# ANDERSON SERANGOON JUNIOR COLLEGE
## JC2 Preliminary Examination 2023
## Higher 2

**FURTHER MATHEMATICS** **9649/02**

**Paper 2** **18 September 2023**

**3 hours**

**Additional Materials:** **Answer Booklets**

**List of Formulae (MF26)**

### READ THESE INSTRUCTIONS FIRST

An answer booklet will be provided with this question paper. You should follow the instructions on the front cover of the booklet. If you need additional answer booklet, ask the invigilator for a continuation booklet.

Write your name and class on the cover page and on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

**1** The points $P$ and $P'$ in the Argand diagram represent the complex numbers $z$ and $z'$ respectively. The transformation $\mathbf{T}: P \to P'$, where $\mathbf{T}(P) = P'$, is given by $z' = \frac{k^2}{z^*}$ where $k$ is a non-zero real constant and $z^*$ is the conjugate of $z$. Show that the line $PP'$ passes through the origin $O$. [2]

Let $Q$ represent the complex number $w$ and $Q' = \mathbf{T}(Q)$. If $P$ and $Q$ lie on the circles with centers at the origin and of radius 2 and 3 respectively, find the value of $k$ for which the ratio of the area of triangle $OP'Q'$ to area of triangle $OPQ$ is $4:1$. Give your answer in exact form. [3]

**2** Let $F_n$ be the sequence of Fibonacci numbers :

$$F_1 = 1, F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for all } n \in \mathbb{Z}, n \geq 3$$

and $L_n$ be the sequence of Lucas numbers:

$$L_1 = 1, L_2 = 3 \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for all } n \in \mathbb{Z}, n \geq 3$$

It can be proven that $L_n = F_{n-1} + F_{n+1}$ for all $n \in \mathbb{Z}, n > 1$.

**(a)** List down the first 6 Fibonacci numbers and Lucas numbers and verify that $L_n = F_{n-1} + F_{n+1}$ is true for $n = 5$. [2]

**(b)** For a fixed $m$ where $m \geq 2$, prove by induction on $n$ that

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1}.$$ [5]

**(c)** Hence or otherwise, show that $F_{2n} = F_n L_n$. [2]

**3** **(a)** Show that $\left(1 - e^{i\theta}\right)^2 = -4e^{i\theta} \sin^2 \frac{\theta}{2}$. [2]

**(b)** For a positive integer $n$, series $C$ and $S$ are given by

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots.. + \cos 2n\theta,$$

$$S = -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots.. + \sin 2n\theta.$$

Show that $C = (-4)^n \cos n\theta \sin^{2n}\left(\frac{1}{2}\theta\right)$, and find a similar expression for $S$. [4]

**(c)** Given that $w = e^{i\theta}$ is a sixth root of 1, where $-\pi < \theta \leq \pi$, find the possible values of $(1-w)^6$ without using a graphing calculator. [5]

**4** The terms of a sequence are denoted by $u_1, u_2, u_3, ....$ and are related by the equation

$$u_{r+1} = 4u_r - u_{r-1} - 6u_{r-2} \;, \; r \geq 3$$

Denoting $\begin{pmatrix} u_r \\ u_{r-1} \\ u_{r-2} \end{pmatrix}$ by $x_r$, the above relation may be written as $x_{r+1} = \mathbf{M}x_r$ where $\mathbf{M}$ is a $3 \times 3$ matrix. Find $\mathbf{M}$ and verify that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 3 \\ 1 \end{pmatrix}$ are eigenvectors of $\mathbf{M}$. [4]

Express $\mathbf{M}$ in the form $\mathbf{PDP}^{-1}$, where $\mathbf{D}$ is a diagonal matrix and $\mathbf{P}$ is a $3 \times 3$ matrix. Hence, obtain a formula for $u_r$ in terms of $r$, given that $u_1 = -7, u_2 = 1$ and $u_3 = -1$. [7]

**5** Consider the equation $\mathrm{f}(x) = 0$ where $\mathrm{f}(x) = \sin x + \ln x, \; x > 0.$

**(i)** Show, without the use of a graph, that there is exactly one root in the interval $(0.5, 1)$. [2]

**(ii)** Find an estimation for the root using linear interpolation once and determine, with clear explanation, whether it is an underestimation or overestimation of the actual root. [5]

**(iii)** Explain why using linear interpolation a second time, with the estimated root in (ii) and 1 as the interval limits, will be unwise. [1]

A student wants to use an iterative formula to find an estimate for the root of $\mathrm{f}(x) = 0$ using $x_0 = 0.5$ as the initial estimate.

**(iv)** By considering a necessary condition, explain why the iterative formula $x_{n+1} = \sin^{-1}\left(\ln \frac{1}{x}\right)$ may not work. [2]

**(v)** By using an appropriate iterative formula, find an estimate for the root, correct to 2 decimal places. Explain why, using standard series from MF26, the initial estimate of $x_0 = 0.5$ is an appropriate one. [4]

**Section B: Probability and Statistics [50 marks]**

**6** The sine distribution, $X$, is defined by the probability density function given by $\mathrm{f}(x)=\frac{\pi}{2}\sin(\pi x), \quad x\in[0,1]$.

**(i)** Find the cumulative distribution function of the sine distribution. [2]

**(ii)** Show that, if $Y=\frac{1}{2}\left(1-\cos(\pi X)\right)$, then $Y$ is the standard uniform distribution i.e. $Y\sim U(0,1)$. [3]

**7** Drink driving is one of the main causes of traffic accidents. Many drivers do not think that their reaction is impaired after drinking 2 cans of beer. To study whether this is true, a researcher randomly selected 20 drivers and got them to complete a reaction test before and after drinking 2 cans of beer. The results (timings in seconds) are as shown in the table below.

| Before | 4.2 | 4.3 | 4.4 | 4.4 | 4.5 | 4.7 | 4.8 | 5.0 | 5.1 | 5.2 |
|---|---|---|---|---|---|---|---|---|---|---|
| After | 4.1 | 4.4 | 4.8 | 4.9 | 4.7 | 5.1 | 5.0 | 5.3 | 5.3 | 5.5 |

| Before | 5.3 | 5.4 | 5.8 | 6.0 | 6.1 | 6.1 | 6.3 | 6.5 | 6.6 | 6.6 |
|---|---|---|---|---|---|---|---|---|---|---|
| After | 5.5 | 5.7 | 5.7 | 5.8 | 6.1 | 6.6 | 6.5 | 6.5 | 6.8 | 6.9 |

Conduct an appropriate $t$-test to determine if drinking 2 cans of beer impairs reaction time, stating clearly a necessary assumption. [5]

**8** Twelve pairs of identical twins are allocated to two training programmes, A and B, with one twin of each pair to each programme. At the end of the training the twins are observed by a panel of expert teachers and, for each pair, the better performer is determined. The results are as follows:

| Pair | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Better Performer | A | A | B | A | A | A | B | A | A | B | A | A |

**(a)** Do these observations provide evidence that programme A is a better training programme than programme B at the 5% level of significance? [4]

From the scoresheets of the panel of expert teachers in which the better performer is given a higher score, the differences in performance score for each pair were tabulated and the order of the pairings based on the magnitude of the absolute differences in descending order is as follows:

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pair | 5 | 9 | 4 | 3 | 1 | 8 | 12 | 2 | 7 | 11 | 6 | 10 |

**(b)** Making use of this additional information, perform another test on whether there is evidence that programme A is a better training programme than programme B at the 5% level of significance. [4]

**9** An experiment was carried out to test if experienced fun claw machine players have a higher chance of 'catching' the prizes as compared to novice players. Two groups of 20 players each were randomly selected, with one group consisting of experienced players and the other consisting of novice players. Each player played a total of 10 games.

The experienced players 'caught' $m$ prizes in total and the novice players 'caught' $r$ prizes in total. A 95% confidence interval for $p_e$, the proportion of prizes 'caught' by the experienced players, was found.

**(i)** Find the largest possible width for the 95% symmetric confidence interval for $p_e$, leaving your answer to 4 decimal places and state the corresponding value of $m$. [4]

**(ii)** It is now known that the 95% symmetric confidence interval found for $p_e$ is $(0.2365, 0.3635)$. Find the value of $m$. [2]

**(iii)** Given that $r = 45$, find a 95% symmetric confidence interval for $p_n$, the proportion of prizes 'caught' by the novice players and state what is meant by "a 95% confidence interval for $p_n$". [4]

**(iv)** State an assumption necessary for the 95% symmetric confidence interval for $p_n$ and $p_e$ to be found. [1]

**(v)** The experienced players claimed that they are better in 'catching' the prizes as compared to the novice players. Discuss briefly whether the confidence intervals support this claim. [1]

**10** A population of N objects is classified according to two attributes *A* and *B*. The numbers $n_1, n_2, n_3, n_4$ in each category, shown in the following table, are such that $n_1 + n_2 + n_3 + n_4 = N$.

| | $A$ | not $A$ |
|---|---|---|
| $B$ | $n_1$ | $n_2$ |
| Not $B$ | $n_3$ | $n_4$ |

Show that if the attributes *A* and *B* are independent, then $n_1 n_4 = n_2 n_3$. [2]

Each item of a random sample of 100 items selected from a day's production at a factory was classified according to quality and the shift during which it was made. The results are given in the following contingency table.

| | | Shift | |
|---|---|---|---|
| | | Day | Night |
| Quality | Good | 48 | 34 |
| | Defective | 8 | 10 |

It is required to test whether the attributes of quality and shift are independent in this day's production. State why it is incorrect to deduce from the fact that $48 \times 10 \neq 34 \times 8$ that quality and shift in the day's production are not independent. [1]

Suppose the contingency table were as follows.

| | | Shift | |
|---|---|---|---|
| | | Day | Night |
| Quality | Good | $48 - m$ | $34 + m$ |
| | Defective | $8 + m$ | $10 - m$ |

Determine the range of positive values of *m* in order that the data would provide significant evidence at the 10% level that quality and shift are not independent. [5]

**11** A store owner claims that the number of customers entering the store in a randomly chosen hour can be modelled using a Poisson distribution with $\lambda$ as the average rate at which the customers are entering the store per hour.

**(i)** State two assumptions that the store owner has to make in order for the claim to be true. [2]

**(ii)** Given that $m$ is the modal number of customers entering the store in a randomly chosen hour, show that $\lambda - 1 \leq m \leq \lambda$. If $\lambda$ is an integer, deduce the modal number, leaving your answer in terms of $\lambda$. [4]

**(iii)** There is at least a 20% chance that there will be exactly $m$ customers entering the store in a randomly chosen hour. Find the range of values for $\lambda$ if $\lambda$ is an integer. [3]

**(iv)** The store opens for 24 hours. Given that $\lambda = 3$, find the probability that there are no new customers entering the store for more than 5 hours if there were no new customers entering the store for more than 2 hours. [3]

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