent or not. Then we need to check that whether there are solutions to the following equations:

Theorem

(a) $\{v_1, v_2,\ldots, v_n\}$ is a **linearly dependent** set of elements if and only if there exists real scalars $\alpha_1, \alpha_2,\ldots, \alpha_n$, **not all of them are zero**, such that

$$(\alpha_1 \otimes v_1) \oplus (\alpha_2 \otimes v_2) \oplus \ldots \oplus (\alpha_n \otimes v_n) = \mathbf{0}.$$

(b) $\{v_1, v_2,\ldots, v_n\}$ is a **linearly independent** set of elements whenever there exists real scalars $\alpha_1, \alpha_2,\ldots, \alpha_n$ such that $(\alpha_1 \otimes v_1) \oplus (\alpha_2 \otimes v_2) \oplus \ldots \oplus (\alpha_n \otimes v_n) = \mathbf{0}$, then $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$.

Proof

Suppose that $\{v_1, v_2 \ldots v_n\}$ is linearly dependent set. ~~The~~

4.2 Method to Check If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is Linearly Independent or Dependent in V

Let $(\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \ldots \oplus (\alpha_n \otimes \mathbf{v}_n) = \mathbf{0}$.

If upon solving for $\alpha_1, \alpha_2, \ldots, \alpha_n$ and **ALL** $\alpha_1, \alpha_2, \ldots, \alpha_n$ are zeros **ONLY IS THE ONLY SOLUTION**, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly independent** in V.

If some of the α_i's are **not zeros**, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly dependent in** V.

Example 11
Which of the following sets are **linearly dependent** in the respective vector spaces ?

(a) $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ in $\mathbb{R}^3$;

(b) $\left\{\begin{pmatrix}1\\0\\1\\2\end{pmatrix}, \begin{pmatrix}2\\4\\2\\10\end{pmatrix}, \begin{pmatrix}0\\2\\0\\3\end{pmatrix}\right\}$ in $\mathbb{R}^4$;

(c) $\left\{\begin{pmatrix}0\\0\\0\\0\end{pmatrix}\right\}$ in $\mathbb{R}^4$.

Solution :

(a) **Method 1 (by hand):**

Suppose $\alpha\begin{pmatrix}1\\1\\0\end{pmatrix} + \beta\begin{pmatrix}1\\0\\1\end{pmatrix} + \gamma\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

Then $\alpha + \beta = 0 \quad \ldots (1)$

$\alpha + \gamma = 0 \quad \ldots (2)$

$\beta + \gamma = 0 \quad \ldots (3)$

$(1) - (2) \qquad \beta - \gamma = 0 \quad \ldots (4)$

$(3) + (4) \qquad 2\beta = 0$, i.e. $\beta = 0$

From (4) & (1), $\gamma = 0$ and $\alpha = 0$.

Hence $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is linearly independent in $\mathbb{R}^3$, i.e. **not linearly dependent** in $\mathbb{R}^3$.

Method 2 (by GC):

Suppose $\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

$2\alpha+\beta=0$

$\gamma=0$

$2\alpha+\beta+\gamma=0$

Then $\alpha+\beta=0$ … (1)

$\alpha+\gamma=0$ … (2)

$\beta+\gamma=0$ … (3)

From GC, $\alpha=0$, $\beta=0$ and $\gamma=0$ is the only solution.

Hence $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is linearly independent in $\mathbb{R}^3$, i.e. **not linearly dependent** in $\mathbb{R}^3$.

Method 3 (by determinant) :

Suppose $\alpha\begin{pmatrix}1\\1\\0\end{pmatrix}+\beta\begin{pmatrix}1\\0\\1\end{pmatrix}+\gamma\begin{pmatrix}0\\1\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$.

Consider $\det\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}$.

Since $\det\begin{pmatrix}1&1&0\\1&0&1\\0&1&1\end{pmatrix}=$ -2 $\therefore$ unique soln

$\left\{\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$ is linearly independent in $\mathbb{R}^3$, i.e. **not linearly dependent** in $\mathbb{R}^3$.

(b) **Method 1 (by hand):**

show

Suppose $\alpha\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+\beta\begin{pmatrix}2\\4\\2\\10\end{pmatrix}+\gamma\begin{pmatrix}0\\2\\0\\3\end{pmatrix}\neq\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$. to be linearly ind.

Then,
$$\begin{aligned} \alpha+2\beta &= 0 \quad \text{-----(1)}\\ 4\beta+2\gamma &= 0 \quad \text{-----(2)}\\ \alpha+2\beta &= 0\\ 2\alpha+10\beta+3\gamma &= 0 \quad \text{-----(3)} \end{aligned}$$

From (1), $\alpha=-2\beta$ -----(4)

Subst (4) into (3), $6\beta+3\gamma=0 \Rightarrow \gamma=-2\beta$ -----(5)

In fact, (2) and (5) are the same.

So, $\alpha\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+\beta\begin{pmatrix}2\\4\\2\\10\end{pmatrix}+\gamma\begin{pmatrix}0\\2\\0\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Rightarrow -2\beta\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+\beta\begin{pmatrix}2\\4\\2\\10\end{pmatrix}-2\beta\begin{pmatrix}0\\2\\0\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

In particular, if we choose $\beta=1$, we get $-2\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+\begin{pmatrix}2\\4\\2\\10\end{pmatrix}-2\begin{pmatrix}0\\2\\0\\3\end{pmatrix}=\begin{pmatrix}-2+2-0\\0+4-4\\-2+2-0\\-4+10-6\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

Hence $\left\{\begin{pmatrix}1\\0\\1\\2\end{pmatrix},\begin{pmatrix}2\\4\\2\\10\end{pmatrix},\begin{pmatrix}0\\2\\0\\3\end{pmatrix}\right\}$ is linearly dependent in $\mathbb{R}^4$.

Method 2 (by GC):

Suppose $\alpha\begin{pmatrix}1\\0\\1\\2\end{pmatrix}+\beta\begin{pmatrix}2\\4\\2\\10\end{pmatrix}+\gamma\begin{pmatrix}0\\2\\0\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

Then,
$$\begin{aligned} \alpha+2\beta &= 0 \quad \text{-----(1)}\\ 4\beta+2\gamma &= 0 \quad \text{-----(2)}\\ \alpha+2\beta &= 0\\ 2\alpha+10\beta+3\gamma &= 0 \quad \text{-----(3)} \end{aligned}$$

From GC, $\alpha=\gamma$, $\beta=-\frac{1}{2}\gamma$ and $\gamma=\gamma$.

Hence $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 2 \\ 10 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 3 \end{pmatrix} \right\}$ is linearly dependent in $\mathbb{R}^4$.

Method 3 (by Row Operations):

Method 4 (by Observation)

(c)

Useful Results

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be elements in a real vector space V.

(a) If one of the $\mathbf{v}_i$'s is $\mathbf{0}$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly dependent in V.**

(b) In particular, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if $\mathbf{v}_1$ or $\mathbf{v}_2$ can be written as a scalar multiple of the other.

(c) The columns or rows of a square matrix are linearly independent if and only if the determinant is non-zero.

5 Basis and Dimension of a Real Vector Space

5.1 Basis

Let $(V, \oplus, \otimes)$ be a vector space over $\mathbb{R}$.

Suppose that there exists a **finite set of** elements $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ such that

(a) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ **spans** V **and**

(b) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is **linearly independent.**

Then the set of elements $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is said to be a **finite basis** for V.

Note: For the current syllabus, we consider **finite basis** only.

Example 12

Show that $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ and $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ are bases (plural for basis) for $\mathbb{R}^3$ with respect to the standard operations.

Solution :

Consider $\begin{pmatrix}x\\y\\z\end{pmatrix} \in \mathbb{R}^3$, then $\begin{pmatrix}x\\y\\z\end{pmatrix} = x\begin{pmatrix}1\\0\\0\end{pmatrix} + y\begin{pmatrix}0\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\0\\1\end{pmatrix}$

Thus, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ spans $\mathbb{R}^3$.

Consider $a\begin{pmatrix}1\\0\\0\end{pmatrix} + b\begin{pmatrix}0\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Then, the only solution to this eqn is $a=b=c=0$

So, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is linearly independent.

So, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is a basis for $\mathbb{R}^3$ with respect to the standard operations.

Consider $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \left(\tfrac{1}{2}x+\tfrac{1}{2}y-\tfrac{1}{2}z\right)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left(-\tfrac{1}{2}x+\tfrac{1}{2}y+\tfrac{1}{2}z\right)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + \left(\tfrac{1}{2}x-\tfrac{1}{2}y+\tfrac{1}{2}z\right)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

So, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans $\mathbb{R}^3$.

Consider $a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then, using GC, the only soln is $a=b=c=0$

So, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

So, $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is another basis for $\mathbb{R}^3$ with respect to the standard operations.

Note

1) Basis of a vector space is **<u>not</u>** unique.
2) A basis for a vector space V contains the **smallest** possible number of vectors that span V.
3) $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is known as the **standard basis** for $\mathbb{R}^3$.
4) The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}$ is a (finite) **standard basis** for $\mathbb{R}^n$ with respect to the standard operations.
   In particular, $\{1\}$ is a basis for the set of real numbers with respect to the usual real number addition and scalar multiplication.

Theorem 5.1 (Uniqueness of Basis Representation)

Suppose that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis of a vector space V. Then every vector V can be expressed **uniquely** as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
i.e. if $\mathbf{v} \in V$ such that

$$\mathbf{v} = (\alpha_1 \otimes \mathbf{v}_1) \oplus (\alpha_2 \otimes \mathbf{v}_2) \oplus \cdots \oplus (\alpha_n \otimes \mathbf{v}_n) = (\beta_1 \otimes \mathbf{v}_1) \oplus (\beta_2 \otimes \mathbf{v}_2) \oplus \cdots \oplus (\beta_n \otimes \mathbf{v}_n),$$

then $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \ldots, \alpha_n = \beta_n$.

5.2 Dimension of a Vector Space

Theorem 5.2

Let V be a real vector space with basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. Let $S=\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of k vectors in V.
(a) If $k > n$, then S is linearly dependent.
(b) If $k < n$, then S does not span V.

Theorem 5.3 (Uniqueness of Dimension)

Let V be a real vector space with two bases $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$, where m and n are positive integers. Then $m = n$.

By virtue of the above Theorem, the following is well-defined:
Let V be a real vector space spanned by a (finite) basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$.
The **dimension of** V is defined as **the number of spanning elements in a basis**, i.e. n , and is denoted by **dim** V.
We define the dimension of the trivial vector space $\{\mathbf{0}\}$ to be zero.

Example 13

$(\mathbb{R}^n, +, \cdot)$, with the usual addition and scalar multiplication, is a vector space of dimension n over $\mathbb{R}$.
In particular, $\mathbb{R}$ with the standard operations is a vector space of dimension 1 over $\mathbb{R}$;
$\mathbb{R}^2$ with the standard operations is a vector space of dimension 2 over $\mathbb{R}$;
$\mathbb{R}^3$ with the standard operations is a vector space of dimension 3 over $\mathbb{R}$.

Example 14

Let $U=\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 : b+c+d=0\right\}$ and $W=\left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 : a+b=0\ ,\ c=2d\right\}$ be subspaces of $\mathbb{R}^4$. Find a basis for U and W, and determine their dimensions.

✓ span
✓ linearly ind

Solution:

$$U = \left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 : b = -c-d\right\}$$

$$= \begin{pmatrix} a \\ -c-d \\ c \\ d \end{pmatrix}$$

$$= a\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \ \cancel{a,c,d}\ a, c, d \in \mathbb{R}$$

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \in \mathbb{R}^4 : a+b=0,\ c=2d \right\}$$

$$= \left\{ \begin{pmatrix} -b \\ b \\ 2d \\ d \end{pmatrix} \in \mathbb{R}^4 \right\}$$

$$= b\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

Ws basis is $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$ and $\dim W = 2$.

Useful Result

Let U be a subspace of a vector space V. Then $0 \le \dim U \le \dim V$.

If $U \ne \{\mathbf{0}\}$ or $U \ne V$, then $0 < \dim U < \dim V$.

Theorem 5.4

A vector space $(V, \oplus, \otimes)$ has dimension n, and S is a set of vectors in V with exactly n vectors. Then, S is a basis for V if and only if either S spans V or S is linearly independent. That is,
(a) If S spans V, then S is linearly independent.
(b) If S is linearly independent, then S spans V.

For example, $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$ is linearly independent in $\mathbb{R}^4$.

$\mathbb{R}^4$ is a vector space of dimension 4 with respect to the standard operations.

Hence from the above theorem, $\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}\right\}$ which are linearly independent, can also be a basis for $\mathbb{R}^4$ although we did not check whether the set spans $\mathbb{R}^4$.

Theorem 5.5

Let V be a nonzero vector space.
(a) Every set of linearly independent vectors in V can be enlarged to a basis of V.
(b) Every spanning set for V can be reduced to a basis of V.

Example 15

Find a basis of $\mathbb{R}^3$ that contains the vector (1,2,3). Justify your answer.

Summary