National Junior College
2016 – 2017 H2 Mathematics
Complex Numbers **Tutorial Solutions**

Basic Mastery Questions

1 (a) $(3-8\text{i})(5+7\text{i})=15+21\text{i}-40\text{i}+56$
$=71-19\text{i}$

(b) $\dfrac{7+5\text{i}}{4-3\text{i}}=\left(\dfrac{7+5\text{i}}{4-3\text{i}}\right)\left(\dfrac{4+3\text{i}}{4+3\text{i}}\right)$
$=\dfrac{28+21\text{i}+20\text{i}-15}{16+9}$
$=\dfrac{13}{25}+\dfrac{41}{25}\text{i}$

2 (a) $\left|-2\text{i}\right|=2$, $\arg(-2\text{i})=-\dfrac{\pi}{2}$
$-2\text{i}=2\text{e}^{\text{i}\left(-\frac{\pi}{2}\right)}=2\left[\cos\left(-\dfrac{\pi}{2}\right)+\text{i}\sin\left(-\dfrac{\pi}{2}\right)\right]$

(b) $\left|-1+\sqrt{3}\text{i}\right|=\sqrt{1+3}=2$
$\arg\left(-1+\sqrt{3}\text{i}\right)=\pi-\dfrac{\pi}{3}=\dfrac{2}{3}\pi$
$-1+\sqrt{3}\text{i}=2\text{e}^{\text{i}\frac{2\pi}{3}}$
$=2\left(\cos\dfrac{2\pi}{3}+\text{i}\sin\dfrac{2\pi}{3}\right)$

(c) $\left|(-1-\text{i})\left(-1+\sqrt{3}\text{i}\right)\right|=\left|-1-\text{i}\right|\left|-1+\sqrt{3}\text{i}\right|=2\sqrt{2}$
$\arg\left[(-1-\text{i})(-1+3\text{i})\right]$
$=\arg(-1-\text{i})+\arg(-1+3\text{i})$
$=-\dfrac{3\pi}{4}+\dfrac{2\pi}{3}$
$=-\dfrac{\pi}{12}$
$(-1-\text{i})\left(-1+\sqrt{3}\text{i}\right)=2\sqrt{2}\text{e}^{-\text{i}\frac{\pi}{12}}$
$=2\sqrt{2}\left(\cos\left(-\dfrac{\pi}{12}\right)+\text{i}\sin\left(-\dfrac{\pi}{12}\right)\right)$

(d) $\left|\dfrac{-1+\sqrt{3}\text{i}}{-1-\text{i}}\right|=\dfrac{\left|-1+\sqrt{3}\text{i}\right|}{\left|-1-\text{i}\right|}=\dfrac{2}{\sqrt{2}}=\sqrt{2}$

$$\arg\left(\frac{-1+\sqrt{3}\mathrm{i}}{-1-\mathrm{i}}\right)$$
$$=\arg\left(-1+\sqrt{3}\mathrm{i}\right)-\arg\left(-1-\mathrm{i}\right)$$
$$=\frac{2}{3}\pi+\frac{3\pi}{4}$$
$$=\frac{17}{12}\pi$$
$$\text{Principal}\arg(z)=\frac{17}{12}\pi-2\pi$$
$$=-\frac{7\pi}{12}$$
$$\frac{-1+\sqrt{3}\mathrm{i}}{-1-\mathrm{i}}=\sqrt{2}\mathrm{e}^{\mathrm{i}\left(-\frac{7\pi}{12}\right)}$$
$$=\sqrt{2}\left[\cos\left(-\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{12}\right)\right]$$

(e) $$1+\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}=1+\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}=1+\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}=\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}$$
$$\left|\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right|=\sqrt{\frac{9}{4}+\frac{3}{4}}=\sqrt{\frac{12}{4}}=\sqrt{3}$$
$$\arg\left(\frac{3}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)=\tan^{-1}\frac{\sqrt{3}}{3}=\frac{\pi}{6}$$
$$1+\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}=\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$
$$=\sqrt{3}\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$$

Alternative Method for 2(e):
$$1+\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}=\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\left(\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}+\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\right)$$
$$=\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\left(\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)+\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$$
$$=\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}\left(2\cos\frac{\pi}{6}\right)=(2)\left(\frac{\sqrt{3}}{2}\right)\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}=\sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$

(f) $$\left|3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right|=3,\ \arg\left(3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}\right)=\frac{\pi}{3},\ 3\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}=3\left(\cos\frac{\pi}{3}+\mathrm{i}\sin\frac{\pi}{3}\right)$$

(g) $$|-100|=100,\ \arg(-100)=\pi,\ -100=100(\cos\pi+\mathrm{i}\sin\pi)=100\mathrm{e}^{\mathrm{i}\pi}$$

(h) $$-3\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}=3(-1)\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}=3\mathrm{e}^{\mathrm{i}\pi}\times\mathrm{e}^{\mathrm{i}\frac{5\pi}{6}}=3\mathrm{e}^{\mathrm{i}\frac{11\pi}{6}}\equiv3\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{6}-2\pi\right)}=3\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{6}\right)}$$
$$=3\left[\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]$$

Alternatively,

$$\left|-3e^{i\frac{5\pi}{6}}\right| = |-3|\left|e^{i\frac{5\pi}{6}}\right| = 3$$

$$\arg\left(-3e^{i\frac{5\pi}{6}}\right) = \arg(-3) + \arg\left(e^{i\frac{5\pi}{6}}\right) = \pi + \frac{5\pi}{6} = \frac{11\pi}{6} \equiv -\frac{\pi}{6}$$

$$\therefore -3e^{i\frac{5\pi}{6}} = 3\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

3 (a) $2\left[\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right] = 2\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\sqrt{3} - i$

(b) $2e^{i\left(-\frac{\pi}{3}\right)} = 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

4 $\sqrt{-1}\sqrt{-1} \neq \sqrt{1}$

Practice Questions

1 $z = -\sqrt{3} + i \qquad w = 4 + 4i$

$|z| = 2 \qquad |w| = 4\sqrt{2}$

$\arg(z) = \frac{5\pi}{6} \qquad \arg(w) = \frac{\pi}{4}$

(i) $\left|-\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{2}$

$\arg\left(-\frac{1}{z}\right) = \arg(-1) - \arg(z) = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$

$-\frac{1}{z} = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

(ii) $\left|\frac{1}{z^*}\right| = \frac{1}{|z^*|} = \frac{1}{|z|} = \frac{1}{2}$

$\arg\left(\frac{1}{z^*}\right) = \arg(1) - \arg(z^*) = 0 + \arg(z) = \frac{5\pi}{6}$

$\frac{1}{z^*} = \frac{1}{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

(iii) $\left|(w^*)^3\right| = |w^*|^3 = |w|^3 = \left(4\sqrt{2}\right)^3 = 128\sqrt{2}$

$\arg(w^*)^3 = 3\arg(w^*) = -3\arg(w) = -3\left(\frac{\pi}{4}\right) = -\frac{3\pi}{4}$

$$(w^*)^3 = 128\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{3\pi}{4}\right)\right)$$

(iv) $$\left|\frac{z^*}{w}\right| = \frac{|z|}{|w|} = \frac{2}{4\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

$$\arg\left(\frac{z^*}{w}\right) = \arg(z^*) - \arg(w) = -\arg(z) - \arg(w) = -\frac{5\pi}{6} - \frac{\pi}{4} = -\frac{13\pi}{12}$$

$$\text{Principal } \arg\left(\frac{z^*}{w}\right) = -\frac{13\pi}{12} + 2\pi = \frac{11\pi}{12}$$

$$\frac{z^*}{w} = \frac{\sqrt{2}}{4}\left(\cos\frac{11\pi}{12} + \mathrm{i}\sin\frac{11\pi}{12}\right)$$

(v) $$\left|z^2w^3\right| = |z|^2|w|^3 = 2^2(128\sqrt{2}) = 512\sqrt{2}$$

$$\arg(z^2w^3) = 2\arg(z) + 3\arg(w) = 2\left(\frac{5\pi}{6}\right) + 3\left(\frac{\pi}{4}\right) = \frac{29\pi}{12}$$

$$\text{Principal } \arg(z^2w^3) = \frac{29\pi}{12} - 2\pi = \frac{5\pi}{12}$$

$$z^2w^3 = 512\sqrt{2}\left(\cos\frac{5\pi}{12} + \mathrm{i}\sin\frac{5\pi}{12}\right)$$

2 (i) $$\left|\frac{2p}{w^2}\right| = \frac{2|p|}{|w|^2} = \frac{3}{2}$$

$$\arg\left(\frac{2p}{w^2}\right) = \arg(2) + \arg(p) - 2\arg(w)$$

$$= 0 + \frac{7\pi}{8} - 2\left(-\frac{5\pi}{8}\right) = \frac{17\pi}{8} \equiv \frac{\pi}{8}$$

(ii) $$\left(\frac{2p}{w^2}\right)^n = \left(\frac{3}{2}\right)^n\left(\cos\frac{n\pi}{8} + i\sin\frac{n\pi}{8}\right)$$

For $\left(\frac{2p}{w^2}\right)^n$ to be purely imaginary,

$$\cos\left(\frac{n\pi}{8}\right) = 0$$

$$\frac{n\pi}{8} = \ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\frac{n\pi}{8} = \frac{(2k+1)\pi}{2}, k \in \mathbb{Z}$$

$$n = 4(2k+1), k \in \mathbb{Z}$$

$$= \ldots, -12, -4, 4, 12, \ldots$$

Thus, smallest positive value of $n = 4$.

3 (i) $|p| = \left|\dfrac{w}{w^*}\right| = \dfrac{|w|}{|w^*|} = \dfrac{|w|}{|w|} = 1$; $\arg p = \arg\dfrac{w}{w^*} = \arg w - \arg w^* = 2\arg w = 2\theta$

(ii) $p = e^{2i\theta} \Rightarrow p^5 = e^{10i\theta}$

Since p^5 is real, $\sin 10\theta = 0$ and $\cos 10\theta > 0$. This implies 10θ is of a multiple of 2π.

Solving $\sin 10\theta = 0$, we have $\theta = \dfrac{k\pi}{5}, k = 1, 2, 3, 4$. Thus, $\theta = \dfrac{\pi}{5}$ or $\dfrac{2\pi}{5}$.

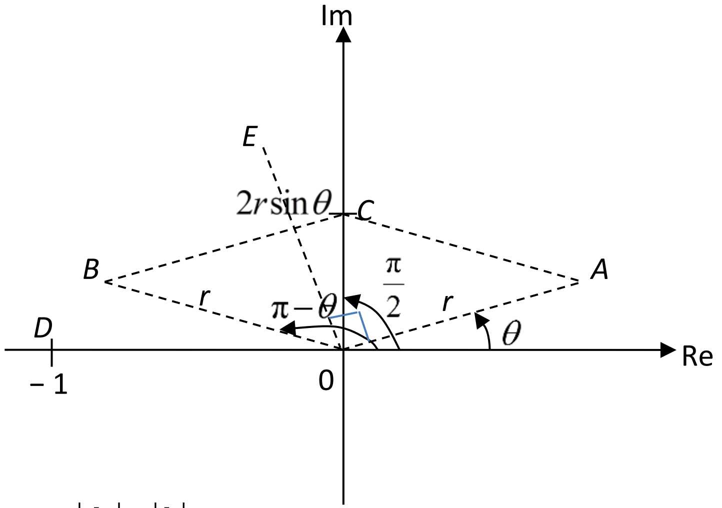
Alternatively,

p^5 is real and positive $\Rightarrow \arg(p^5)$ is of a multiple of 2π.

$10\theta = 2k\pi,\ k \in \mathbb{Z}$

$$\theta = \frac{k\pi}{5} = \frac{\pi}{5} \text{ or } \frac{2\pi}{5} \text{ since } 0 < \theta < \frac{1}{2}\pi$$

4



$$\left|\frac{b}{a^*}\right| = \left|\frac{b}{a}\right| = 1$$

$$\arg\left(\frac{b}{a^*}\right) = \arg(b) + \arg(a) = \pi$$

5

$$\mathrm{e}^{2\alpha\mathrm{i}}+\mathrm{e}^{-2\alpha\mathrm{i}}$$
$$=\cos 2\alpha+\mathrm{i}\sin 2\alpha+\cos(-2\alpha)+\mathrm{i}\sin(-2\alpha)$$
$$=2\cos 2\alpha \text{ ,which is real for all } \alpha$$
$$w=\frac{2}{1+\mathrm{e}^{4\alpha\mathrm{i}}}=\frac{2}{\mathrm{e}^{2\alpha\mathrm{i}}\left(\mathrm{e}^{-2\alpha\mathrm{i}}+\mathrm{e}^{2\alpha\mathrm{i}}\right)}$$
$$=\frac{2\mathrm{e}^{-2\alpha\mathrm{i}}}{2\cos 2\alpha}$$
$$=\frac{\cos 2\alpha-\mathrm{i}\sin 2\alpha}{\cos 2\alpha}$$
$$=1-\mathrm{i}\tan 2\alpha$$
$$\therefore \mathrm{Re}(w)=1$$

Alternatively,

$$w=\frac{2}{1+\mathrm{e}^{4\alpha\mathrm{i}}}\cdot\frac{1+\mathrm{e}^{-4\alpha\mathrm{i}}}{1+\mathrm{e}^{-4\alpha\mathrm{i}}}$$
$$=\frac{2\left(1+\mathrm{e}^{-4\alpha\mathrm{i}}\right)}{1+\mathrm{e}^{-4\alpha\mathrm{i}}+\mathrm{e}^{4\alpha\mathrm{i}}+1}$$
$$=\frac{2\left(1+\mathrm{e}^{-4\alpha\mathrm{i}}\right)}{2+2\cos 4\alpha}$$
$$=\frac{1+\cos(-4\alpha)+\mathrm{i}\sin(-4\alpha)}{1+\cos 4\alpha}$$
$$=\frac{1+\cos 4\alpha-\mathrm{i}\sin 4\alpha}{1+\cos 4\alpha}=1-\mathrm{i}\frac{\sin 4\alpha}{1+\cos 4\alpha}$$
$$=1-\mathrm{i}\frac{2\sin 2\alpha\cos 2\alpha}{1+\left(2\cos^2 2\alpha-1\right)}$$
$$=1-\mathrm{i}\frac{\sin 2\alpha}{\cos 2\alpha}=1-\mathrm{i}\tan 2\alpha \quad \therefore \mathrm{Re}(w)=1$$

6 Let $w=a+b\mathrm{i}$

$$ww^*+2w=3+4i$$
$$\left(a^2+b^2\right)+2\left(a+b\mathrm{i}\right)=3+4\mathrm{i}$$
$$\begin{cases}a^2+b^2+2a=3\\2b=4\end{cases}\Rightarrow\begin{cases}b=2\\a=-1\end{cases}\Rightarrow w=-1+2\mathrm{i}$$

7 $\mathrm{i}z+2w=1\Rightarrow -z+2\mathrm{i}w=\mathrm{i}\Rightarrow z=2\mathrm{i}w-\mathrm{i}---(1)$

$4z+(2-i)w^*=-6---(2)$

Substitute (1) into (2),

$4(2\mathrm{i}w-\mathrm{i})+(3-\mathrm{i})w^*=-6$

Let $w=x+\mathrm{i}y$

$8\mathrm{i}(x+\mathrm{i}y)+(3-\mathrm{i})(x-\mathrm{i}y)=-6+4\mathrm{i}$

$8ix-8y+3x-3iy-ix-y=-6+4i$
$(-8y+3x-y)+(8x-x-3y)i=-6+4i$

Compare real and imaginary parts,
$-9y+3x=-6 \Rightarrow -3y+x=-2---(3)$
$7x-3y=4----(4)$
Solving (3) & (4)
$7(3y-2)-3y=4 \Rightarrow 18y=18$
$\Rightarrow y=1 \Rightarrow x=1$

So $w=1+i$ $\Rightarrow z=2i(1+i)-i=-2+i$

8 $(x+iy)^2=12i-5$
$x^2+2iy-y^2=12i-5$
Comparing real and imaginary parts,
$x^2-y^2=-5$ --- (1)
$2xy=12$ --- (2)
From (2), $x=\frac{6}{y}$. Substitute $x=\frac{6}{y}$ into (1), we get
$\frac{36}{y^2}-y^2=-5$ $\therefore y^4-5y^2-36=0$
$y^2=9$ or -4 (N.A. because $y^2 \geq 0$ as y is a real number)
$y=3, x=2$
$y=-3, x=-2$
$x+iy=2+3i$ or $-2-3i$

$z^2+4z=12i-9=12i-5-4$
$z^2+4z+4=12i-5$
$(z+2)^2=12i-5$
$z+2=\pm(2+3i)$
$z=3i$ or $-4-3i$

9 (i) Since $1+i$ is a root of the equation $2w^3+aw^2+bw-2=0$,
$2(1+i)^3+a(1+i)^2+b(1+i)-2=0$
$2(-2+2i)+a(2i)+b(1+i)-2=0$
$(b-6)+(4+2a+b)i=0+0i$

Comparing real parts, Comparing imaginary parts,

$$b-6=0$$
$$b=6$$

$$4+2a+b=0$$
$$a=\frac{-b-4}{2}$$
$$\therefore a=\frac{-6-4}{2}=-5$$

(ii) Since the polynomial equation has real coefficients, $1+\mathrm{i}$ and $1-\mathrm{i}$ are roots to the equation. $2w^3-5w^2+6w-2=\left(w-(1+\mathrm{i})\right)\left(w-(1-\mathrm{i})\right)(2w-A)$

Comparing constants,

$$-A(1+\mathrm{i})(1-\mathrm{i})=-2$$
$$A\left(1-\mathrm{i}^2\right)=2$$
$$A\left(1-(-1)\right)=2$$
$$A=1$$

$$2w^3-5w^2+6w-2=0$$
$$\left(w-(1+\mathrm{i})\right)\left(w-(1-\mathrm{i})\right)(2w-1)=0$$
$$w=1+\mathrm{i},\quad 1-\mathrm{i},\quad \frac{1}{2}.$$

<u>**Alternative solutions** to parts (ii) and (iii)</u>

Since coefficients are real, if first root is 1 + i, then second root is <u>**1 – i**</u>

Quadratic factor is $(w-1-\mathrm{i})(w-1+\mathrm{i})=$ <u>$\mathbf{w^2-2w+2}$</u>

$2w^3+aw^2+bw-2=(w^2-2w+2)(2w-1)$

$=(2w^3-4w^2+4w)+(-w^2+2w-2)$

$=2w^3-5w^2+6w-2$

giving <u>$\mathbf{a=-5}$</u> and <u>$\mathbf{b=6}$</u>

And third root is <u>$\mathbf{w=1/2}$</u>

10 Let $\mathrm{P}(z)=z^3-2z^2+az+1+3\mathrm{i}$

$\Rightarrow \mathrm{P}(\mathrm{i})=i^3-2i^2+ai+1+3\mathrm{i}=0$

$-i+2+ai+1+3\mathrm{i}=0$

$1+2i-a+i-3=0$

$\Rightarrow a=-2+3\mathrm{i}$

Use long division or by comparing coefficient method,

$\mathrm{P}(z)=z^3-2z^2+(-2+3\mathrm{i})z+1+3\mathrm{i}$

$=(z-\mathrm{i})\left[z^2+(-2+\mathrm{i})z-3+\mathrm{i}\right]$

$$z^2+(-2+\mathrm{i})z-3+\mathrm{i}=0 \Rightarrow z=\frac{-(-2+\mathrm{i})\pm\sqrt{(-2+\mathrm{i})^2-4(-3+\mathrm{i})}}{2}$$

$$\Rightarrow z=\frac{(2-\mathrm{i})\pm(4-\mathrm{i})}{2} \Rightarrow z=-1 \text{ or } 3-\mathrm{i}$$

11 $z=\cos\theta+\mathrm{i}\sin\theta=\mathrm{e}^{\mathrm{i}\theta}$

$$3+\mathrm{i}(3z+1)=\mathrm{e}^{\mathrm{i}(\pi+\theta)}$$

$$=\mathrm{e}^{\mathrm{i}\pi}\cdot\mathrm{e}^{\mathrm{i}\theta}=-z$$

$$3+\mathrm{i}=-z-3z\mathrm{i}=z(-1-3\mathrm{i})$$

$$z=-0.6+0.8\mathrm{i}$$

Challenging Questions

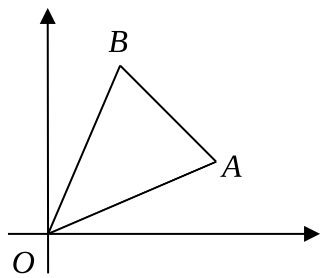
1 Point A represents $2\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}$

Point B represents $2\mathrm{e}^{\mathrm{i}\frac{5\pi}{12}}$

$$\measuredangle BOA=\frac{5\pi}{12}-\frac{\pi}{12}=\frac{\pi}{3}$$

$$|OA|=2=|OB|$$

We can now conclude that triangle ABC is isosceles.



In addition,

$$\therefore \measuredangle OBA=\measuredangle OAB=\frac{1}{2}\left(\pi-\frac{\pi}{3}\right)=\frac{\pi}{3}$$

$\therefore \triangle OAB$ is an equilateral triangle.

2 Since $z=\mathrm{i}$ is a root of $z^3+(1-3\mathrm{i})z^2-(2+2\mathrm{i})z-2=0$, we have

$$(z-\mathrm{i})(z^2+az+b)=0$$

$$z^3+(a-\mathrm{i})z^2+(b-a\mathrm{i})z-\mathrm{i}b=0$$

Comparing coefficient,

$$a-\mathrm{i}=1-3\mathrm{i} \quad \& \quad \mathrm{i}b=2$$

$$a=1-2\mathrm{i} \quad \& \quad b=-2\mathrm{i}$$

$z^2 + az + b = z^2 + (1-2\text{i})z - 2\text{i} = 0$

$$z = \frac{-(1-2\text{i}) \pm \sqrt{(1-2\text{i})^2 - 4(-2\text{i})}}{2}$$

$$= \frac{-1+2\text{i} \pm (1+2\text{i})}{2}$$

$$= 2\text{i} \text{ or } -1$$

$\therefore$ The other roots are $z = 2\text{i}$ & $z = -1$.

$$\left[z^3 + (1-3\text{i})z^2 - (2+3\text{i})z - 2\right]^* = 0^*$$

$$(z^*)^3 + (1-3\text{i})^*(z^*)^2 + (-2-3\text{i})^* z^* - 2 = 0$$

$$(z^*)^3 + (1+3\text{i})(z^*)^2 + (-2+3\text{i})z^* - 2 = 0$$

Note that $w = z^*$, the roots of the equation $w^3 + (1+3\text{i})w^2 + (3\text{i}-2)w - 2 = 0$ are $w = -\text{i}, -2\text{i}, -1$.

3 (i) Substitute $w = x + y\text{i}$ into the first equation to obtain:

$x^2 + y^2 - 16\sqrt{3}\text{i} + 8\text{i}(x + y\text{i}) = 0$

Imaginary parts: $-16\sqrt{3} + 8x = 0 \quad \Rightarrow x = 2\sqrt{3}$

Real parts: $x^2 + y^2 - 8y = 0$

Substitute $x = 2\sqrt{3}$: $y^2 - 8y + 12 = 0$

$(y-6)(y-2) = 0 \quad \Rightarrow y = 2$ or 6 (NA $\because y < 5$)

$w = 2\sqrt{3} + 2\text{i}$

(ii) <u>Method 1:</u>

$w = 2\left(\sqrt{3} + \text{i}\right) = 4e^{\frac{\pi}{6}\text{i}}$

$w^n = \left(4e^{\frac{\pi}{6}\text{i}}\right)^n = 4^n e^{\frac{n\pi}{6}\text{i}} = 4^n\left(\cos\frac{n\pi}{6} + \text{i}\sin\frac{n\pi}{6}\right)$ is real if $\sin\frac{n\pi}{6} = 0$.

$\frac{n\pi}{6} = k\pi$ where $k \in \mathbb{Z}$.

w^n is real provided $n = 6k$ where $k \in \mathbb{Z}$.

<u>Method 2:</u>

$\arg(w) = \frac{\pi}{6} \quad \Rightarrow \quad \arg(w^n) = \frac{n\pi}{6}$

Since w^n is real, $\frac{n\pi}{6} = k\pi$

$n = 6k$ where $k \in \mathbb{Z}$.

(iii) $1+\left(\frac{w}{4}\right)^3+\left(\frac{w}{4}\right)^6+\left(\frac{w}{4}\right)^9+....+\left(\frac{w}{4}\right)^{21}=\frac{\left(\left(\frac{w}{4}\right)^3\right)^8-1}{\left(\frac{w}{4}\right)^3-1}=\frac{\left(e^{\frac{\pi}{6}i}\right)^{24}-1}{e^{\frac{\pi}{2}i}-1}=\frac{e^{4\pi i}-1}{i-1}=0$

(iv)

$\arg(\frac{zi}{1+i})=\arg(z)+\arg i-\arg(1+i)=\frac{3}{4}\pi$

$\Rightarrow \arg(z)=\frac{\pi}{2}$ since $\arg i-\arg(1+i)=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

$\arg(z)=\frac{\pi}{2}$ & $|z|=4 \quad \Rightarrow z=4i$

(v) <u>Method 1:</u>

Area of triangle required $=\frac{1}{2}$ base $\times$ height $=\frac{1}{2}(4)(2\sqrt{3})=4\sqrt{3}$ unit^2

<u>Method 2:</u>

$\angle ZOW=\frac{\pi}{3}$ (using $\arg z-\arg w=\frac{\pi}{2}-\frac{\pi}{6}$)

Area of triangle required

$=\frac{1}{2}(OZ)(OW)\sin\angle ZOW$

$=\frac{1}{2}(4)(4)\sin\frac{\pi}{3}$

$=4\sqrt{3}$ unit^2